SOLUTIONS 2022-2023

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2022-2023

Primer Parcial de Matemáticas: grupo E

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos. No se corregirá nada que no esté escrito en este cuadernillo.

P1 [2 pt] Sea $f(x) = \arctan(\log \sin x)$ definida en $(0, \pi)$.

1) Calcular los límites de $f(x)$ cuando $x \to 0^+$ y cuando $x \to \pi^-$.
2) Esbozar la gráfica de f indicando si hay simetría en torno a algún punto.
3) Hallar $R\,f$, el recorrido de f.
4) Discutir si f tiene un mínimo absoluto en su dominio.
5) Calcular $f'(x)$, la derivada de f con respecto a x.

P2 [2 pt] Sea la parábola $f(x) = Ax^2 + Bx + C$ donde A, B, C son constantes y A es differente de cero. Determinar si se puede aplicar el *Teorema del Valor Medio* a $f(x)$ en el intevalo $[a, b]$. 1) Hallar los posibles valores de $c \in [a, b]$ que satisfacen el teorema. 2) Comentar brevemente (dos líneas) el resultado.

P3 [2pt+2pt] a) Calcular el límite cuando $n \to \infty$ de las sucesiones:

1) $\sqrt{n \log n + 4n^2} - 2n\, e^{1/n}$, 2) $\dfrac{\sqrt{n \log n + 4n^2} - 2n\, e^{1/n}}{\log n}$.

Si alguno de ellos es cero (ó ∞) añada también cómo tiende a cero, por ejemplo, *tiende a* 0 *como* $1/n^2$.

b) Demostrar que

$$\tan\left(\frac{\pi}{4} + a\right) - \tan\left(\frac{\pi}{4} - a\right)$$

con a un número real arbitrario se puede escribir como $b \tan ca$ para unos ciertos valores de b y c.

P4 [2 pt] Sea $f(x) = \dfrac{1}{\cos x}$ definida en $[0, \pi] - \{\pi/2\}$. a) Encontrar el dominio y recorrido de $g \equiv f^{-1}$. b) Encontrar la ecuación de una recta tangente a $g(x)$ en $x = \pi$.

(P2) MVT : [Mean value theorem :] [or Lagrange theorem]

Let f be a function that satisfies:

1) f cont on $[a,b]$,
2) f differentiable on the open interval (a,b),

then there is a number $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Apply the theorem to the parabola Ax^2+Bx+C, $A \neq 0$ which is continuous on every $[a,b] \subset \mathbb{R}$ and derivable on (a,b):

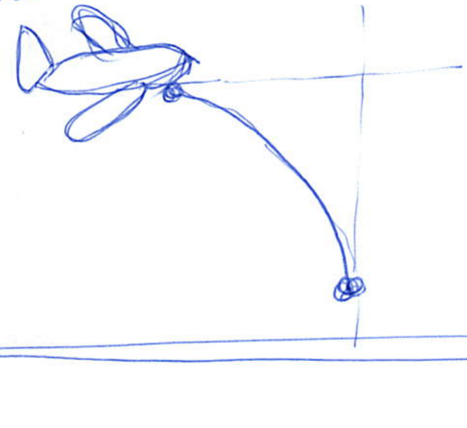
$$\frac{f(b)-f(a)}{b-a} = \frac{A(b^2-a^2)+B(b-a)}{b-a} = A(b+a)+B$$

$$f'(c) = 2Ac+B$$

From , $A(b+a)+B = 2Ac+B$

$$\Rightarrow \boxed{c = \frac{1}{2}(a+b).}$$

$[c=\frac{1}{2}(a+b)]$ is the <u>middle point</u> of the interval $[a,b]$. This is a <u>most surprising result</u>, because the 'instantaneous' mean value is attained at the central point of $[a,b]$ independently of A, B, C. If the function were a straight line, $Ax+B$ or a polynomial of order three, for example, Ax^3+Bx^2+Cx+D, c would depend <u>on</u> the particular curve. Not with a parabola...



SOLUTIONS 2022-2023

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2022-2023

First Mid-term Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**

P1 [2 pt] The function $f(x) = \arctan(\log \sin x)$ is defined on the interval $(0, \pi)$ exclusively.

