2023-2024- Final - SOLUTIONS

January

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UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2023-2024

2023-2024 Final Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**

P1 [2pt] Sketch the graph of $f(x) = \dfrac{x^3}{2x-1}$, showing clearly on your sketch any asymptotes.

[Please, draw the asymptotes in bright red/green if you have these color pens and reserve the usual blue/black ink for the graph of the function]

P2 [2pt] a) Find the real values of x for which the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converges.

b) Find the exact sum of the previous series when $x = 1, 3, -3, -4, 9$.

c) From your answer in b) write a function $f(x)$ that exactly matches the given power series when i) $x \geq 0$, ii) $x < 0$.

P3 [2pt] a) Calculate

$$\int_0^{\log 3} \frac{dx}{\sqrt{e^x+1}}.$$

(b) Determine whether $\int_0^\infty \dfrac{dx}{\sqrt{e^x+1}}$ is convergent or divergent.

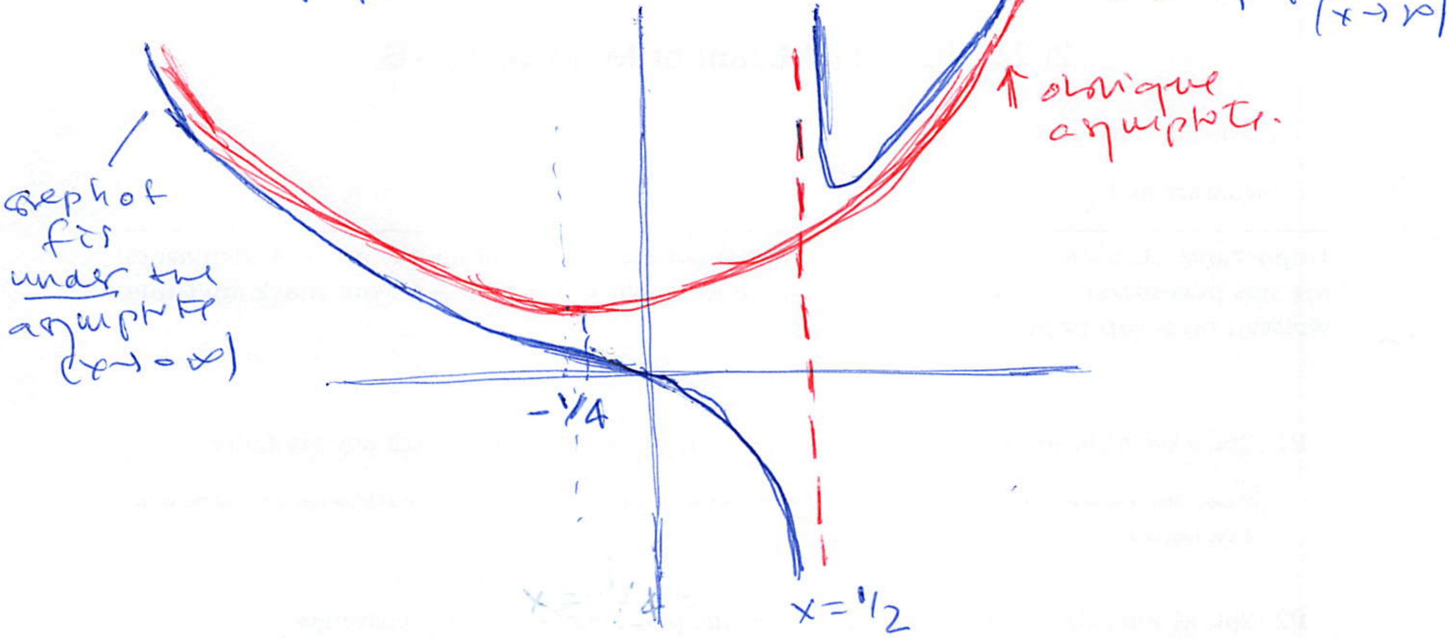
P4 [2pt] Find the first **five non zero** terms of the power series in x for $\dfrac{1-\sin x}{1-x}$ and state clearly its *radius of convergence*.

P5 [1.20+0.8pt] a) Calculate $\lim_{n\to\infty} n^2\left(\sqrt{n^4+11} - \sqrt{n^4+1}\right)$.

b) In interval notation the solution set for the inequality $\dfrac{x+1}{x-2} \leq \dfrac{x+2}{x+3}$ is $(-\infty, a) \cup [b, 2)$. Find a, b.

