SOLUTIONS
Extraordinary Exam
June 2024

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2023-2024

2023-2024 Examen Extraordinario de Matemáticas: grupo E

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos. No se corregirá nada no incluido en este cuadernillo. Más claramente: si se entregan hojas en sucio ésas no se corrigen.

P1 [2pt] a) Calcular los puntos de inflexión de la gráfica de la función $f(x) = \dfrac{1}{\cosh^2 x}$ expresando su valor mediante un logaritmo. b) Representar gráficamente $f(x)$.

P2 [2pt] Sean a, b, c números reales arbitrarios. Demostrar mediante el *Teorema del Valor Medio* que

$$4a\,x^3 + 3b\,x^2 + 2c\,x = a + b + c$$

tiene siempre una solución entre 0 y 1

P3 [2.30pt] Sean las funciones

$$f(x) = e^x \log(1 + 2x), \qquad g(x) = e^{-x} \log(1 + 2x).$$

El logaritmo es neperiano por supuesto. Se pide hallar:

a) los **tres** primeros términos no nulos del desarrollo en serie de potencias de $f(x)$ y de $g(x)$ en torno a $x = 0$,

b) los límites

$$\lim_{x\to 0} \frac{f(x) - 2x}{x^3}, \qquad \lim_{x\to 0} \frac{g(x) - 2x}{x^3}.$$

c) el ángulo existente entre las rectas tangentes a las gráficas de $f(x)$ y de $g(x)$ en $x = 0$,

d) el signo (si es positiva o si es negativa o si es cero) de la integral

$$\int_{-0.01}^{0.01} dx\, \frac{\log(1 + 2x)}{e^x}.$$

P4 [0.4 + 0.2 + 0.2 + 1.5 + 1.4 = 3.7pt]

a) Encontrar el coeficiente de x^{14} en el desarrollo de $(1 + x^5 + x^9)^{10}$.

b) Encontrar el ángulo b que satisface la identidad $4\cos a + 3\sin a = 5\cos(a + b)$. Indicar en qué cuadrante está dicho ángulo.

c) "Si la sucesión $\{a_n\}$ no converge, la sucesión $\{a_n/n\}$ no converge". ¿Es la afirmación anterior cierta o falsa? Si es cierta, pruébela, si es falsa encuentre un contraejemplo.

d) Discutir en función del valor del parámetro b la convergencoa o divergencia de la serie

$$\sum_{n=1}^{\infty} \left(1 - b\,\cos\frac{1}{n}\right).$$

e) Completando cuadrados en el denominador encontrar la integral indefinida

$$\int \frac{dx}{\sqrt{x^2 + 2x + 5}}.$$

(P4) a) Use the "Newton binomial formula" (or theorem)

$$(1+a)^m = \sum_{n=0}^{m} \binom{m}{n} a^n, \qquad (*)$$

which is special, since one with a one in the left hand side.

$$(1+x^5+x^9)^{10} \overset{(*)}{=} \sum_{n=0}^{10} \binom{10}{n} (x^5+x^9)^n$$

use

$$= \sum_{n=0}^{10} \binom{10}{n} x^{5n} (1+x^4)^n$$

use again

$$\overset{(*)}{=} \sum_{n=0}^{10} \sum_{m=0}^{n} \binom{10}{n}\binom{n}{m} x^{5n+4m}$$

n	m	$5n+4m$
0	0	0
1	0	5
1	1	9
2	0	10
2	1	14
2	2	18
3	0	15

: all too big (>14)

$$= \dots + \underbrace{\binom{10}{2}\binom{2}{1}}_{\text{this is the coefficient!}} x^{14} + \dots$$

$$\binom{10}{2}\binom{2}{1} = \frac{10!}{\cancel{2}\cdot 8!}\cdot\cancel{2} = 10\cdot 9 = \boxed{90}$$

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b) $4\cos a + 3\sin a = 5\cos(a+b)$

is an identity ($\equiv$ meaning that is valid for all a).