1) Calculate the límit of $f(x)$ when $x \to 0^+$ and when $x \to \pi^-$.
2) Sketch the graph of f and indicate point symmetries (if any).
3) Find $R\,f$, the range of f.
4) Discuss whether f has an absolute minimum in its domain.
5) Calculate $f'(x)$, the derivative of f with respect to x.

P2 [2 pt] Consider the parabola $f(x) = Ax^2 + Bx + C$ with $A \neq 0, B, C$ constants. Determmine if the *Mean Value Theorem* can be applied to $f(x)$ on the inteval $[a, b]$ and

1) find all possible values of $c \in [a, b]$ that satisfy the theorem.
2) Comment briefly (two lines) the result obtained in 1).

P3 [2pt+2pt] a) Calculate the limits when $n \to \infty$ of the sequences:

1) $\sqrt{n \log n + 4n^2} - 2n\, e^{1/n}$, 2) $\dfrac{\sqrt{n \log n + 4n^2} - 2n\, e^{1/n}}{\log n}$.

If the limit is 0 (or ∞) add also a sentence of the type "and goes to 0 as $1/n^2$", for instance.

b) Show that

$$\tan\left(\frac{\pi}{4} + a\right) - \tan\left(\frac{\pi}{4} - a\right)$$

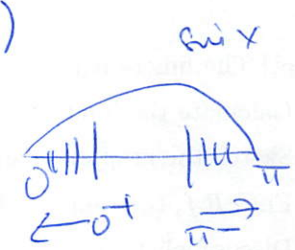
where a is an arbitrary real number can be written as $b \tan ca$ with given numbers b, c.

P4 [2 pt] Consider $f(x) = \dfrac{1}{\cos x}$ defined on $[0, \pi] - \{\pi/2\}$. a) Find the domain and range of $g \equiv f^{-1}$. b) Find the equation of the straight line which is tangent to $g(x)$ at the point $x = \pi$.

(P1) The function $f(x) = \arctan(\log \sin x)$ is studied exclusively on $(0, \pi)$. We do not consider any other interval.

$f(x)$ is a composition of functions continuous on every interval under consideration: $\sin x$ is cont on $(0,\pi)$, $\log$ is cont on $(0,1)$ and $\arctan$ is cont on $\mathbb{R}$, therefore
$\arctan(\log(\sin x))$ is cont on $(0,\pi)$

1) $\lim_{x \to 0^+} f(x) = \arctan(\log 0^+)$
$\to \arctan(-\infty)$
$= -\pi/2$



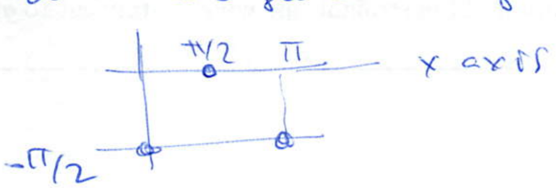
$\lim_{x \to \pi^-} f(x) = \arctan(\log 0^+)$
$= -\pi/2$ too

principal branch of tan is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

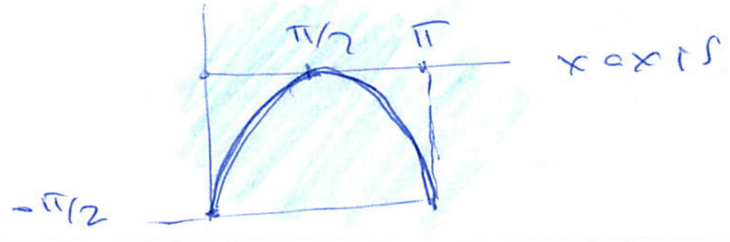
2) $$f'(x) = \frac{\frac{\cos x}{\sin x}}{1 + (\log(\sin x))^2}$$

The denominator of $f'(x)$ is a strictly positive number larger (or equal) to 1, so $f'(x)$ exists for all x in $D_f = (0,\pi)$, meaning that f has no sharp points or corners: f is smooth.