(P1) The graph of $f(x)$ has an oblique asymptote and a vertical asymptote.

The graph is



$$\frac{x^3}{2x-1} = \underbrace{\frac{x^2}{2}+\frac{x}{4}+\frac{1}{8}}_{\shortparallel} + \frac{1/8}{2x-1} \quad : \text{an identity}$$

$$\frac{1}{2}\left(\left(x+\frac{1}{4}\right)^2+\frac{3}{16}\right) \quad : \text{this is the oblique asymptote} \; : \text{is in red in the plot}$$

Has a minimum at $x=-1/4$

At $x=1/2$ there is a vertical asymptote: is in red color in the plot

Near $x=0$, $f(x) \simeq -x^3$ which is

No more things to be considered.

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Curso 2023-2024

Examen Final de Matemáticas: grupo E

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos. No se corregirá nada no incluido en este cuadernillo. Más claramente: si se entregan hojas en sucio ésas no se corrigen.

P1 [2pt] Esbozar la gráfica de $f(x) = \dfrac{x^3}{2x-1}$, indicando claramente en el didujo todas las asíntotas.

[Emplee por favor tinta de color llamativo rojo/verde para las asíntotas y reserve el color habitual azul/negro para la gráfica de la función.]

P2 [2pt] a) Calcular para qué valores reales de x converge la serie de potencias $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$.

b) Encontrar la suma exacta de la serie cuando $x = 1, 3, -3, -4, 9$.

c) A la vista de la respuesta del apartado anterior escribir una función $f(x)$ que coincida con la serie de potencias dada si i) $x \geq 0$, ii) $x < 0$.

P3 [2pt] a) Calcular

$$\int_0^{\log 3} \frac{dx}{\sqrt{e^x+1}}.$$

(b) Determinar si la integral $\int_0^{\infty} \dfrac{dx}{\sqrt{e^x+1}}$ es convergente o divergente

P4 [2pt] Encontrar los **cinco** primeros términos no nulos de la serie de potencias en x de $\dfrac{1-\sin x}{1-x}$ y diga claramente cuál es el *radio de convergencia* de la serie.

P5 [1.20+0.8pt]

a) Calcular $\lim_{n\to\infty} n^2\left(\sqrt{n^4+11} - \sqrt{n^4+1}\right)$.

b) El conjunto de soluciones de la desigualdad $\dfrac{x+1}{x-2} \leq \dfrac{x+2}{x+3}$ en notación habitual de intervalos es $(-\infty, a) \cup [b, 2)$. Encontar a, b.

P2

a) $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \dots$

Ratio test applied to $\sum^{\infty} b_n$, $b_n = \frac{x^n}{(2n)!}$ to know for which x's the series converges:

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{\frac{x^{n+1}}{(2n+2)!}}{\frac{x^n}{(2n)!}}\right| = \frac{|x|}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = |x| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = |x|\cdot 0 = 0$$

doesn't depend on n — if x is finite

$\Rightarrow \boxed{\sum_0^{\infty} \frac{x^n}{(2n)!} \text{ is convergent \underline{for all} } x \in \mathbb{R}}$

b) $x=1$:
$1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots = \cosh 1$

famous series

Famous because we know well that

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots, \quad R=\infty$$

$x=3$:
$1 + \frac{(\sqrt{3})^2}{2!} + \frac{(\sqrt{3})^4}{4!} + \frac{(\sqrt{3})^6}{6!} + \dots = \cosh\sqrt{3}$

$x=9=3^2$:
$1 + \frac{3^2}{2!} + \frac{3^4}{4!} + \frac{3^6}{6!} + \dots = \cosh 3$

$x=-3$:
$1 - \frac{3}{2!} + \frac{3^2}{4!} - \frac{3^3}{6!} + \frac{3^4}{8!} - \dots =$

$= 1 - \frac{(\sqrt{3})^2}{2!} + \frac{(\sqrt{3})^4}{4!} - \frac{(\sqrt{3})^6}{6!} + \frac{(\sqrt{3})^8}{8!} - \dots = \cos\sqrt{3}$

trigonometric, not hyperbolic

$x = -4$

$$1 - \frac{4}{2!} + \frac{4^2}{4!} - \frac{4^3}{6!} + \cdots$$

$$= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \cdots = \cos 2$$

[Because we also know well that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad R = \infty.$$
]

c) From b)