Use the identity

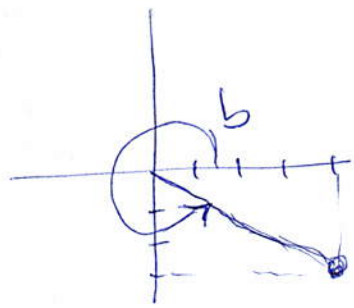
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

valid for all a, b real or complex, and write

$$4\cos a + 3\sin a = 5\cos(a+b)$$
$$= 5(\cos a \cos b - \sin a \sin b)$$

$$\Rightarrow \begin{cases} \cos b = 4/5 & (\text{take } a=0 \text{ above}) \\ \sin b = -3/5 & (\text{take } a = \pi/2), \end{cases}$$

which is an angle (no matter which!!) in the 4th quadrant



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c) The problem asks about sequences, not series.
The statement is FALSE.
Counter example: The sequence $\{a_n\}_1^\infty = \{1, 2, 3, \dots, n, \dots\}$ has no limit when $n \to \infty$ but $\{a_n/n\}_1^\infty = \{1, 1, 1, \dots 1\}$ has limit $_{n\to\infty}$ 1.

[to be convergent is to have a finite limit]

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UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2023-2024

2023-2024 Extraordinary Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**

P1 [2pt] a) Find the points of inflection of the graph of $f(x) = \dfrac{1}{\cosh^2 x}$, and express your result in terms of logarithms. b) Sketch the graph of $f(x)$.

P2 [2pt] Let a, b, c be real numbers. Show using the *Mean Value Theorem* (MVT) that

$$4a\,x^3 + 3b\,x^2 + 2c\,x = a + b + c$$

always has a solution between 0 and 1.

P3 [2.30pt] Given the functions

$$f(x) = e^x \log(1 + 2x), \qquad g(x) = e^{-x} \log(1 + 2x),$$

where log means Napierian logarithm, find:

a) the first **three non zero** terms of the power series in x for $f(x)$ and $g(x)$,

b) the limits

$$\lim_{x\to 0} \frac{f(x) - 2x}{x^3}, \qquad \lim_{x\to 0} \frac{g(x) - 2x}{x^3},$$

c) the angle between the tangent line to the graph of $f(x)$ and the tangent line to the graph of $g(x)$ at $x = 0$,

d) whether the next integral is positive, negative or zero

$$\int_{-0.01}^{0.01} dx\, \frac{\log(1 + 2x)}{e^x}.$$

P4 [0.4 + 0.2 + 0.2 + 1.5 + 1.4] = 3.7 pt

a) Find the coefficient of x^{14} in the expansion of $(1 + x^5 + x^9)^{10}$.

b) Find the angle b that satisfies the identity $4\cos a + 3\sin a = 5\cos(a + b)$. Indicate in which quadrant b is.

c) 'If the sequence $\{a_n\}$ does not converge then the sequence $\{a_n/n\}$ does not converge'. Is the previous statement true or false? If true prove it, if false give a counterexample.

d) Discuss in terms of the parameter b the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(1 - b\,\cos\frac{1}{n}\right).$$

e) By completing the square in the denominator find the indefinite integral

$$\int \frac{dx}{\sqrt{x^2 + 2x + 5}}.$$

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d) $\sum_{n=1}^{\infty}\left(1-b\cos\frac{1}{n}\right)$:

Only convergent if $b=1$. For all other values of b is divergent

We see it.

$$a_n = 1-b\cos\frac{1}{n}.$$

Preliminary test (necessary condition for convergence):

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum^{\infty} a_n \text{ diverges.}$$

$$\lim_{n\to\infty}\left(1-b\cos\frac{1}{n}\right) = 1-b.$$

If $b\neq 1$ the series is divergent. When $b=1$,

$$\begin{aligned} a_n &= 1-\cos\frac{1}{n} \\ &= \frac{1}{2!n^2}-\frac{1}{4!n^4}+\frac{1}{6!n^6}-\cdots \\ &= \frac{1}{2!n^2}\underbrace{\left(1-\frac{2}{4!n^2}+\frac{2}{6!n^4}-\cdots\right)}_{\text{tends to } 1 \text{ if } n\to\infty}, \end{aligned}$$

then

$$\lim_{n\to\infty}\frac{1-\cos\frac{1}{n}}{\frac{1}{2!}\frac{1}{n^2}} = 1$$

meaning that $\sum^{\infty}\left(1-\cos\frac{1}{n}\right)$ and $\sum^{\infty}\frac{1}{2!}\frac{1}{n^2}$ have the same character. Since

$$\frac{1}{2!}\underbrace{\sum^{\infty}\frac{1}{n^2}}_{\text{Basel's series}} \text{ is convergent}$$

$$\Rightarrow \sum^{\infty}\left(1-\cos\frac{1}{n}\right) \text{ is convergent.}$$

