UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2021-2022

Primer Parcial de Matemáticas: grupo F

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos.

P1 [2 pt] Sea $f(x) = e^{-x/\sqrt{3}} \sin x$.

a) Hallar los puntos que cumplen $f'(x) = 0$ y los puntos donde $f'(x)$ no existe (si los hubiera).

b) Los puntos donde $f(x) = 0$.

c) Calcular, si existen, $\lim_{x\to\pm\infty} f(x)$.

d) Representar $f(x)$ gráficamente

P2 [2 pt] Sea $g(x) = \sqrt{x^2 + x^3}$. Estudiar la existencia de máximo y mínimo absolutos: i) en el intervalo $[-1/2, 1/2]$, ii) en el dominio de g, indicando claramente si en este caso se viola el *Teorema de Valores Extremos*.

P3 [1 pt] Sea el polinomio $f(x) = x^5 + 2x^3 + 7x + 1$.

i) Probar que f es inyectiva (one-to-one)

ii) Encontrar $g'(1)$ donde g denota la *función inversa* de f.

P4 [1 pt] Sea $f(x) = \log x$ en el intevalo $[1/3, 3]$.

i) Acotar f en dicho el intervalo por $f_{\min} \le f(x) \le f_{\max}$, donde $f_{\max}, f_{\min}$ denotan el valor máximo y mínimo de f en el intervalo.

ii) Deducir de la acotación en i) que para todo a, b tal que $1/3 \le a < b \le 3$ se cumple que

$$\frac{\log b - \log a}{b - a} < 3.$$

P5 [1.5+1.5pt] a) Determinar si la sucesión dada por

$$a_{n+1} = \frac{1}{2}\Big(a_n + \frac{7}{a_n}\Big), \qquad a_1 = 2,$$

cumple $a_n^2 > 7$.

b) Discutir si tiene líimite (sólo se aceptan como respuestas o bien una prueba empleando la definición de límite o bien la cita de algún teorema de sucesiones bien establecido).

P6 [1pt] Encontrar $\sum_{m=1}^{2n} \log(m^2)$ en términos de n. (No hace falta prueba por inducción).

SOLUTIONS
PARCIAL 1 - MATEMÁTICAS
202 - 2021

(1) The function $f(x) = e^{-x/\sqrt{3}} \sin x$ is a continuous and derivable function everywhere (product of continuous derivable functions on $\mathbb{R}$). There are no points x's where f' does not exist.

$$f(x) = e^{-x/\sqrt{3}} \sin x$$

$$f'(x) = e^{-x/\sqrt{3}} \left(-\frac{\sin x}{\sqrt{3}} + \cos x \right)$$

a) $f'(x) = 0$ if and only if

$$\sin x = \sqrt{3} \cos x$$

solutions in $[0, 2\pi]$

$x = \frac{\pi}{3}, \ \frac{\pi}{3} + \pi$

1st quadrant, 3rd

obligatorio ponerlo

solutions in $\mathbb{R}$.

$$\boxed{x = \frac{\pi}{3} + n\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots}$$

(ó $n \in \mathbb{Z}$)