Also at $x = \pi/2$, $f(\frac{\pi}{2}) = \arctan(\log 1) = 0$
so our continuous function goes through the points



f smooth then there is a maximum at $x = \frac{\pi}{2}$
[⌒]: the graph of f is



[It is impossible

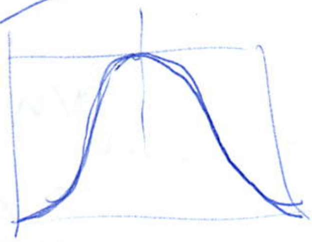
because there is no vertical asymptote

because there f' DNE (does not exist)

and $\arctan(\underbrace{\log 0^+}_{\to -\infty})$ is a vertical asymptote..

same reason here.

It is impossible



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3) From the graph of f on $(0, \pi)$

$R_f = \left(-\frac{\pi}{2}, 0\right]$

4) No. The absolute minimum would be at $x=0$ or $x=\pi$ with value $f_{min} = -\frac{\pi}{2}$, but $0, \pi \notin (0, \pi)$. There is no an absolute minimum.

[Last thing: from the graph of f one would say that there is a symmetry around $x = \frac{\pi}{2}$. We check algebraically that this is the case, because

$$f\left(\frac{\pi}{2}+x\right) = f\left(\frac{\pi}{2}-x\right)$$

$\sin\left(\frac{\pi}{2}+x\right) = \cos x$, $\quad \sin\left(\frac{\pi}{2}-x\right) = \cos x$

and

$$f\left(\frac{\pi}{2}+x\right) = \arctan\left(\log\left(\sin\left(\frac{\pi}{2}+x\right)\right)\right)$$
$$= \arctan \log(\cos x)$$
$$f\left(\frac{\pi}{2}-x\right) = \arctan\left(\log\left(\sin\left(\frac{\pi}{2}-x\right)\right)\right)$$
$$= \arctan \log(\cos x)$$

$$\Rightarrow f\left(\frac{\pi}{2}+x\right) = f\left(\frac{\pi}{2}-x\right)$$

End of the story.

P3 a) Result:

$$\sqrt{n\log n + 4n^2} - 2ne^{1/n} \underset{n \text{ large}}{\simeq} \frac{\log n}{4} \xrightarrow[n\to\infty]{} \infty$$

b) $$\frac{\sqrt{n\log n + 4n^2} - 2ne^{1/n}}{\log n} \underset{n \text{ large}}{\simeq} \frac{1}{4}$$

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a) The limit

$$\lim_{n\to\infty} \sqrt{n\log n + 4n^2} - 2ne^{1/n} = \infty - \infty$$

indetermination form.

$$\sqrt{n\log n + 4n^2} - 2ne^{1/n}$$
$$= \sqrt{4n^2\left(\frac{\log n}{4n}+1\right)} - 2ne^{1/n}$$
$$= 2n\left[\sqrt{1+\frac{\log n}{4n}} - e^{1/n}\right]$$

$\sqrt{n^2} = |n| = n$

$$= 2n\left[\sqrt{1+\frac{\log}{4n}} - e^{1/n}\right] \cdot \frac{\left[\sqrt{1+\frac{\log}{4n}} + e^{1/n}\right]}{\left[\sqrt{1+\frac{\log n}{4n}} + e^{1/n}\right]}$$

$$= 2n\,\frac{1+\frac{\log}{4n} - e^{2/n}}{\sqrt{1+\frac{\log n}{4n}} + e^{1/n}}$$

↑ up to here all are identities

$$\underset{n\,\text{large}}{\sim} \frac{2n\left[\cancel{1} + \frac{\log n}{4n} - \cancel{1}\right]}{\sqrt{1+\frac{\log n}{4n}} + e^{4/n}}$$

tends to 2 if $n \to \infty$

Notice: $\infty - \infty$ (we cancel one ∞ with another: $2n$ cancels with $2n$)

$\infty - \infty$

$\cancel{2n} - \cancel{2n}$

(and then $\frac{\log n}{n}$ becomes the dominant term)

$\hookrightarrow$ has been promoted !!

$$\underset{n\,\text{large}}{\approx} \frac{2n \frac{\log n}{4n}}{2}$$

$$= \frac{\log n}{4} \to \infty.$$

Conclusion: $\sqrt{n\log n + 4n^2} - 2ne^{4/n} \simeq \frac{\log n}{4} \xrightarrow[n\to\infty]{} \infty$

b) obvious from the previous result:

$$\frac{\sqrt{n\log n + 4n^2} - 2ne^{4/n}}{\log n} \simeq \frac{1}{4} \quad \text{when } n \to \infty$$