[This is the 'Test of comparison by the limit' explained in our lectures (*) If $\sum^{\infty} a_n$

(*) In Spanish: 'Test de comparación por paso al límite'.

and $\sum^{\infty} b_n$ are series of positive terms ($a_n, b_n > 0$ for all n) and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C, \quad C \text{ a finite number}$$

then either both series converge or both series diverge.

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e) You only need to remember (or to obtain in the exam the exact primitive if you have a vague idea)

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

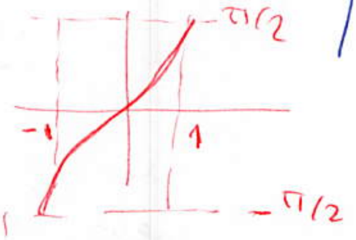
$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}}$$

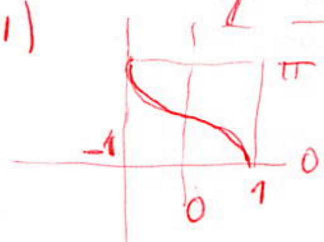
$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}}$$

In fact this is, to be precise but not necessary right now,

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1)$$

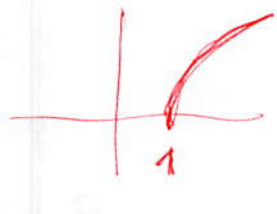


$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1)$$



(no plot here) $\quad (\operatorname{arcsinh} x)' = \dfrac{1}{\sqrt{x^2+1}}, \quad x \in (-\infty, \infty)$

$\operatorname{arccosh} x = \dfrac{1}{\sqrt{x^2-1}}, \quad x \in (1, \infty)$ principal branch



always a 1 there (green)

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{4+(x+1)^2}}$$

$$x^2+2x+5 = (x+1)^2 + 4$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{1+\left(\frac{x+1}{2}\right)^2}}$$

notice: we manipulate to get a 1 there.

$$= \int \frac{du}{\sqrt{1+u^2}}$$

change of variables
$u = \frac{x+1}{2}$
$du = \frac{dx}{2}$

$$= \operatorname{arcsinh} u$$

$$= \operatorname{arcsinh}\left(\frac{x+1}{2}\right) + C$$

hyperbolic sine (totally different of $\arcsin x$)

integration constant

Comment: Some students know that
$\operatorname{arcsinh} x = \log\left(x+\sqrt{x^2+1}\right)$, $x \in (-\infty, \infty)$.
Here it is only information. In coming years you will need to know too (I hope).

(P1) $\cosh x \overset{\text{def}}{=} \frac{1}{2}(e^x+e^{-x})$

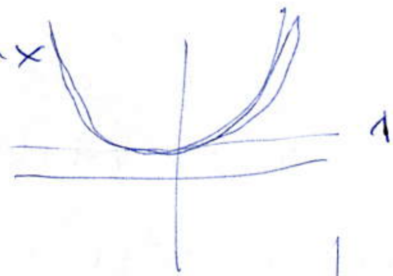
$\sinh x \overset{\text{def}}{=} \frac{1}{2}(e^x-e^{-x})$

From the definition the very useful relations are

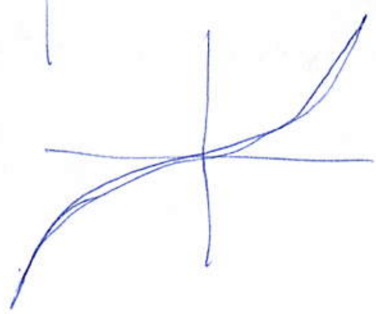
$$\cosh^2 x - \sinh^2 x = 1$$
$$(\cosh x)' = \sinh x \qquad , \quad ' \equiv \frac{d}{dx}$$
$$(\sinh x)' = \cosh x$$

that do not include exponentials that are very sometimes.

