2022-2023 - Mathematics
Solutions

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2022-2023

# Examen Final Ordinario de Matemáticas E

Nombre y Apellidos:

Firma y DNI:

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**Nota:** En esta prueba no se permiten libros o apuntes ni **calculadora**. La nota total de este examen son 10 puntos. El profesor no calificará nada escrito en hojas en sucio.

P1 [2pt] Sea $f(x) = \tanh(x+1) - \tanh(x-1)$ definida para todo $x$ real (o sea, diferencia de tangentes *hiperbólicas*)

a) Determinar si $f(x)$ es par, impar o no tiene paridad definida en torno a $x = 0$.

b) Calcular los límites de $f(x)$ cuando $x \to \pm\infty$

c) Determinar y clasificar los extremos absolutos y relativos de $f(x)$

d) Representar $f(x)$

P2 [3pt] Calcular los **cuatro** primeros términos no nulos del desarrollo en serie de potencias en torno a $x = 0$ de las funciones

a) $-1 + \sqrt{1+x}$

b) $\sin(-1 + \sqrt{1+x})$.

Nota: El problema puntuará [2pt] si sólo se calculan los **tres** primeros términos no nulos de ambas funciones.

P3 [1pt] Suponiendo que $|\cos x| \neq 1$, calcular la suma de la serie $\sum_{n=1}^{\infty} (\cos x)^{2n}$

P4 [2pt] Sea $\{a_n\}$ la secuencia definida recursivamente por $a_1 = 1$ y

$$a_{n+1} = \frac{1 + 4\,a_n + \sqrt{1 + 24\,a_n}}{16}, \quad n \geq 1.$$

i) Suponiendo que esta secuencia tuviera límite, calcular su valor. ii) ¿Cuál sería el valor del límite si $a_1 = 0$ en lugar de $a_1 = 1$?

P5 [2pt] Calcular el valor de la integral

$$I = \int_0^{\pi/6} dx\, \frac{1}{\cos^3 x}.$$

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UNIVERSIDAD COMPLUTENSE DE MADRID
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Academic Year 2022-2023

# 2022-2023-Final Exam of Mathematics-B

Name and surname:

Signature and DNI:

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**Important:** Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**.

P1 [2pt] Consider the function $f(x) = \tanh(x+1) - \tanh(x-1)$ defined for every real number $x$ (note that the function is a difference of *hyperbolic* tangents)

a) Determine if $f(x)$ is even, odd or neither around $x = 0$.

b) Find the limits of $f(x)$ when $x \to \pm\infty$

c) Determine and classify the local and global extrema of $f(x)$

d) Plot $f(x)$ over the whole real axis.

P2 [3pt] Find the first **four** non-zero terms of the power series expansion about $x = 0$ of

a) $-1 + \sqrt{1+x}$

b) $\sin(-1 + \sqrt{1+x})$.

Notice that the problem mark will be [2pt] and not [3pt] if you calculate only the first **three** non-zero terms.

P3 [1pt] Suppose $|\cos x| \neq 1$. Evaluate the sum of the series $\sum_{n=1}^{\infty} (\cos x)^{2n}$

P4 [2pt] Define the sequence $\{a_n\}$ recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1 + 4\,a_n + \sqrt{1 + 24\,a_n}}{16}, \quad n \geq 1.$$

i) Assuming that has a limit, calculate it.

ii) What would be the value of the limit if $a_1 = 0$ instead of $a_1 = 1$?

P5 [2pt] Calculate the value of the integral

$$I = \int_0^{\pi/6} dx \, \frac{1}{\cos^3 x}.$$

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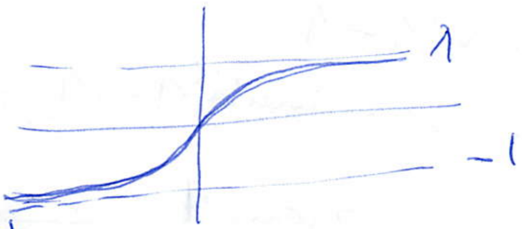
# SOLUTIONS – Final Exam Mathematics 2022-2023

(P1) $f(x) \equiv \tanh(x+1) - \tanh(x-1)$

a) $f(x)$ is even because $\tanh x$ is an odd function:




