SOLUCIONES

Extra-Maths

2022-2023

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UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2022-2023

Examen Extraordinario de Matemáticas E

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros o apuntes ni **calculadora**. La nota total de este examen es de 10 puntos. El profesor no calificará nada escrito en hojas en sucio.

P1 [2pt] Sea

$$f(x) = \begin{cases} (x-1)^4 \cos^2 \dfrac{1}{x-1}, & x \neq 1 \\ 0 & x = 1 \end{cases}$$

a) Calcular $f'(1)$ y $f''(1)$. b) Indicar razonadamente si la función $f(x)$ tiene o no un extremo local en $x = 1$.

P2 [2pt] a) Calcular $\int_0^\pi dx\, xe^{-x^2}$.

b) Demostrar mediante el *Teorema del Valor Medio* que la ecuación $2\pi e^{-x^2} = \dfrac{1-e^{-\pi^2}}{x}$ tiene al menos una solución en el intervalo $(0, \pi)$.

P3 [2pt] a) Calcular descomponiendo en fracciones simples el valor de la integral

$$I = \int_1^a dx\, \frac{1}{(1+x+x^2)\, x^2}$$

donde $a > 1$ es una constante. b) Calcular $\lim_{a\to+\infty} I$.

Nota: Obviamente el resultado que se obtiene en b) es exactamente $\int_1^\infty dx\, \dfrac{1}{(1+x+x^2)\, x^2}$.

P4 [1pt+1pt+0.5pt+1.5pt]

a) ¿Es cierto que $2 \arctan \dfrac{2}{\pi} = \arctan \dfrac{4\pi}{\pi^2 - 4}$? Contestar *sí* o *no* sin justificar la respuesta no pumtúa.

b) Encontrar todos los x reales que satisfacen $|x^2 + x - 2| = -x^2 - x + 2$. *Nota*: $|x|$ es el valor absoluto de x.

c) Escribir **dos** ejemplos sencillos de series de potencias que converjan en sólo un punto.

d) Hallar los **cuatro** primeros términos no nulos del desarrollo en serie de potencias de $\dfrac{2x^2+1}{x^3-x+1}$ en torno a $x = 0$.

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1 $$f(x) = \begin{cases} (x-1)^4 \cos^2 \frac{1}{x-1}, & x \neq 1 \\ 0 & x = 1 \end{cases}$$

a) Calculate $f'(1)$, $f''(1)$
b) Indicate if $f(x)$ has a local extreme value at $x=1$

The problem is <u>equivalent</u> to $\boxed{\begin{matrix} x=1 \text{ is} \\ u=0 \end{matrix}}$

$$g(u) \equiv \begin{cases} u^4 \cos^2 \frac{1}{u}, & u \neq 0 \\ 0 & u = 0 \end{cases}$$

if you take $u \equiv x-1$. We study problem [instead.

ooo

$g(u)$ is continuous at $u=0$ because

$$\lim_{u\to 0} u^4 \cos^2 \frac{1}{u} = 0 \times \underset{\text{function:}}{\text{bounded}} = \underline{\underline{0}};$$

$0 \leq |\cos^2 u| \leq 1$

that is, the limit ($\underline{\underline{0}}$) coincides with the definition: $g(0) = \underline{\underline{0}}$.

a) $g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$

Newton's formula for the derivative at a point

$g(0)$ as given in the problem

$$\frac{g(0+h)-g(0)}{h} = \frac{1}{h}\left[h^4 \cos^2 \frac{1}{h} - 0\right] = h^3 \cos^2 \frac{1}{h}$$

but

$$\lim_{h\to 0} h^3 \cos^2 \frac{1}{h} = 0$$

by the "squeeze theorem"

$$0 \leq \Big|h^3 \underbrace{\cos^2 \frac{1}{h}}\Big| \leq h^3 \xrightarrow[h\to 0]{} 0$$

bounded by 0 and 1
$\Rightarrow$ tends to zero <u>too</u> when $h \to 0$

$\Rightarrow \boxed{g'(0) = 0}$ or $\boxed{f'(1) = 0}$

calculating $g''(0)$

$$g'(u) = \begin{cases} 4u^3 \cos^2 \frac{1}{u} + 2u^2 \cos\frac{1}{u} \sin\frac{1}{u}, & u \neq 0 \\ 0 & u = 0 \end{cases}$$

↑ we just have calculated it.