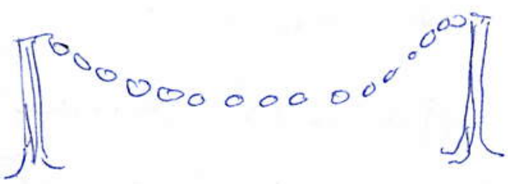
Graph of $\cosh x$



Graph of $\sinh x$

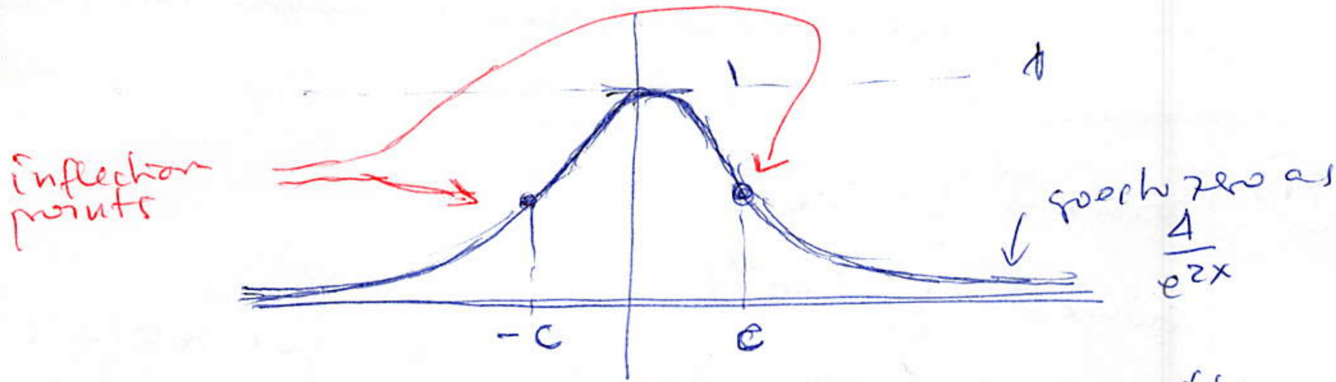


$\cosh x$ is a continuous function, derivable everywhere, never zero (so $f(x) = 1/\cosh^2 x$ is a continuous and derivable function everywhere), even and positive. It is the "catenary curve", the form that a chain hanging from two points has under its own weight in a uniform gravitational field $\downarrow\downarrow\downarrow\downarrow\downarrow$ $\vec{g}$ = constant vector



then $1/\cosh^2 x$ has graph:

b)



Has inflection points at $x = c$ (and consequently at $x = -c$ has another) where $1/\cosh^2 x$ changes curvature.

a) Determination of c: abscisse of the inflection point: $f''(c) = 0$.

$$f(x) = (\cosh x)^{-2}$$

$$f'(x) = -2(\cosh x)^{-3} \sinh x$$

$$f''(x) = 2\left[3(\cosh x)^{-4} \sinh^2 x, - \cosh^{-2} x\right]$$

$$= 2\underbrace{(\cosh x)^{-4}}_{\text{never zero}} \left[3 \sinh^2 x - \cosh^2 x\right]$$

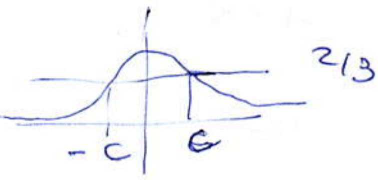
$f''(x) = 0$ inflection point condition $\Rightarrow 3 \sinh^2 x = \cosh^2 x$

that it is equivalent to

$$\sinh^2 x = \frac{1}{2}$$

or

$$\cosh^2 x = 3/2 : \longrightarrow$$



or

$$\tanh^2 x = \frac{1}{3}.$$

[Now you solve $\sinh^2 x = 1/2$, or $\cosh^2 x = 3/2$ or $\tanh^2 x = 1/3$, your choice, but you are not allowed to answer

$$c = \operatorname{arcsinh} \frac{1}{\sqrt{2}}$$

$$= \operatorname{arccosh} \sqrt{\frac{3}{2}}$$

$$= \operatorname{arctanh} \frac{1}{\sqrt{3}}$$

because you are asked explicitly for "logarithms in your result".]