$\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ ↑ if needed

We see it,

$$f(-x) = \tanh(-x+1) - \tanh(-x-1)$$
$$= -\tanh(x-1) + \tanh(x+1)$$
$$= f(x).$$

oddness of $\tanh x$

b) To calculate $\lim_{x\to\infty} f(x)$ we can do it directly,

$$\lim_{x\to\infty}\left(\tanh(x+1) - \tanh(-x-1)\right)$$
$$= \lim_{x\to\infty}\left(\tanh x - \tanh x\right)$$
$$= 1 - 1$$
$$= 0$$

or rewriting $f(x)$ as

$$f(x) = \frac{2\tanh 1}{\cosh^2 x} \cdot \frac{1}{1 - \tanh^2 x \tanh^2 1} \quad (*)$$

never zero

(now it is very clear that f is even). The constant $2\tanh 1$ is bounded (of course!)

and $\tanh^2 x \tanh^2 1$ is always a positive finite number in $[0,1)$ for finite x.

so ↙ all the constants are

$$\lim_{x\to\infty} \frac{\text{bounded}}{\cosh^2 x} = \underline{\underline{0}}.$$ //

By parity $\lim_{x\to-\infty} f(x) = \underline{0}$ as well.

$\ulcorner$ $0 \leq \tanh^2 x \tanh^2 1 < 1$

because

$$-1 < \tanh x < 1 \qquad \forall x$$
$$0 \leq \tanh^2 x < 1$$
$$0 \leq \tanh^2 1 < 1$$
$$\Longleftrightarrow \quad 0 \leq \tanh^2 x \tanh^2 1 < 1$$

$\llcorner$

$\ulcorner$ writing $f(x)$ as $\dfrac{2\tanh 1}{\cosh^2 x} \cdot \dfrac{1}{1-\tanh^2 x \tanh^2 1}$

is immediate to conclude that

$f(x)$ is strictly positive for finite $x$ real. (never 0).

Hardly read this in exams... (why?)

very useful property to plot $f(x)$.

A conclusion that is reached also knowing that $\tanh x$ is an increasing fction everywhere:

$$x+1 > x-1 \qquad \forall x$$
$$\tanh(x+1) > \tanh(x-1)$$
$$\Longleftrightarrow \quad \tanh(x+1) - \tanh(x-1) > 0$$

$\llcorner$

$\ulcorner$ Deduction of [*]. Use the identity

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1+\tanh a \tanh b}$$

$$f(x) = \frac{\tanh x + \tanh 1}{1+\tanh x \tanh 1} - \frac{\tanh x - \tanh 1}{1-\tanh x \tanh 1}$$

$$= \frac{\text{numerator}}{1-\tanh^2 x \tanh^2 1} \quad //$$

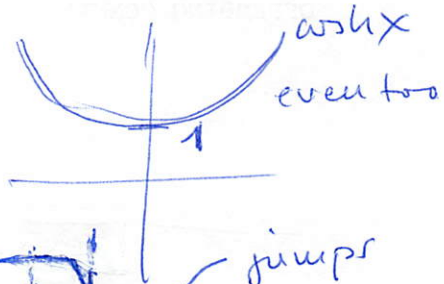
numerator $= \cancel{\tanh x} + \tanh 1 - \tanh x \tanh 1 (\tanh x + \cancel{\tanh 1})$

$\quad - (\cancel{\tanh x} - \tanh 1 + \tanh x \tanh 1 (\tanh x - \cancel{\tanh 1}))$

$= 2\tanh 1 - 2\tanh^2 x \tanh 1$

$= 2\tanh 1 (1 - \tanh^2 x)$

$= \dfrac{2\tanh 1}{\cosh^2 x}$

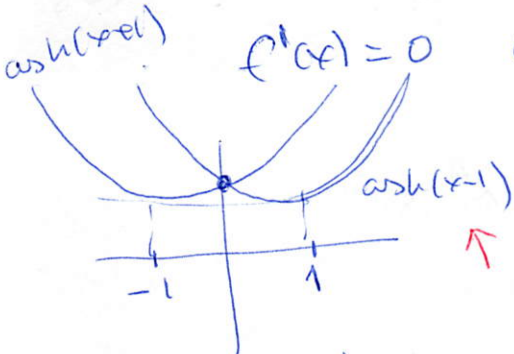



meaning: no  or jumps

c) f(x) is continuous and derivable everywhere.