Again $g'(u)$ is continuous at $u=0$ because $\lim_{u \to 0} g'(u) = 0$.

(the limit coincides with the definition)

$$g''(0) = \lim_{h \to 0} \frac{g'(0+h) - g'(0)}{h} \text{''}$$

$$\frac{g'(0+h) - g'(0)}{h} = \frac{1}{h}\left[4h^3\cos^2\frac{1}{h} + 2h^2 \cos\frac{1}{h} \sin\frac{1}{h}\right]$$

$$= 4h^2 \underbrace{\cos^2\frac{1}{h}}_{\text{bounded for all } h} + 2h \underbrace{\cos\frac{1}{h}\sin\frac{1}{h}}_{\text{bounded for all } h}$$

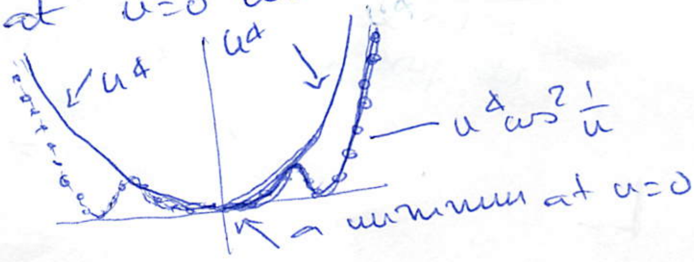
$$\xrightarrow[h \to 0]{} 0 \quad \left(\text{tends to zero as } h \times \text{a constant}\right)$$

$\Rightarrow \boxed{g''(0) = 0}$ or $\boxed{f''(1) = 0}$

b) By the definition of $g(u)$: it is even around $u=0$ and always non-negative,

$u^4 \cos^2 \frac{1}{u} \geq 0$ near $u=0$

and continuous at $u=0$ with value 0, so



$\Rightarrow$ yes, $g(u)$ has a local minimum of value 0 at $u=0$.

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② a) $\int_0^\pi dx\, e^{-x^2} \cdot x = -\frac{1}{2}\left[e^{-x^2}\right]_0^\pi$

direct integration

$= -\frac{1}{2}\left[e^{-\pi^2} - 1\right]$

$= \frac{1}{2}\left(1 - e^{-\pi^2}\right).$

Solution: $\boxed{\int_0^\pi dx\, x e^{-x^2} = \frac{1}{2}\left(1 - e^{-\pi^2}\right)}$

b) Consider $f(x) = e^{-x^2}$ cont everywhere and in particular on $[0,\pi]$ and derivable on $(0,\pi)$, ~~then~~ the MVT states that $\exists\, c \in (0,\pi)$ s.t.

$$\frac{f(\pi) - f(0)}{\pi - 0} = f'(c)$$

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{e^{-\pi^2} - 1}{\pi}$$

$f'(c) = e^{-c^2}(-2c)$, $c \in (0,\pi)$ by theorem.

$\Rightarrow \quad \frac{1 - e^{-\pi^2}}{\pi} = 2c e^{-c^2}$, $c \in (0,\pi)$: at least "one c".

this is exactly the equation to be solved in the problem (change c by x if you wish or do not change anything):

$$\frac{1 - e^{-\pi^2}}{c} = 2\pi e^{-c^2}.$$

(3) $$\int_1^a \frac{1}{(1+x+x^2)x^2} \quad , \; a>1$$

First decompose the integrand in simple fractions as

$$\frac{1}{(1+x+x^2)x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x+x^2}$$

where A, B, C, D are constants, this is the reason of using simple fractions, to be found and $1+x+x^2$ has no real roots (will you tell me this when writing your exam?). From