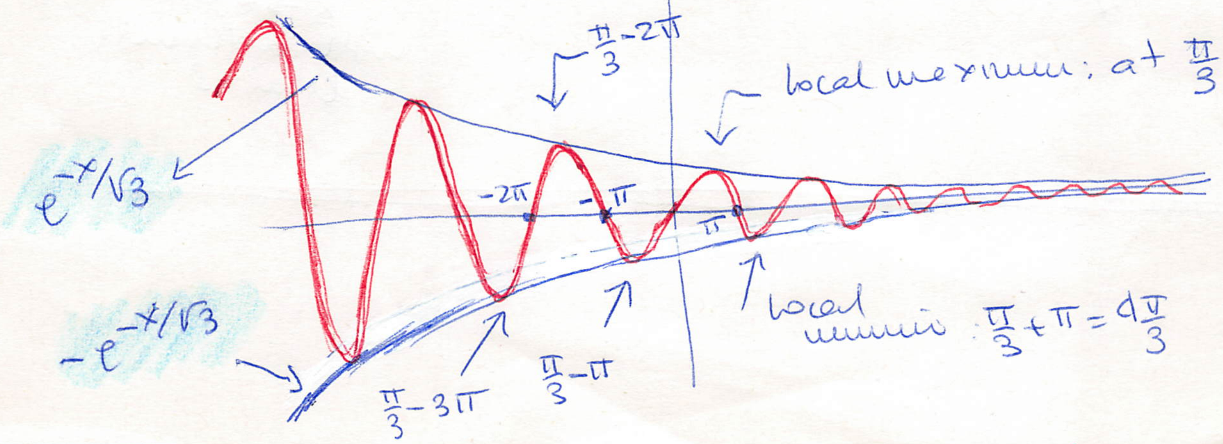
b) $f(x) = 0$ only when

$$\sin x = 0,$$

or equivalently,

$$\boxed{x = 0 + n\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots}$$

c) The graph of the function is (red)



Thus,

modes at $x = n\pi$

local maxima : at $x = \frac{\pi}{3} + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

local minima : at $x = \frac{\pi}{3} + \pi + 2n\pi$

The limits at $x \to \pm\infty$

$$\lim_{x \to \infty} \frac{\sin x}{e^{x/\sqrt{3}}} = \frac{\text{bounded}}{\infty} = \underline{\underline{0}}$$

$$\lim_{x \to -\infty} e^{-x/\sqrt{3}} \cdot \sin x = \not\exists = \infty .$$ oscillant function between $[-1, 1]$

Take the sequence $x_1, x_2, x_3, \ldots$ given by

$x_1 = \frac{\pi}{3} - 2\pi$

$x_2 = \frac{\pi}{3} - 4\pi$ ↙ local maxima points in $x < 0$

$x_3 = \frac{\pi}{3} - 6\pi$

$\vdots \qquad \vdots$

$$\lim_{x \to -\infty} f(x_n) = +\infty .$$

Take now the sequence $x_1', x_2', x_3', \ldots$ with

$x_1' = \frac{\pi}{3} - \pi$

$x_2' = \frac{\pi}{3} - 3\pi$ ↙ local minima points in $x < 0$

$x_3' = \frac{\pi}{3} - 5\pi$

$\vdots \qquad \vdots$

$$\lim_{x \to -\infty} f(x_n') = -\infty$$

two subsequences towards $x \to -\infty$ with different limits

(2) $g(x) = \sqrt{x^2+x^3}$ i) in $[-\frac{1}{2}, \frac{1}{2}]$

ii) in its domain

The function $g(x)$ is continuous where $x^2+x^3 \geq 0$, that is, in $[-1, \infty)$. Square roots have in general graphs with sharp points (like

or) and vertical asymptotes.

Both are points where $g'(x)$ is not defined.

In our case,

$$g'(x) = \frac{1}{2} \frac{2x+3x^2}{(x^2+x^3)^{1/2}}$$

$$= \frac{1}{2} \frac{x(2+3x)}{\sqrt{x^2(1+x)}} \qquad \sqrt{x^2} = |x|$$

$$= \frac{1}{2} \frac{x}{|x|} \frac{2+3x}{\sqrt{1+x}}.$$

The points were $g'(x)$ DNE (does not exits) are

$$x = 0 \quad \& \quad x = -1.$$

<u>Reasons:</u>

At $x=0$, $\frac{x}{|x|}$ is a discontinuous function:

$$\frac{x}{|x|} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

At $x=-1$, $g(x)$ has a vertical asymptote (the denominator of $g'(x)$ vanishes but not the numerator)

i) Fermat theorem says that <u>absolute</u> maxima and minima are to be found among the points:

a) $x = -1/2$, $g(-1/2) = \frac{1}{2\sqrt{2}}$

$x = 1/2$, $g(1/2) = \frac{1}{2}\sqrt{\frac{3}{2}}$

$\frac{1}{4} - \frac{1}{8} = \frac{1}{8}$"

b) points where $g'(x)=0$. None at $[-1/2, 1/2]$.