$|||$
smooth graph
(suave cito...)

$$f'(x) = \frac{1}{\cosh^2(x+1)} - \frac{1}{\cosh^2(x-1)}$$




$f'(x) = 0 \iff \cosh(x+1) = \cosh(x-1)$ – is impossible. (cosh is always a +ve function)

$\iff x = 0$

↑ graphic manner to resolve $\cosh(x+1) = \cosh(x-1)$

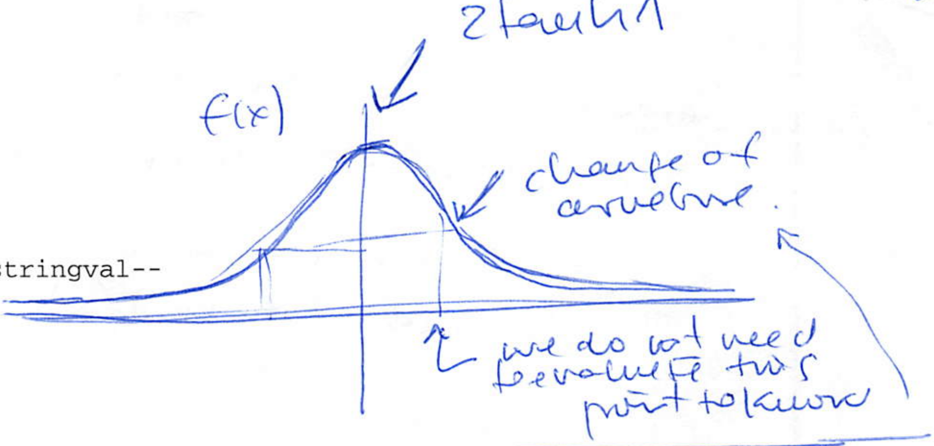
| | | | | | |
|---|---|---|---|---|---|
| $x$ | $0$ | | $1$ | | $\infty$ |
| $f(x)$ | $2\tanh 1$ | $\searrow$ | $\tanh 2$ | $\searrow$ | $0^+$ |
| $f'(x)$ | $0$ | | | | |

notice that

$$\underbrace{\tanh 2}_{\text{is } f(1)} = \frac{2\tanh 1}{1+\tanh^2 1} < \underbrace{2\tanh 1}_{\text{is } f(0)}$$

$\Rightarrow$ At $x=0$ there is a maximum (absolute) of value $2\tanh 1$. The function has no absolute minima (is defined on $(-\infty, \infty)$ so EVT does not apply)

The plot is

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P2

$$-1+\sqrt{1+x} = \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \binom{1/2}{4}x^4 + \cdots$$

$$\binom{1/2}{1} = 1/2 \qquad \binom{1/2}{3} = \frac{1/2(-1/2)(-3/2)}{3!} = 1/16$$

$$\binom{1/2}{2} = \frac{1/2(-1/2)}{2!} = -\frac{1}{8} \qquad \binom{1/2}{4} = \frac{1/2(-1/2)(-3/2)(-5/2)}{4!} = -\frac{5}{128}$$

Thus,

$$-1+\sqrt{1+x} = \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

— o0o —

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$$ around $u \to 0$

where (see the previous part)

$$u = \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \cdots \quad \xrightarrow[x\to 0]{} 0$$

Thus

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$$

$$= x\left(\frac{1}{2} - \frac{x}{8} + \frac{x^2}{16} - \frac{5}{128}x^3 + \cdots\right)$$

$$-\frac{x^3}{3!}\left(\frac{1}{2} - \frac{x}{8} + \frac{x^2}{16} - \frac{5}{128}x^3 + \cdots\right)^3$$

$$+\frac{x^5}{5!}\left(\frac{1}{2} - \frac{x}{8} + \frac{x^2}{16} \cdots\cdots\right)^5$$

in red are the terms that need to be considered for the "four non zero terms".

written but not necessary for the first "four" terms

$$= \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} - \frac{3}{128}x^4 + \dots$$

Term by term:

$x\left[\frac{1}{2}\right]$

$x^2\left[-\frac{1}{8}\right]$ $\quad \frac{1}{8}\left(\frac{1}{2}-\frac{1}{6}\right) = \frac{1}{8}\cdot\frac{4}{12} = \frac{1}{8}\cdot\frac{1}{3} = \frac{1}{24}$

$x^3\left[\frac{1}{16} - \frac{1}{3!}\frac{1}{8}\right]$

$+x^4\left[-\frac{5}{128} - \frac{1}{3!}\left(\frac{1}{2}\right)^2\left(-\frac{3}{8}\right)\right]$

$$-\frac{5}{128} + \frac{1}{\cancel{3}\cdot 2}\cdot\frac{1}{4}\,\frac{\cancel{3}}{8} = -\frac{5}{128} + \frac{1}{64}$$

$$= \frac{-5+2}{128} = -\frac{3}{128}.$$

Result:

$$\sin\left(-1+\sqrt{1+x}\right) = \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} - \frac{3}{128}x^4 + \dots$$