I will retrieve

$$\sinh x = \pm \frac{1}{\sqrt{2}}$$

Take

$$\sinh x = \frac{1}{\sqrt{2}} \quad (\text{we calculate } c > 0)$$

$$\sinh x = \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \quad \frac{e^{x}-e^{-x}}{2} = \frac{1}{\sqrt{2}} \quad , \text{ define } \quad u \equiv e^{x}$$

$$\Leftrightarrow \quad u - \frac{1}{u} = \sqrt{2}$$

$$\Leftrightarrow \quad u^2 - \sqrt{2}u - 1 = 0 \quad , \quad u \begin{cases} \frac{1+\sqrt{3}}{\sqrt{2}} \\ \cancel{\frac{1-\sqrt{3}}{2}} \end{cases}$$

: not valid because e^x is never negative for real x

$$u = e^{x} = \frac{1+\sqrt{3}}{2}$$

$$\Rightarrow \quad \boxed{c = \log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)}$$

This is an ugly result, a better way to write c is

$$\begin{aligned} c &= \log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) \\ &= \frac{1}{2}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^2 \\ &= \frac{1}{2}\log\left(\frac{4+2\sqrt{3}}{2}\right) \\ &= \frac{1}{2}\log(2+\sqrt{3}) \quad // \end{aligned} \qquad \boxed{c = \frac{1}{2}\log(2+\sqrt{3})}$$

Last comment: from P4e) there are some students that know that $\operatorname{arcsinh} x = \log(x+\sqrt{x^2+1})$ and

$$c = \operatorname{arcsinh}\frac{1}{\sqrt{2}} = \log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$$

but here you were asked to deduce the result, not to write it by heart.

(P2) Let $f(x)$ be the continuous and derivable function everywhere defined by

$$f(x) = ax^4 + bx^3 + cx^2,$$

then

$$f(1) = a+b+c,$$
$$f(0) = 0.$$

By the MVT there is a number r between 0 and 1 [f is considered at $x = 0, 1$] such that

$$\frac{f(1)-f(0)}{1-0} = f'(r) = 4ar^3 + 3br^2 + 2cr$$

$$\Rightarrow \quad 4ar^3 + 3br^2 + 2cr = a+b+c, \quad 0 < r < 1$$

as desired. Peculiar fact, a, b, c are arbitrary numbers!! (o see, $\forall a, b, c$ real)

(P3) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1$

a) then

$$\log(1+2x) = 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \cdots \quad |x| < 1/2.$$

and

$$e^x \log(1+2x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \cdots\right)$$

$$\begin{aligned}
&= 2x \\
&\quad + x^2[-2+2] \\
&\quad + x^3\left[\frac{8}{3} - 2 + 1\right] \\
&\quad + x^4\left[-4 + \frac{8}{3} - 1 + \frac{1}{3}\right] + \cdots \\
&= 2x + \frac{5}{3}x^3 - 2x^4 + \cdots \qquad (*)
\end{aligned}$$

convergent for all $|x| < 1/2$

there are three non zero terms

similarly

$$e^{-x}\log(1+2x) = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\left(2x - 2x^2 + \frac{8}{3}x^3 - \cdots\right)$$

$$= 2x - 4x^2 + \frac{17}{3}x^3 + \cdots \quad (**)$$

convergent for all $|x| < 1/2$ too.

the coef 2 is irrelevant to calculate this limit

b) $\frac{f(x)-2x}{x^3} = \frac{5}{3} - 2x + \cdots$, $\lim_{x\to 0} \frac{f(x)-2x}{x^3} = \frac{5}{3}$ "