c) points where $g'(x)$ DNE:

$x=0 \quad g(0)=0$ (el pico)

$x=-1$ but $-1 \notin [-\frac{1}{2}, \frac{1}{2}]$

<u>conclusion</u>: In the interval $[-1/2, 1/2]$ there is

absolute maximum of value $\frac{1}{2}\sqrt{\frac{3}{2}}$ at $x = 1/2$
absolute minimum of value 0 at $x=0$

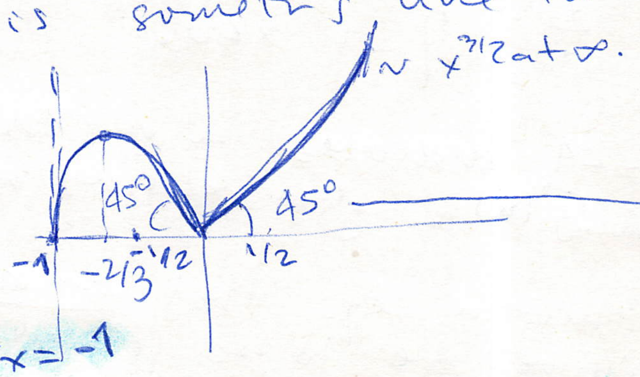
ii) Better we sketch the graph of $\sqrt{x^2+x^3}$

x	-1	$-2/3$	0	1		∞
$f'(x)$	∞	0	$+$	$+$	$+$	$+$
$f(x)$	0	$\frac{2}{3\sqrt{3}}$	0	$\nearrow$	$\nearrow$	$\nearrow$

$f(-2/3) = \sqrt{\frac{4}{9} - \frac{8}{27}} = \frac{2}{3\sqrt{3}}$:

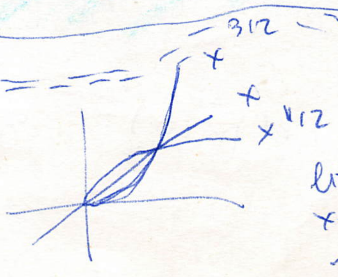
The graph is something like this:

OJO! En el examen no se pedía el dibujo...



The function has in $[-1, \infty)$ absolute minima at $x=0$ with value 0 and at $x=-1$ and <u>has</u> no absolute maxima.

$\frac{12}{27} - \frac{8}{27} = \frac{4}{27}$ " $\frac{2}{3\sqrt{3}}$



$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} \frac{x}{2|x|} \frac{2+3x}{\sqrt{1+x}} = +1$

$\lim_{x\to 0^-} g'(x) = -1$.

This fact, not to have absolute maximum, does not violate the EVT since $[-1,\infty)$ is not a bounded closed interval of $\mathbb{R}$

---

(3) $f(x) = x^5 + 2x^3 + 7x + 1$

i) $f'(x) = 5x^4 + 6x^2 + 7 > 0$ for all $\underline{x \in \mathbb{R}}$,
then $f(x)$ is a strictly increasing function
$\Rightarrow$ f is injective

$\exists f^{-1} = g$: $\quad D_g = (-\infty, \infty)$
$\qquad\qquad\qquad R_g = (-\infty, \infty)$

[Another proof of injectivity, less direct than the previous one, is at the end]

ii) ~~$y = x^5 + 2x^3 + 7x + 1$~~ : direct function
$x = g^5 + 2g^3 + 7g + 1$: inverse function
use implicit derivation together with