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(P3) $\displaystyle\sum_{n=1}^{\infty}(\cos x)^{2n} = \cos^2 x + \cos^4 x + \cos^6 x + \dots$

$= \cos^2 x\left(1 + \cos^2 x + \cos^4 x + \dots\right)$

geometric series of ratio $\cos^2 x \neq 1$ (so it's convergent)

$$= \cos^2 x \frac{1}{1-\cos^2 x} = \frac{\cos^2 x}{\sin^2 x} = \cot g^2 x.$$

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(P4) $a_{n+1} = \dfrac{1 + 4a_n + \sqrt{1+24a_n}}{16}, \quad n \geq 1$

a) obviously, if $\{a_n\}_{n\to\infty} \to l$ also $\{a_{n+1}\}_{n\to\infty} \to l$ (one is a subsequence of the other), so the limit $l$ has to satisfy

I explain why.

$$l = \frac{1+4l+\sqrt{1+24l}}{16} \quad [**]$$

Another reason: def of limit: $\forall n \geq N \dots$

$$12l-1=\sqrt{1+24l}$$

$$(12l-1)^2=1+24l \quad , \quad l \begin{cases} 0 \\ \frac{1}{3} \end{cases}$$

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But $l=0$ is a spurious solution and does not satisfies [$**$], so $\boxed{l=1/3}$

b) If $a_1=0$, $l=1/3$ as well. We have not used $a_1$ to calculate $l$. → I give a reason...

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(P5) $$\int_0^{\pi/6}\frac{dx}{\cos^3 x}$$

$t=\sin x$
$dt=\cos x\,dx$

$$\frac{dx}{\cos^3 x}=\frac{\cos x\,dx}{\cos^4 x}=\frac{dt}{(1-\sin^2 x)^2}=\frac{dt}{(1-t^2)^2},$$

then

$$\int_0^{\pi/6}dx\,\frac{1}{\cos^3 x}=\int_0^{1/2}\frac{dt}{(1-t^2)^2}$$

decomposition of $\frac{1}{(1-t^2)^2}$ in simple fractions is

$$\frac{1}{(1-t^2)^2}=\frac{1}{4}\left[\frac{1}{1-t}+\frac{1}{1+t}+\frac{1}{(1-t)^2}+\frac{1}{(1+t)^2}\right]$$

[why?

$$\frac{1}{(1-t^2)^2}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{(1-t)^2}+\frac{D}{(1+t)^2}$$

Tevenint

then $A=B$, $C=D$

$$=A\underbrace{\left[\frac{1}{1-t}+\frac{1}{1+t}\right]}_{\frac{2}{1-t^2}}+C\underbrace{\left[\frac{1}{(1-t)^2}+\frac{1}{(1+t)^2}\right]}_{\frac{2(1+t^2)}{(1-t^2)^2}}$$

$$= \frac{2A}{1-t^2} + \frac{2C(1+t^2)}{(1-t^2)^2}$$

$$= \frac{2A(1-t^2) + 2C(1+t^2)}{(1-t^2)^2} = \frac{1}{(1-t^2)^2}$$

at the beginning

$t=1$, $4C=1$, $\boxed{C=1/4}$

coef $t^2$ $2C-2A=0$, $\boxed{A=1/4}$

—ooo—

$$\int_0^{1/2} \frac{dt}{(1-t^2)^2} = \frac{1}{4}\left[-\log(1-t) + \log(1+t) + \frac{1}{1-t} - \frac{1}{1+t}\right]_0^{1/2}$$

$$= \frac{1}{4}\left[-\log 1/2 + \log 3/2 + 2 - 2/3\right]$$

$$= \frac{1}{4}\log 3 + \frac{1}{3}$$

Result: $\boxed{\int_0^{\pi/6} \frac{dt}{\cos 3x} = \frac{1}{4}\log 3 + \frac{1}{3}}$

—ooo—

Quick comments

1) I dond use it in P1 but also

$$f'(x) = \left(\tanh(x+1) - \tanh(x-1)\right)'$$
$$= \tanh^2(x-1) - \tanh^2(x+1),$$

that is an odd function because is the derivative of an even function. No need to prove that it is odd: it is. And continuous. therefore $f'(0)=0$.

2) P1 can be done <u>without</u> exponentials.
If you introduce them, you will open Pandora's box... with <u>no necessity</u>.

3) $f(x) = \tanh(x+1) - \tanh(x-1)$
is even, continuous and derivable everywhere. This means that $f'(x)$ is odd and it happens that is also continuous. By the continuity of an <u>odd</u> function we have that

$$f'(0) = 0,$$

so, one of the solutions of $f'(x)=0$ is $x=0$.