$\frac{g(x)-2x}{x^3} = \frac{-4x^2 + \frac{17}{3}x^3 + \cdots}{x^3}$ "

irrelevant to answer

$$\lim_{x\to 0} \frac{g(x)-2x}{x^3} = \lim_{x\to 0}\left(-\frac{4}{x} + \frac{17}{3} + \cdots\right) \begin{cases} -\infty & \text{if } x \to 0^+ \\ \infty & \text{if } x \to 0^- \end{cases}$$

c) no new calculations are needed. From

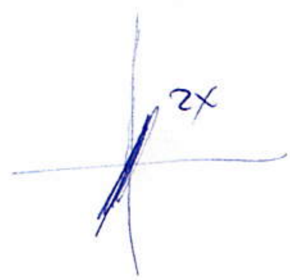
$f(x) \sim 2x$ at $x \to 0$

$g(x) \sim 2x$ at $x \to 0$

2x is the line:

$f(x) = 2x + \frac{5}{3}x^3 - 2x^4 + \cdots$

$g(x) = 2x - 4x^2 + \frac{17}{3}x^3 + \cdots$



The angle between 2x and 2x is $\underline{0}$, both tangent lines at $x=0$ to $f(x)$ and $g(x)$ are the same line: $2x$.

d) You have to answer negative with almost no justification at all, only

$$\int_{-0.01}^{0.01} dx\, g(x) = \int_{-0.01}^{0.01} dx \left(2x - 4x^2 + \frac{17}{3}x^3 + \dots\right),$$

convergent if $|x| < \pi/2$

$[-0.01, 0.01] \in |x| < \pi/2$

$$\int_{-0.01}^{0.01} 2x = \underline{\underline{0}} \quad \text{by symmetry,}$$

$$\int_{-0.01}^{0.01} dx\, \frac{17}{3}x^3 = \underline{\underline{0}} \quad \text{by symmetry,}$$

$$\vdots$$

so

$$\int_{-0.01}^{0.01} dx\, g(x) \simeq -4\int_{-0.01}^{0.01} dx\, x^2$$

$$= -8\int_{0}^{0.01} dx\, x^2$$

$$\simeq -\frac{8}{3}\frac{1}{10^6},$$

i.e., negative.

[now go to page 14 to see all explanation of this if you wish, but no explanation was requested in the exam... nor any of you provided one...]

d) $\int_{-0.01}^{0.01} dx \; \frac{\log(1+2x)}{e^x} =$ write the power series for the integrand that is convergent when $|x|<1/2$

The interval $[-0.01, 0.01]$ is

$$= \int_{-0.01}^{0.01} dx \left[2x - 4x^2 + \frac{17}{3}x^3 + \cdots \right]$$

zero the integral by symmetry

$$= 2\int_{0}^{0.01} dx \left[-4x^2 + a x^4 + b x^6 + \cdots \right]$$

we do not need these values... read more.

$$= 2\left[-\frac{4}{3}x^3 + \frac{a}{5}x^5 + \frac{b}{7}x^7 + \cdots \right]_0^{1/100}$$

$$= 2\left[-\frac{4}{3}\frac{1}{10^6} + \frac{a}{5}\frac{1}{10^{10}} + \frac{b}{7}\frac{1}{10^{14}} + \cdots \right]$$

$$= 2\left[-\frac{4}{3}\frac{1}{10^6} \right]\left[1 - \frac{3a}{20}\frac{1}{10^4} - \frac{3b}{28}\frac{1}{10^8} - \cdots \right] \quad (*)$$

dominant term, so irrespective of the values of $a, b, \ldots$

the integral is negative

[why "irrespective of the value of $a, b, \ldots$"?]
Because

$$e^{-x}\log(1+2x) \simeq 2x - 4x^2 + \frac{17}{3}x^3 + ax^4 + cx^5 + bx^6 + \cdots$$

is convergent $\forall\, x$ in $|x|<1/2$, and divergent otherwise. Take $x=1/2$:

$$2\cdot\frac{1}{2} - 4\frac{1}{2^2} + \frac{17}{3}\frac{1}{2^3} + a\frac{1}{2^4} + c\frac{1}{2^5} + d\frac{1}{2^6} + \cdots$$

means that $|a| \sim 16$, $|c| \sim 32$, $|b| \sim 64$, ...
that inserted in $(*)$ are only "small corrections to 1 in blue pencil"