———— o O o ———— o O o ————

2) Use

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

that comes from

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B}.$$

Thus

$$\tan\left(\frac{\pi}{4}+a\right) = \frac{\tan\frac{\pi}{4} + \tan a}{1 - \tan\frac{\pi}{4}\tan a} = \frac{1+\tan a}{1-\tan a},$$

$\tan\frac{\pi}{4} = 1$

and

$$\tan\left(\frac{\pi}{4}+a\right) - \tan\left(\frac{\pi}{4}-a\right) = \frac{1+\tan a}{1-\tan a} - \frac{1-\tan a}{1+\tan a}$$

$$= \frac{(1+\tan a)^2-(1-\tan a)^2}{1-\tan^2 a}$$

$$= \frac{4\tan a}{1-\tan^2 a}$$

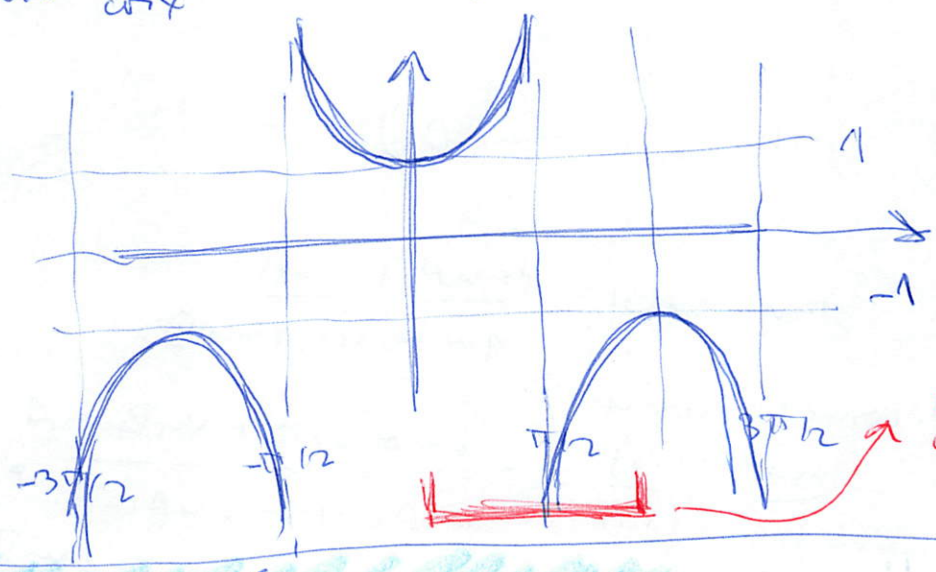
$$= 2\tan 2a \,,$$

$\tan 2a = \dfrac{2\tan a}{1-\tan^2 a}$ is a very well known formula that always comes to one's brain if you study it. [*]

Result:

$$\tan\left(\frac{\pi}{4}+a\right)-\tan\left(\frac{\pi}{4}-a\right) = 2\tan 2a \quad \forall a$$

(P4) $f(x) = \dfrac{1}{\cos x}$ is a very simple function



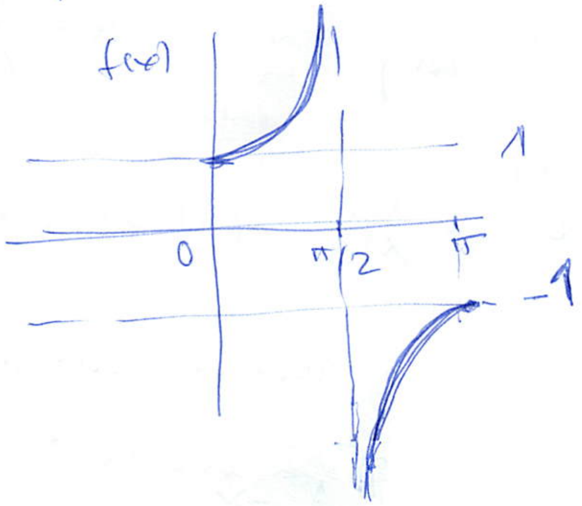
we study it only in this interval

- Characteristic of ignoramus

[*] Disobey the now frequent voices that say that "memory is not important", "do not learn things by heart" and some other stupidities. A mathematician / Physicist without memory is nothing: never will achieve anything.

