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2021-2022

Solutions:

Segundo Parcial

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2021-2022

Segundo Parcial de Matemáticas: grupos D y F

Nombre y Apellidos:

Firma, DNI y **Grupo:**

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos.

P1 [2 pt] Encontrar el intervalo y radio de convergencia de la serie $\sum_{n=2}^{\infty} \dfrac{x^n}{n\, 2^n \log n}$. Discuta, como siempre, la convergencia en los extremos.

P2 [2pt] Sea

$$f(x) = x^2 \arctan x - x \log(1 + x^n)$$

donde $n = 1, 2, 3 \ldots$. i) Encontrar

$$\lim_{x \to 0} f(x)/x^3$$

para los casos $n = 1, 2, 3, 4$. ii) Determinar **todos** los $n's$ para los que dicho límite es un valor finito **no** nulo e indicar qué valor es éste.

P3 [2pt] a) Calcular $\int_0^{\pi/6} dx\, \cos^3 x$. b) Encontrar $g(0)$ y $g'(0)$ si

$$g(x) = \int_{-1}^{e^{2x}} dt\, \arctan(t^3).$$

c) Encontrar $f(0)$ sabiendo que $f(\pi) = 2$ y que el valor de la integral que sigue es 5.

$$\int_0^{\pi} dx\, (f(x) + f''(x)) \sin x$$

P4 [1pt+1pt] a) Calcular la suma $\sum_{n=2}^{\infty} \dfrac{n-1}{n!}$

b) Determinar la convergencia o divergencia de las siguientes series (diga claramemte el criterio que usted emplea para responder): a) $\sum_{n=1}^{\infty} \dfrac{6^n + \log n}{n! + n^3}$ b) $\sum_{n=1}^{\infty} 3^{n \cos 2}$

P5 [1.5pt] Sea $f(x) = \dfrac{1 + 3x}{1 - x^2}$ i) Determinar el coeficiente de x^8 en su serie de potencias centrada en $x = 0$. ii) Encontrar $f^{(21)}(0)$.

P6 [0.25pt+0.25pt] a) Escribir los cuatro primeros términos no nulos de la serie de potencias de $J_1(x)$ en torno a $x = 0$.

b) Dar el valor exacto de la suma $\sum_{n=0}^{\infty} (-1)^n \dfrac{9^{n-2}}{n!}$

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1 By the ratio test applied to $\sum_{n=2}^{\infty} b_n$ with

$$b_n \equiv \frac{x^n}{n 2^n \log n} \quad , \quad x : \text{arbitrary but fixed.}$$

we have

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{x^{n+1}}{x^n} \frac{n}{n+1} \frac{2^n}{2^{n+1}} \frac{\log n}{\log(n+1)} \right|$$

$$= \frac{|x|}{2} \frac{n}{n+1} \frac{\log n}{\log(n+1)}$$

$$\rho = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \frac{|x|}{2}$$

the best possible type of convergence

If $|x| < 2$ the power series converges absolutely (students say "converges" only). If $|x| > 2$ the series diverges either by oscillation or by being unbounded. For $x = 2, -2$ the test says nothing, so we check explicitly

$x=2$: $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ = divergent by the integral test

$$\int^{\infty} \frac{dx}{x \log x} = \int^{\infty} \frac{d(\log x)}{\log x} = \log(\log x)\Big]^{\infty}$$

$$= \log(\log \infty)) \to \infty.$$ divergent (the integral and the series have the same character)

$x=-2$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ convergent by the AST.

1) $a_n \equiv \frac{1}{n \log n} > 0 \quad \forall n \geq 2$

2) $a_n > a_{n+1}$: because

$$(n+1) \log(n+1) > n \log n \quad \forall n \geq 2$$

so

$$\frac{1}{(n+1) \log(n+1)} < \frac{1}{n \log n}$$

3) $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n \log n} = 0$. No more justification is needed.

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Solutions: Partial 2 Exam

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(2) The problem asks for which values of n the function

$$f(x) = x^2\arctan x - x\log(1+x^n)$$

grows at the same rate than x^3 when $x \to 0$ or, in equivalent words, the values of n for which f(x) has an expansion in power series of the form

$$f(x) = a_3x^3 + a_4x^4 + \dots$$

with <u>a_3 different from zero</u>. In that case,

$$\lim_{x\to 0}\frac{f(x)}{x^3} = a_3 \quad , \quad a_3 \neq 0.$$

as an obvious result.

According to the value of n, the "first powers" of f(x) are:

$x^2\arctan x$	$x\log(1+x^n)$	n
$x^3 - \frac{x^5}{3} + \dots$	$1x^2 - \frac{x^3}{2} + \dots$	1
$1x^3 - \frac{x^5}{3} + \dots$	$1x^3 - \frac{x^5}{2} + \dots$	2
$1x^3 - \frac{x^5}{3} + \dots$	$x^4 - \frac{x^7}{2} + \dots$	3
$x^3 - \frac{x^5}{3} + \dots$	$x^5 - \frac{x^9}{2} + \dots$	4
$x^3 - \frac{x^5}{3} + \dots$	$x^6 - \frac{x^{11}}{2} + \dots$	5
independent of n	clear for other n's...	

