2022-2023: SOLUTIONS

P2: Matemáticas

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2022-2023

2022-2023-Segundo Examen Parcial de Matemáticas-E

Includes the English version of the exam.

Nombre y Apellido:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos. No se corregirá nada que no esté escrito en este cuadernillo.

P1 [2pt] Encontrar la suma en términos de funciones elementales de cada una de siguientes series de potencias

i) $1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\dfrac{x^6}{7!}+\dfrac{x^8}{9!}\cdots$ ii) $1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\dfrac{x^8}{8!}\cdots$

iii) $1-\dfrac{x^2}{3}+\dfrac{x^4}{5}-\dfrac{x^6}{7}+\dfrac{x^8}{9}\cdots$ iv) $\dfrac{x^2}{2}+\dfrac{x^4}{4}+\dfrac{x^6}{6}+\dfrac{x^8}{8}\cdots$

P2 [2pt] Sea la curva de ecuación $y^2 = x^4\,(1-x)$. Dibujar sus dos ramas y_+ e y_- cuando $x > 0$ y calcular el área que encierran.

Nota: A las dos ramas juntas se las conoce como perla de Sluze o *teardrop curve.*

P3 [2pt] a) Hallar los coefficientes a, b, c en el desarrollo de $f(x)$ alrededor de $x = 0$ dado por

$$\Big(1+\frac{x}{2}\Big)^{-2/3}-\Big(1+\frac{2x}{3}\Big)^{-1/2}=a+b\,x+c\,x^2+\cdots .$$

b) Cuál es el *radio de convergencia* R de la serie?

c) La integral $\displaystyle\int_0^1 dx\,\frac{f(x)}{x^{5/2}}$ de los tres primeros términos, ¿tiene un valor finito? Si es así calcule el valor.

P4 [2pt] Determinar si las siguiientes series numéricas son absolutamente convergentes, convergentes o divergentes Diga claramente qué test o tests está usted utilizando para responder.

i) $\sum_{n=1}^{\infty}(\sqrt{1+n^2}-n)$, ii) $\displaystyle\sum_{n=1}^{\infty}\sqrt[n]{n\log n}$, iii) $\displaystyle\sum_{n=0}^{\infty}\binom{5}{n}(-3)^n$.

P5 [2pt] Probar que $\sum_{n=2}^{\infty}\dfrac{1}{n^{3/2}}$ es convergente (vale con un test bien elegido). ¿Dónde está mal la siguiente "demostración" de que es divergente?

$$\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{27}}+\frac{1}{\sqrt{64}}+\frac{1}{\sqrt{125}}+\cdots>\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{36}}+\frac{1}{\sqrt{81}}+\frac{1}{\sqrt{144}}+\cdots$$

que es

$$\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\cdots=\frac{1}{3}\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\Big).$$

Como la serie armónica diverge, la serie original diverge.

Ⓟ2 $y^2 = x^4(1-x)$

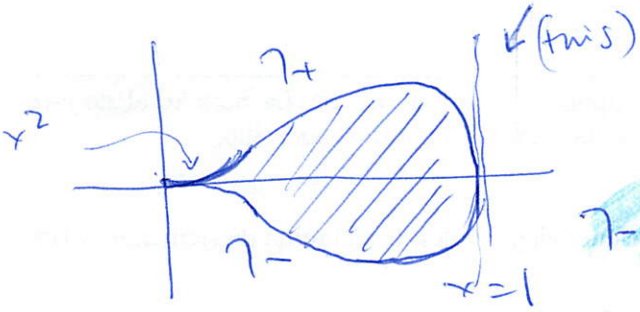
$y_+ = x^2\sqrt{1-x}$, $y_- = -x^2\sqrt{1-x}$

in obvious notation.

y_+: Sketch: No derivative is necessary:

x near 0 , $y_+ \simeq x^2$: $\smile$ plot

x = 1 : has a vertical tangent



y_- is the negative of y_+

$$\text{Area} = 2\int_0^1 dx\, x^2\sqrt{1-x}$$

$t=\sqrt{1-x}$: Good change

[Here $t = 1-x$ also results in a simple integral but $t=\sqrt{1-x}$ is easier.