Then $f(x) = \frac{1}{\cos x}$ restricted to $Df = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

is



$Df = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Rf = from the picture = $[1, \infty) \cup (-\infty, -1]$

f is one-to-one in its domain (passes the horizontal line test)

[Immediately after this we can establish that

$g \equiv f^{-1}$, $\quad Dg = (-\infty, -1] \cup [1, \infty)$

$\quad Rg = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

and that, for instance,

$f(0) = 1 \iff g(1) = 0$

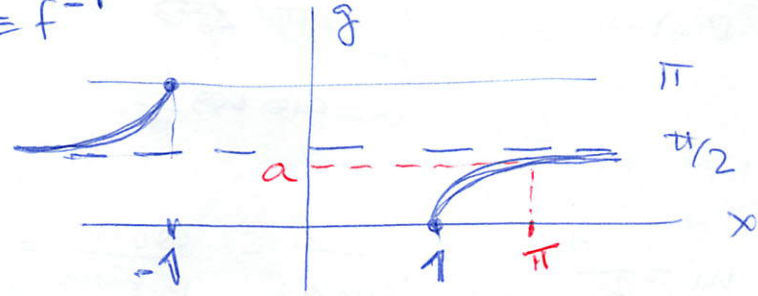
$f(\pi) = -1 \iff g(-1) = \pi$

$f(\frac{\pi}{2}^-) = \infty \iff g(\infty) = \frac{\pi}{2}^-$

$f(\frac{\pi}{2}^+) = -\infty \iff g(-\infty) = \frac{\pi}{2}^+$

$f(\pi/4) = \sqrt{2}$, $f(\frac{\pi}{6}) = \frac{1}{\cos \pi/6} = \frac{2}{\sqrt{3}}$.

Graph of $g \equiv f^{-1}$



More about g:

$\cancel{y = \frac{1}{\cos x}}$, $x = \frac{1}{\cos y}$ or $x = \frac{1}{\cos g}$

to avoid errors

Then,

$$g(x) = \arccos \frac{1}{x} \quad \text{(if we want to)}.$$

$x \in D_g$.

It is true. I do not use it but it is true

equivalent expressions

Anyway

$$\cos g = \frac{1}{x} \iff x \cos g = 1 \iff g = \arccos \frac{1}{x},$$

the derivative of g wrt x is (you may derive any of the previous three expressions, you always get the same g')

$$-\sin g \cdot g' = -\frac{1}{x^2} \implies g' = \frac{1}{x^2 \sin g}$$

$$\cos g - x \sin g \cdot g' = 0 \implies g' = \frac{\cos g}{x \sin g}$$

$$g'(x) = -\frac{-1/x^2}{\sqrt{1-\frac{1}{x^2}}} = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{|x|\sqrt{x^2-1}}$$

even even

The straight line

$$y - y_0 = m(x - x_0)$$

is $x_0 = \pi$, $y_0 = g(\pi) = a$ (picture) $\iff f(a) = \pi$, a a number in $(0, \pi/2)$:

$$f(a) = \pi \iff \frac{1}{\cos a} = \pi \iff \cos a = \frac{1}{\pi}$$

$$\iff a = \arccos \frac{1}{\pi}$$

take what you prefer.

$$m = \frac{1}{\pi^2 \sin g(\pi)} = \frac{1}{\pi^2 \sin a} = \frac{\cos a}{\pi \sin a} = \frac{1}{\pi\sqrt{\pi^2-1}}$$ [many options]

The line is (take what you prefer)

$$\text{straight line} = a + \frac{\cos a}{\pi \sin a}(x-\pi)$$

I like these more...

$$\text{straight line} = a + \frac{1}{\pi^2 \sin a}(x-\pi)$$

$$\text{straight line} = \underbrace{\arccos \frac{1}{\pi}}_{\text{is in } \left(0, \frac{\pi}{2}\right)} + \frac{x-\pi}{\pi\sqrt{\pi^2-1}}$$

simple but ugly

$$\left[\begin{array}{l} \cos a = \frac{1}{\pi} \\ \sin a = \frac{1}{\pi}\sqrt{\pi^2-1} \end{array}\right.$$