$$f(0) = 1 \iff g(1) = 0$$

and write

$$1 = (5g^4 + 6g^2 + 7)\, g'(x),$$

~~from~~ where

$$g'(x) = \frac{1}{5g^4 + 6g^2 + 7},$$

$$g'(1) = \frac{1}{0 + 0 + 7} = \frac{1}{7}.$$

$$\boxed{g(1) = 0}, \quad \boxed{g'(1) = \frac{1}{7}}$$

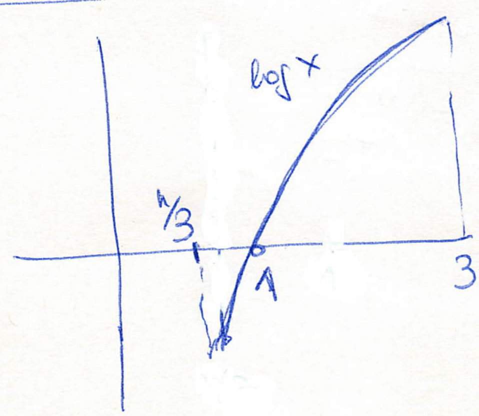
To prove injectivity via
$$f(a)=f(b) \Rightarrow a=b \quad \forall a,b \in \mathbb{R},$$
that is, to prove that
$$a^5+2a^3+7a=b^5+2b^3+7b \qquad [*]$$
has the unique solution $a=b$, is more involved. One possibility is using the MVT applied to
$$h(x)=x^5+2x^3+7x,$$
in $[a,b]$. For example, MVT states that $\exists c \in (a,b)$ s.t.
$$\frac{h(b)-h(a)}{b-a} = \underbrace{5c^4+6c^2+7}$$
↑ this is a number greater than 7

with $[a,b] \subset (-\infty,\infty)$.

and
$$b^5+2b^3+7b-(a^5+2a^3+7a)$$
$$=\left(\begin{array}{c}\text{a number}\\ \text{greater}\\ \text{than } 7\end{array}\right) * (b-a).$$

The only possibility for
$$b^5+2b^3+7b-(a^5+2a^3+7a)=0,$$
which is $[*]$ above, is that
$$b-a=0 \quad \forall a,b$$

P4



$\log x$ is strictly increasing function in $[\frac{1}{3},3]$

a) By the EVT the absolute maximum value of $\log x$ is at $x=3$ and the absolute minimum value at $x=-1/3$ and

$$-\log 3 \leq \log x \leq \log 3 \qquad \forall x \in [\tfrac{1}{3}, 3]$$

b) By the MVT there exist $c \in (a,b)$ with $[a,b] \subset [\frac{1}{3}, 3]$ s.t.

$f(x) = \log x$

$$\frac{\log b - \log a}{b-a} = \frac{1}{c} \qquad , \; f'(x) = \frac{1}{x}$$

Since $\quad c \in (a,b) \subset [\frac{1}{3}, 3]$

$$\frac{1}{3} < c < 3$$

and $\quad 3 > \frac{1}{c} > \frac{1}{3}$.

Consequently,

$$\frac{1}{3} < \frac{\log b - \log a}{b-a} < 3 \qquad \forall a,b \text{ in } [\tfrac{1}{3}, 3].$$

(P5) I write first the result.
The sequence

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right), \quad a_1 = 2$$

is of positive terms, $a_1 = 2$, $a_2 = \frac{11}{4}$, $a_3 = \frac{233}{88}, \ldots$

It is <u>strictly decreasing</u>,

$$a_n > a_{n+1} \qquad \forall n \geq 2$$

(proved in b)) and bounded below

$$a_n^2 > 7 \qquad n \geq 2$$

so it has a limit as the "Theorem of monotonic sequences" states. The limit, is of no importance in the problem (the value of the limit) and the unique possible candidate is $l=\sqrt{7}$, but it is not used anywhere here).

a) $a_1^2 = 4$ is not greater than 7

$a_2 = \frac{11}{4}$ and $a_2^2 > 7$

$a_3 = \frac{233}{88}$ and $a_3^2 > 7$

$a_2^2 = \frac{11}{4} \cdot \frac{11}{4} = \frac{121}{16} > 7$ because $121 > 112 = 16 \cdot 7$

$a_3^2 = \frac{233}{88} \cdot \frac{233}{88} > 7$ because $\underbrace{233 \cdot 233}_{54289} > \underbrace{88 \cdot 88 \cdot 7}_{54208}$