$t^2 = 1-x$, $x = 1-t^2$

$2t\,dt = -dx$

$$\text{Area} = 2\int_0^1 dx\, x^2\sqrt{1-x}$$

$$= -2\int_1^0 dt\, 2t\underbrace{(1-t^2)^2}_{=x^2}\underbrace{t}_{=\sqrt{1-x}} = 4\int_0^1 dt\, t^2(1-t^2)^2$$

$$= 4\int_0^1 dt\,(t^2-2t^4+t^6)$$

$$= 4\underbrace{\left[\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right]}_{= 8/105}$$

$$= \frac{32}{105}$$

Area of

$= \frac{32}{105}$

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2022-2023

2022-2023-Second Mid-term Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**

P1 [2pt] Find the exact sum of each of the following power series in terms of elementary functions

i) $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\cdots$ ii) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}\cdots$

iii) $1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\frac{x^8}{9}\cdots$ iv) $\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\frac{x^8}{8}\cdots$

P2 [2pt] Consider the curve $y^2 = x^4\,(1-x)$. Sketch the graph of its two branches y_+ and y_- when $x > 0$ and find the area inside.

Note: The two branches together are known as Sluze Pearl or as teardrop curve.

P3 [2pt] a) Find the coefficients a, b, c in the series expansion around $x = 0$ of $f(x)$ given by

$$\left(1+\frac{x}{2}\right)^{-2/3}-\left(1+\frac{2x}{3}\right)^{-1/2}=a+bx+cx^2+\cdots .$$

b) What is the value of the *radius of convergence* R?

c) Is the integral $\int_0^1 dx\,\frac{f(x)}{x^{5/2}}$ of the first three terms finite? If so, find its value.

P4 [2pt] Determine whether the following series are absolutely convergent, convergent or divergent. State clearly the test or tests used to answer.

i) $\sum_{n=1}^{\infty}(\sqrt{1+n^2}-n)$, ii) $\sum_{n=1}^{\infty}\sqrt[n]{n\log n}$, iii) $\sum_{n=0}^{\infty}\binom{5}{n}(-3)^n$.

P5 [2pt] Show that $\sum_{n=2}^{\infty}\frac{1}{n^{3/2}}$ is convergent (use a convenient test to prove it). What is wrong with the following "proof" that it diverges?

$$\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{27}}+\frac{1}{\sqrt{64}}+\frac{1}{\sqrt{125}}+\cdots>\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{36}}+\frac{1}{\sqrt{81}}+\frac{1}{\sqrt{144}}+\cdots$$

which is

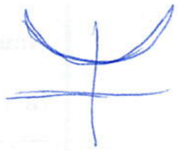
$$\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\cdots=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right).$$

Since the harmonic series diverges, the original series diverges.

P1 i) $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\dots$

$$=\frac{1}{x}\underbrace{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right)}_{\sin x \text{ around } x=0}=\frac{\sin x}{x}$$

equal: 7 & 7!



ii) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots=\cosh x$

↗ like $\cos x$ but all terms positive.

can be deduced from (if needed)

$$e^x=1+\cancel{x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots$$

and

$$\cosh x \underset{\text{definition}}{=} \frac{1}{2}\left(e^x+e^{-x}\right)$$

iii) $1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\dots$ equal digit

$$=\frac{1}{x}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\right)=\frac{\arctan x}{x}.$$

If you do not know that $\arctan x$ is $x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$ integrate term by term the geometric series (and dont forget the integration constant)

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\dots, \quad |x|<1$$

iv) $S\equiv\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots$ same digit but does not sound familiar. But its derivative is very famous $'\equiv\frac{d}{dx}$

call it S

$$S'=x+x^3+x^5+\dots=x\left(1+x^2+x^4+\dots\right)$$

$$=\frac{x}{1-x^2}, \quad R=1$$

The primitive of $\frac{x}{1-x^2}$ is very simple:

$$-\frac{1}{2}\log\left(1-x^2\right)+C$$

and

$$S = -\frac{1}{2}\log(1-x^2) + C_1$$

↗ to be fixed with an easy point inside $|x|<1$ ($x=0$ is the point)

$\Rightarrow C_1 = 0$, because S is also $\frac{x^2}{2} + \frac{x^4}{4} + \dots$

conclusion: $\boxed{S = -\frac{1}{2}\log(1-x^2)}$

comment (aside): All series in this problem are even around $x=0$;

$\frac{\sin x}{x}$, $\cosh x$, $\frac{\arctan x}{x}$, $-\frac{1}{2}\log(1-x^2)$

are also even functions around $x=0$.