- Case $n=1$,

$$f(x) = -x^2 + \frac{3}{2}x^3 + \ldots$$

and

$$\frac{f(x)}{x^3} \underset{\text{when } x\to 0}{\sim} -\frac{1}{x} \underset{x\to 0^+}{\longrightarrow} -\infty$$

Not our n

- Case $n=2$

$$f(x) = x^3[1-1] + x^5\left[-\frac{1}{3}+\frac{1}{2}\right] + x^7(\ldots)$$

$$= \frac{x^5}{6} + x^7(\ldots)$$

and

$$\frac{f(x)}{x^3} \underset{\text{when } x\to 0}{\sim} \frac{x^2}{6} \underset{x\to 0}{\longrightarrow} 0$$

not our n either

- Case $n=3$,

$$f(x) = x^3 - x^4 - \frac{x^5}{3} + \ldots$$

$$\frac{f(x)}{x^3} \underset{\text{when } x\to 0}{\simeq} \frac{x^3}{x^3} \underset{x\to 0}{\longrightarrow} 1$$

our n !!

- case $n=4$

$$f(x) = x^3 + x^5(\ldots),$$

$$\frac{f(x)}{x^3} = 1 + x^2(\ldots) \underset{x\to 0}{\longrightarrow} 1$$

our n too

And so on (see the table above). The conclusion is that for

$$n \geq 3,$$

$f(x)$ behaves as x^3 when x is close to 0 and

$$\lim_{x\to 0} \frac{f(x)}{x^3} = 1, \quad \boxed{n = 3, 4, 5, \ldots}$$

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a) $\int_0^{\pi/6} dx \cos^3 x$ ← odd

Immediate primitive: $\cos^2 x \cos x \, dx$

$$= (1-\sin^2 x)\, d(\sin x)$$

$$\int_0^{\pi/6} dx \cos^3 x = \int_0^{1/2} (1-t^2)\, dt = \frac{1}{2} - \left[\frac{t^3}{3}\right]_0^{1/2}$$

$t = \sin x$

$dt = \cos x \, dx$

$$= \frac{1}{2} - \frac{1}{8\cdot 3}$$

$$= \frac{12}{24} - \frac{1}{24} = \frac{11}{24}$$

$$\boxed{\int_0^{\pi/6} dx \cos^3 x = \frac{11}{24}}$$

b) $g(x) = \int_{-1}^{e^{2x}} dt \arctan(t^3)$

$g(0) = \int_{-1}^{1} dt \arctan t^3 = \underline{\underline{0}}$ by symmetry.

[It is the integral of a continuous odd function over a symmetrical interval around the origin]

$$g'(x) = 2e^{2x} \arctan e^{6x}$$

$$g'(0) = 2 \arctan 1 = \frac{\pi}{2}.$$

c) By parts

$$\int_0^{\pi} dx f(x) \sin x = \int_0^{\pi} d(-\cos x) f(x)$$

$$= -\left[\cos x f(x)\right]_0^{\pi} + \int_0^{\pi} dx \cos x f'(x)$$

$$= f(\pi) + f(0) + \int_0^{\pi} dx \cos x f'(x)$$

parts again

$$\int_0^\pi dx\, f''(x)\sin x = \int_0^\pi df'(x)\sin x$$

$$= \left[f'(x)\sin x\right]_0^\pi - \int_0^\pi dx\, f'(x)\cos x$$

$$= \underbrace{f'(\pi)\sin\pi}_{0} - \underbrace{f'(0)\sin 0}_{0} - \int_0^\pi dx\, f'(x)\cos x$$

$$= -\int_0^\pi dx\, f'(x)\cos x$$

Since

$$5 = \int_0^\pi dx\, f(x)\sin x + \int_0^\pi dx\, f''(x)\sin x$$

$$\underset{\text{two previous results}}{=} f(\pi) + f(0) + \int_0^\pi dx\, \cancel{f'(x)\cos x} - \int_0^\pi dx\, \cancel{f'(x)\cos x}$$

$$= f(\pi) + f(0)$$

$$\underset{\text{in the problem}}{=} 2 + f(0) \Rightarrow \boxed{f(0) = 3}$$

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(4) a) $\sum_{n=2}^{\infty} \frac{n-1}{n!}$ is a telescoping series

$$\sum_{n=2}^{\infty}\left[\frac{1}{(n-1)!} - \frac{1}{n!}\right] = \frac{1}{1!} - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} + \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}} + \cancel{\frac{1}{4!}} - \cancel{\frac{1}{5!}} + \cancel{\frac{1}{5!}} - \cancel{\frac{1}{6!}} + \dots = 1$$

informal

Partial sums

$$S_1 = \frac{1}{1!} - \frac{1}{2!}$$

↙ Forward

$$S_2 = \frac{1}{1!} - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \frac{1}{3!}$$

$$S_3 = \frac{1}{1!} - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} + \cancel{\frac{1}{3!}} - \frac{1}{4!} = \frac{1}{1!} - \frac{1}{4!}$$

$\vdots$

$$S_N = \frac{1}{1!} - \frac{1}{N!} \xrightarrow[N \to \infty]{} 1.$$

Consequently, $\boxed{\sum_{n=2}^{\infty} \frac{n-1}{n!} = 1}$

b) $\sum_{n=1}^{\infty} \frac{6^n + \log n}{n! + n^3}$ convergent

by : - comparison by inequalities
- comparison by the limit: has the same character as

$$\sum_{n=1}^{\infty} \frac{6^n}{n!} = \text{convergent (by the ratio test)} = e^6 - 1$$

↑ by power series.