(*) $$\frac{1}{(1+x+x^2)x^2} \underset{\text{an identity}}{=} \frac{x(1+x+x^2)A + B(1+x+x^2) + (Cx+D)x^2}{(1+x+x^2)x^2}$$

we deduce that

$A=-1$, $B=1$, $C=1$, $D=0$

[why?
$x(1+x+x^2)A + B(1+x+x^2) + (Cx+D)x^2 = 1$ identity

$x=0$, $\boxed{B=1}$

coeff x: $A+B=0$, $\boxed{A=-1}$

coeff x^2: $A+B+D=0$ " $\boxed{D=0}$

L coeff x^3: $A+C=0$ " $C=1$

(*) Con los nuevos "planes de estudio" esto ya no lo van a saber hacer los estudiantes que ahora cursan primaria. Genial!!! Viva el conocimiento!!!

$$\int \frac{dx}{(1+x+x^2)x^2} = \int dx \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{x}{1+x+x^2}\right).$$

primitive is easy: is $-\log x$

primitive is $\frac{-1}{x}$

Let us study

$$\int dx \, \frac{x}{1+x+x^2} = \frac{1}{2}\int dx \, \frac{1+2x}{1+x+x^2} - \frac{1}{2}\int \frac{dx}{1+x+x^2}$$

$$= \frac{1}{2}\log(1+x+x^2) - \frac{1}{2}\int \frac{dx}{1+x+x^2},$$

and now let us calculate the primitive of

$$\int dx \, \frac{1}{1+x+x^2}$$

$$\frac{1}{1+x+x^2} \underset{\nearrow}{=} \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \underset{\nearrow}{=} \frac{4/3}{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2}$$

complete squares as usual

multiply numerator and denominator by 4/3

$$= \frac{4/3}{1+\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2} = \frac{4/3}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}$$

In red: constants that you shall not forget...

$$\int \frac{dx}{1+x+x^2} \underset{\nearrow}{=} \frac{4}{3}\int \frac{\sqrt{3}}{2}\,\frac{dt}{1+t^2} = \frac{2}{\sqrt{3}}\arctan t$$

$t = \frac{2x+1}{\sqrt{3}}$

$dt = \frac{2dx}{\sqrt{3}}$

$$= \frac{2}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}}.$$

it is easy to go back to x: otherwise I would think another option...

Conclusion:

$$\int_1^a \frac{dx}{(1+x+x^2)x^2} = \Big[-\log x - \frac{1}{x} + \frac{1}{2}\log(1+x+x^2) - \frac{1}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} \Big]_1^a$$

$$= -\log a - \frac{1}{a} + \frac{1}{2}\log(1+a+a^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2a+1}{\sqrt{3}}\right) + 1 - \frac{1}{2}\log 3 + \frac{1}{\sqrt{3}}\frac{\pi}{3}.$$

$\arctan \sqrt{3} = \frac{\pi}{3}$

in principal branch $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

[observe that if $a=1$, the value of the integral is zero, as it should be].

b) Now the limit when $a \to \infty$

- $\lim_{a\to\infty} \left(-\log a + \frac{1}{2}\log(1+a+a^2)\right)$

$$= \lim_{a\to\infty} \log \frac{(1+a+a^2)^{1/2}}{a}$$

$$= \lim_{a\to\infty} \log \left(\frac{1}{a^2} + \frac{1}{a} + 1\right)^{1/2}$$

$$= \lim_{a\to\infty} \log 1 = \underline{\underline{0}}$$

$\arctan \infty = \frac{\pi}{2}$

- $\lim_{a\to\infty} \left(-\frac{1}{a} - \frac{1}{\sqrt{3}}\arctan\frac{2a+1}{\sqrt{3}}\right) = 0 - \frac{\pi}{2\sqrt{3}} = \frac{-\pi}{2\sqrt{3}}$

Result: $\displaystyle\int_0^\infty \frac{dx}{(1+x+x^2)x^2} = 1 - \frac{1}{2}\log 3 - \frac{\pi\sqrt{3}}{18}$.