(P3) $f(x) \equiv \left(1+\frac{x}{2}\right)^{-2/3} - \left(1+\frac{2x}{3}\right)^{-1/2}$.

a) $\left(1+\frac{x}{2}\right)^{-2/3} - \left(1+\frac{2x}{3}\right)^{-1/2} = a + bx + cx^2 + \dots$

obviously (no other considerations are needed): $\boxed{a=0}$: substitute $x=0$ on the previous line: $0 = a$

Also from binomial series (best manner to deduce b, c)

$$\left(1+\frac{x}{2}\right)^{-2/3} = 1 + \binom{-2/3}{1}\frac{x}{2} + \binom{-2/3}{2}\left(\frac{x}{2}\right)^2 + \dots$$

$$= 1 - \frac{x}{3} + \frac{5}{36}x^2 - \dots \qquad \boxed{R_1 = 2}$$

since $\binom{-2/3}{1} = -2/3$

$$\binom{-2/3}{2} = \frac{(-2/3)(-2/3-1)}{2!} = \frac{5}{9}$$

following $\binom{m}{2} = \frac{m(m-1)}{2!}$, etc.

$$\left(1+\frac{2x}{3}\right)^{-1/2} = 1 - \frac{1}{2}\left(\frac{2x}{3}\right) + \binom{-1/2}{2}\left(\frac{2x}{3}\right)^2 + \binom{-1/2}{3}\left(\frac{2x}{3}\right)^3 + \dots$$

$$= 1 - \frac{x}{3} + \frac{x^2}{6} + \frac{5}{54}x^3 + \dots$$

$$\binom{-1/2}{3} = \frac{-1/2(-1/2-1)(-\frac{1}{2}-2)}{3!} = -\frac{5}{16}$$

$$\boxed{R_2 = 3/2}$$

$$\left(1+\frac{x}{2}\right)^{-2/3} - \left(1+\frac{2x}{3}\right)^{-1/2} = 1-1$$
$$+ x\left[-\frac{1}{3}+\frac{1}{3}\right]$$
$$+ x^2\left[\frac{5}{36}-\frac{1}{6}\right]$$
$$+ x^3\left[\text{not necessary}\right] + \dots$$
$$= 0 + 0\cdot x + x^2\left(-\frac{1}{36}\right) + \dots$$

$$\boxed{\begin{array}{l} a=b=0 \\ c=-\frac{1}{36} \end{array}}$$

Radius of convergence: $\min(R_1, R_2) = 3/2$ (the intersection is $\min(R_1, R_2)$ too).

Answer: $\left(1+\frac{x}{2}\right)^{-2/3} - \left(1+\frac{2x}{3}\right)^{-1/2} = -\frac{x^2}{36} + \dots$ $\boxed{R = 3/2}$

b)

or $|x| < 3/2$: interval of convergence.

c) $$\int_0^1 dx\, \frac{a+b\cdot x+cx^2}{x^{5/2}} = -\frac{1}{36}\int_0^1 dx\, \frac{1}{x^{1/2}} = -\frac{1}{18}\int_0^1 dx\sqrt{x}$$

$a=b=0$
$c=-1/36$

not singular at $x=0$ because the antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$ that tends to 0 if $x \to 0$.

$$= -\frac{1}{18}$$

(P4) i) $\sum_{n=1}^{\infty} \left(\sqrt{1+n^2} - n\right)$ D by comparision with $\sum_1^{\infty} \frac{1}{2n}$

ii) $\sum_{n=1}^{\infty} \sqrt[n]{n \log n}$ D by the Preliminary test

iii) $\sum_{n=0}^{\infty} \binom{5}{n} (-3)^n$ AC, absolutely convergent
It is a finite sum and not a series.

i) $$\sqrt{1+n^2} - n = \left(\sqrt{1+n^2} - n\right) \frac{\left(\sqrt{1+n^2} + n\right)}{\sqrt{1+n^2} + n}$$

$$= \frac{1 + \cancel{n^2} - \cancel{n^2}}{\sqrt{1+n^2} + n}$$

$$\underset{n \text{ large}}{\simeq} \frac{1}{2n}$$

Since
$$\lim_{n\to\infty} \frac{\sqrt{1+n^2} - n}{1/2n} = 1$$
both series, $\sum^{\infty} \left(\sqrt{1+n^2} - n\right)$ and $\sum \frac{1}{2n}$ have the same character, i.e, divergent.