Inequalities: $n! + n^3 > n! \quad \forall n$

$$6^n + \log n < 6^n + 6^n = 2 6^n$$

$$0 < \frac{6^n + \log n}{n! + n^3} < \frac{2 6^n}{n!}$$

$$0 < \sum^{\infty} \frac{6^n + \log n}{n! + n^3} < 2 \sum^{\infty} \frac{6^n}{n!} = \underline{\text{convergent}}$$

Comparison by the limit:

$$\lim_{n\to\infty} \frac{6^n + \log n}{n! + n^3} = \lim_{n\to\infty} \frac{6^n\left(1 + \frac{\log n}{6^n}\right)}{n!\left(1 + \frac{n^3}{n!}\right)} = \lim_{n\to\infty} \frac{6^n}{n!}$$

$\to 0$ when $n \to \infty$

$\to 0$

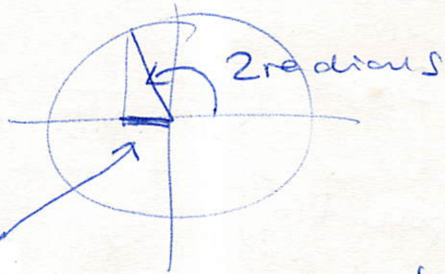
Conclusion: $\sum^{\infty} \frac{6^n+\log n}{n!+n^3}$ behaves as the series $\sum^{\infty} \frac{6^n}{n!}$

Both are convergent.

——o0o——

$\sum^{\infty} 3^{n\cos 2}$ convergent

$\cos 2$ ← radians



$\frac{\pi}{2} < 2 < \pi$

$\cos 2$ is negative

$\cos 2 = -|\cos 2|$

$$\sum^{\infty} 3^{n\cos 2} = \sum^{\infty}\left(\frac{1}{3^{|\cos 2|}}\right)^n$$

= It is a geometric series of ratio $r = \frac{1}{3^{|\cos 2|}} < 1$, therefore convergent

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(P5)

$$\begin{array}{l} \cancel{1} + \cancel{3x} \\ -\cancel{1} \qquad + \cancel{x^2} \\ \quad -\cancel{3x} \qquad + \cancel{3x^3} \\ \qquad -\cancel{x^2} \qquad + x^4 \\ \qquad\quad -\cancel{3x^3} \qquad + 3x^5 \end{array}
\qquad
\begin{array}{|l} 1-x^2 \\ \hline 1+3x+x^2+3x^3+x^4+\dots \end{array}$$

by long division (always)

Enough terms!! No one needs more!

$$\frac{1+3x}{1-x^2} = 1+3x+x^2+3x^3+x^4+3x^5+\dots$$

checking (easy method): $1(1+x^2+x^4+x^6+\dots)$

$+3x(1+x^2+x^4+x^6+\dots)$

$= (1+3x)\underbrace{(1+x^2+x^4+x^6+\dots)}_{=\frac{1}{1-x^2}}$: perfect!!

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[Exámenes: He visto que los alumnos que factorizan el denominador. Nunca, nunca, nunca se hace así. Para qué factorizar si sale siempre la long división. El que no hace la long división es que no conoce el método, que es infalible. Factorizar es para integrales, así no. Hagan en casa

$$1 + x + x^2 \quad \underline{| 1 + x + x^2}$$

y nadie factoriza nada, que son palabras mayores.

]

Consequently,

$$\boxed{\text{coef } x^8 \text{ is } 1}$$

and

$$\boxed{f^{(21)}(0) = 3 \cdot 21!}$$

tres por (veintiuno factorial)

just reading in

$$1 + 3x + x^2 + 3x^3 + x^4 + \ldots$$

P6 c)

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$\cdots + \frac{(-1)^{n+1} x^{2n}}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2} + \cdots \qquad R = \infty$$

close

$$J_1(x) = -\frac{d J_0(x)}{dx}$$

$$= \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots$$

$R = \infty$

b) $$\sum_{n=0}^{\infty} (-1)^n \frac{9^{n-2}}{n!} = \frac{1}{9^2}\Big(\underbrace{1 - 9 + \frac{9^2}{2!} - \frac{9^3}{3!} + \cdots}_{e^{-9}}\Big)$$

$$= \boxed{\frac{e^{-9}}{9^2}}$$

menos nueve.

nueve al cuadrado