Assume that

$$a_n^2 > 7 \qquad \forall n \geq 2.$$

We deduce that

$$a_{n+1} - \sqrt{7} = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right) - \sqrt{7}$$
$$= \frac{1}{2}\left(a_n + \frac{7}{a_n} - 2\sqrt{7}\right)$$
$$= \frac{1}{2a_n}\left(a_n^2 + 7 - 2\sqrt{7}a_n\right)$$
$$= \frac{1}{2a_n}\left(a_n - \sqrt{7}\right)^2 > 0 \ , \ a_n > 0,$$

what indicates that

$$a_{n+1} > \sqrt{7}$$
$$a_{n+1}^2 > 7$$

quiere decir que cualquier a_{n+1} que se calcule con la regla $a_{n+1} = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right)$ es mayor que $\sqrt{7}$ si $a_n > 0$

(and $a_n^2 > 7$).

⌈ Lo mismo si se hace con cuadrados

$$a_{n+1}^2 - 7 = \frac{1}{4}\left(a_n^2 + \frac{49}{a_n^2} + 14 - 28\right)$$

$$= \frac{1}{4}\left(a_n^2 + \frac{49}{a_n^2} - 14\right)$$

$$= \frac{1}{4}\left(a_n - \frac{7}{a_n}\right)^2 > 0 \quad \forall \underline{\underline{n}},$$

$$a_{n+1}^2 > 7$$

⌊

b) $a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right) - a_n$

$$= \frac{1}{2}\left(-a_n + \frac{7}{a_n}\right)$$

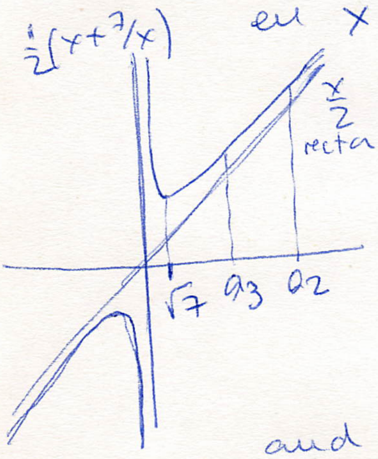
$$= \frac{1}{2}\left(\frac{-a_n^2 + 7}{a_n}\right)$$

< 0 or $\boxed{a_n > a_{n+1}}$

$a_n^2 > 7$
$a_n > 0$

Conclusión: $\{a_n\}_2^\infty$ has a limit. The limit is $l = \sqrt{7}$.

Note: También vale si se demuestra que $f(x) = \frac{1}{2}\left(x + \frac{7}{x}\right)$ tiene un mínimo (resolviendo $f'(x) = 0$) en $x_{min} = \sqrt{7}$ cuyo valor es $\sqrt{7}$. Así pues



$$\frac{1}{2}\left(x + \frac{7}{x}\right) - \sqrt{7} = \frac{1}{2x}\left(x^2 + 7 - 2\sqrt{7}x\right)$$

$$= \frac{1}{2x}\left(x - \sqrt{7}\right)^2$$

If $x > \sqrt{7}$ or $x^2 > 7$
then $\frac{1}{2}\left(x + \frac{7}{x}\right) > \sqrt{7}$

and $\frac{1}{2}\left(x + \frac{7}{x}\right)$ is a_{n+1}.

pero esto mismo que ... más o menos.

(P6) $$\sum_{m=1}^{2n} \log(m^2)$$

Take $n=2$,

$$\sum_{m=1}^{4} \log(m^2) = \log 1^2 + \log 2^2 + \log 3^2 + \log 4^2$$

$$= 2(\log 1 + \log 2 + \log 3 + \log 4)$$

$$= 2 \log (1 \cdot 2 \cdot 3 \cdot 4)$$

$$= 2 \log (4!)$$

Conclusion:

$$\sum_{m=1}^{2n} \log(m^2) = 2 \log \left[(2n)! \right].$$

■