ii) $\lim_{n\to\infty} \sqrt[n]{n \log n} = 1$
Divergent $\sum^{\infty} \sqrt[n]{n \log n}$ by the preliminary test

$$\sqrt[n]{n \log n} = e^{\frac{\log n}{n}} \cdot e^{\frac{\log(\log n)}{n}} \xrightarrow[n\to\infty]{} e^0 \cdot e^0 = \underline{1}$$

Both $\underset{0<n}{0 < \frac{\log n}{n}}$ and $\frac{\log(\log n)}{n} \xrightarrow[n\to\infty]{} 0$
tend to 0 if $n \to \infty$: Sandwich theorem proves ~~thet~~ $\frac{\log(\log n)}{n} \xrightarrow[n\to\infty]{} 0$:

$$0 < \frac{\log(\log n)}{n} < \frac{\log n}{n}$$
→ to zero if $n \to \infty$

$$L \Rightarrow \frac{\log(\log n)}{n} \xrightarrow[n\to\infty]{} 0.$$

(iii) $\sum_{n=0}^{\infty} \binom{5}{n}(-3)^n = 1-\binom{5}{1}3+\binom{5}{2}3^2 - \cdots -\binom{5}{5}3^5 + \underline{\underline{0}}$

the rest of the series is $0+0+0+\ldots$

since $\binom{5}{n}=0$ for $n>5$. We have an ordinary sum, i.e., absolutely convergent.

N.B.: Ratio test does not apply here.

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⑤ $\sum_{1}^{\infty} \frac{1}{n^{3/2}}$ is convergent by the Integral test,

since $f(x) = \frac{1}{x^{3/2}}$ is continuous, positive and decreasing on $[1,\infty)$. Moreover,

please, say it. Are the conditions to apply the test.

$$\int^{\infty} \frac{dx}{x^{3/2}} = -2\left[\frac{1}{\sqrt{x}}\right]^{\infty} = 0 \leftarrow \text{finite}$$

↑ not considered the lower limit.

Primitive of $\frac{1}{x^{3/2}}$ is $-\frac{2}{\sqrt{x}}$

$\Rightarrow \sum_{1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

$\sum_{2}^{\infty} \frac{1}{n^{3/2}} = \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \frac{1}{\sqrt{64}} + \cdots$

The "proof" correctly establishes that

$\frac{1}{\sqrt{8}} > \frac{1}{\sqrt{9}} = \frac{1}{3}$, true

$\frac{1}{\sqrt{27}} > \frac{1}{\sqrt{36}} = \frac{1}{6}$ true

$\frac{1}{\sqrt{64}} > \frac{1}{\sqrt{81}} = \frac{1}{9}$ true and assumes that the inequalities will hold for all n... what is wrong. With numbers...

$\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{27}}, \frac{1}{\sqrt{64}}$ is $\frac{1}{n^{3/2}}$ for $n = 2, 3, 4$

$\frac{1}{3}, \frac{1}{6}, \frac{1}{9}$ is $\frac{1}{3(n-1)}$ again for $n = 2, 3, 4$,

but $\frac{1}{n^{3/2}}$ is not larger than $\frac{1}{3(n-1)}$ if $n \geq 9$ for instance. On the contrary, it is smaller

When is $n^{3/2} = 3(n-1)$? Or easier (and no loss of generality here) when

$$n^{3/2} = 3n \; ?$$

for positive n? Obviously when

$$n^3 = 9n^2,$$

which corresponds to

$$n = 9.$$

Conclusion: a polynomial of order 3, n^3, grows faster when n is larger than any polynomial of order 2, $9n^2$. Here large enough is $n \geq 9$. The comparison is with $3n$ because $3(n-1) < 3n$ for any $n > 1$:

$$n^{3/2} < 3(n-1) \overset{\text{always if } n>1}{<} 3n$$

is

$$\underbrace{\frac{1}{n^{3/2}} > \frac{1}{3(n-1)}}_{\text{this is false if } n \geq 9} \underset{\uparrow \text{ always true.}}{>} \frac{1}{3n}$$

↳ This exercise is from Boas' book.