$$\left[-\frac{\pi}{2\sqrt{3}} + 1 - \frac{1}{2}\log 3 + \frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2}\log 3 - \frac{\pi}{6\sqrt{3}} \right.$$

— o0o —

<u>Comment</u>: Suppose we were asked if the <u>improper</u> integral

$$\int_1^\infty \frac{dx}{(1+x+x^2)x^2}$$

$x=1$ is not a problem.

converges.. It has the same character (Integral test) as the series

$$\sum_1^\infty \frac{1}{(1+n+n^2)n^2}$$

which is convergent since $\sum_1^\infty \frac{1}{n^4}$ is convergent. The result of $\int_1^\infty \frac{1}{(1+x+x^2)x^2}$ is then a <u>finite</u> number. We have found <u>this</u> number.

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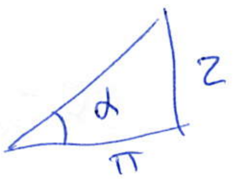
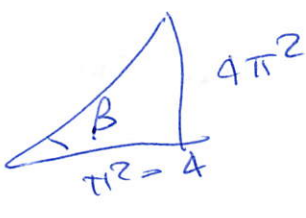
(4) a) Is $\underbrace{2\arctan\frac{2}{\pi}}_{\text{call this angle } \alpha} = \underbrace{\arctan\frac{4\pi}{\pi^2-4}}_{\text{call this angle } \beta}$.

We know that $\tan\alpha = \frac{2}{\pi}$

$\tan\beta = \frac{4\pi}{\pi^2-4}$





The question is : Is $2\alpha = \beta$?

If so

$$\tan 2\alpha = \tan\beta$$

We see if this is true:

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\frac{2}{\pi}}{1-\cancel{4}/\pi^2} = \frac{4/\pi}{\frac{\pi^2-4}{\pi^2}}$$

identity

$$= \frac{4\pi}{\pi^2-4} = \tan\beta .$$

multiply numerator and denominator by π^2

$\boxed{\text{Yes, it is true}}$

[If you have concern about "α is acute"; Is 2α acute? And β? Do not! $\alpha, 2\alpha, \beta$ are all acute angles.]

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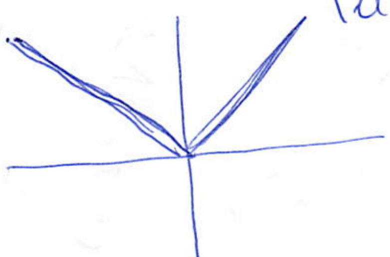
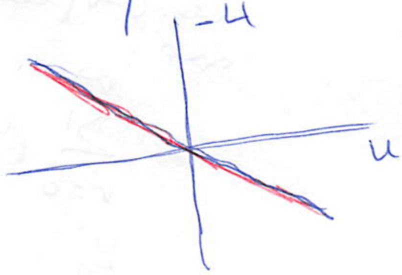
5) $|x^2+x-2| = -x^2-x+2$

is like

$$|u| = -u.$$

where is $|u| = -u$? Graphically

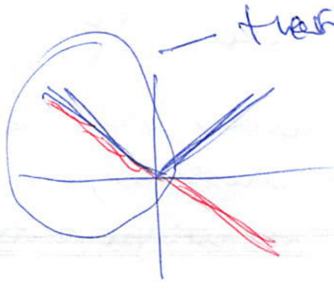




$\Rightarrow$ $|u| = -u$ when $u \le 0$

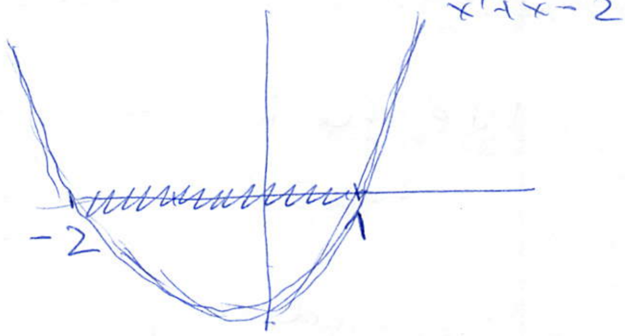
↳ superposition:

— these are equal.



we need to know where $x^2+x-2 \leq 0$

$$x^2+x-2 = (x+2)(x-1)$$



$$\Rightarrow \quad x^2+x-2 \leq 0 \iff \boxed{-2 \leq x \leq 1}$$

— 000 —

c) Power series: $\sum_{n=0}^{\infty} a_n x^n$

By excellence the power series that converges only at $x=0$ is

$$\sum_{n=0}^{\infty} n!\,x^n$$ (vista en clase)

(Euler referred to this series as the divergent series by excellence!). Observe that $R=0$. Other series that are convergent only at $x=0$ derived from this one are:

$$\sum_{n=0}^{\infty} (n!)^2 x^n, \quad \sum_{0}^{\infty} n!\,e^n x^n,$$

$$\sum_{n=0}^{\infty} n!\,e^{-n} x^n, \quad \sum_{0}^{\infty} n!\,(1+n+n^2)\,x^n,$$

$\vdots \qquad \vdots$

do we need to continue?

(Y si no son partiendo de $\sum_0^{\infty} n!\,x^n$ sabemos alguna? Yo ahora mismo, aquí en la clase, no).

$\left(\sum_0^{\infty} e^{n!}\,x^n\right.$ es del mismo tipo...$\left.\right)$

d)

$$
\begin{array}{l}
\cancel{1} \quad\quad\quad + 2x^2 \qquad\qquad \Big|\; 1 - x + x^3 \\
\cancel{-1} + \cancel{x} \quad\quad\quad - x^3 \qquad \overline{1 + x + 3x^2 + 2x^3 + x^4 + \cdots} \\
\hline
\cancel{-x} \quad + x^2 \quad\quad - x^4 \\
\hline
\cancel{3x^2} - x^3 - x^4 \\
\cancel{-3x^2} + 3x^3 + \quad 3x^5 \\
\hline
\cancel{2x^3} - x^4 + 3x^5 \\
\cancel{-2x^3} + 2x^4 \quad - 2x^6 \\
\hline
x^4 + 3x^5 - 2x^6
\end{array}
$$

$$\frac{1+2x^2}{1-x+x^3} = \underbrace{1 + x + 3x^2 + 2x^3}_{\text{first four terms different from zero.}} + \cdots$$

——— o0o ———

V2

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2022-2023

2022-2023-Extraordinary Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper.**

P1 [2pt] Let

$$f(x) = \begin{cases} (x-1)^4 \cos^2 \dfrac{1}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$

a) Calculate $f'(1)$ and $f''(1)$. b) Indicate whether the function $f(x)$ has a local extreme at $x = 1$ or not.

P2 [2pt] a) Find the value of $\int_0^\pi dx\, xe^{-x^2}$.

b) Show using the *Mean Value Theorem* that the equation $2\pi e^{-x^2} = \dfrac{1-e^{-\pi^2}}{x}$ has one solution in the interval $(0, \pi)$ at least.

P3 [2pt] a) Calculate by partial fractions the value of the integral

$$I = \int_1^a dx\, \frac{1}{(1+x+x^2)\, x^2}$$

where $a > 1$ is a constant. b) Find $\lim_{a\to+\infty} I$.

Note: Needless to say that the result of part b) is exactly $\int_1^\infty dx\, \dfrac{1}{(1+x+x^2)\, x^2}$.

P4 [1pt+1pt+0.5pt+1.5pt]

a) Is the relation $2 \arctan \dfrac{2}{\pi} = \arctan \dfrac{4\pi}{\pi^2 - 4}$ true? A mere *Yes* or *No* answer without explanation or justification will be given zero credits.

b) Find all real x numbers that satisfy $|x^2 + x - 2| = -x^2 - x + 2$. *Note*: $|x|$ is the absolute value of x.

c) Give two **simple** examples for power series which converge at a single point only.

d) Calculate the first **four** nonzero terms of the power series expansion for $\dfrac{2x^2+1}{x^3-x+1}$ around $x = 0$.