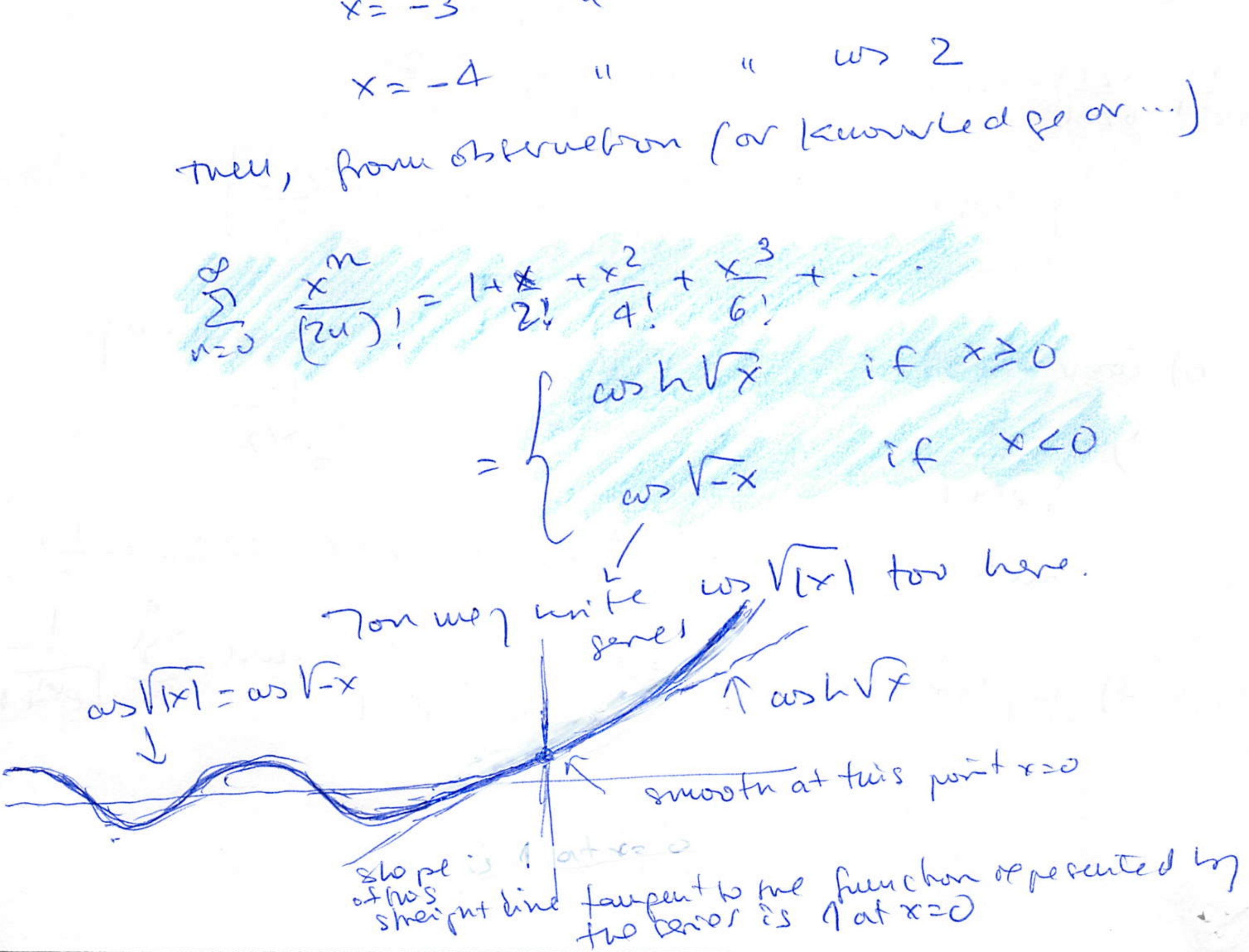
$x = 1$	series sums	$\cosh 1$
$x = 3$	" "	$\cosh\sqrt{3}$
$x = 9$	" "	$\cosh 3$
$x = -3$	" "	$\cos\sqrt{3}$
$x = -4$	" "	$\cos 2$

Then, from observation (or knowledge or...)

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \cdots$$

$$= \begin{cases} \cosh\sqrt{x} & \text{if } x \geq 0 \\ \cos\sqrt{-x} & \text{if } x < 0 \end{cases}$$

You may write series $\cos\sqrt{|x|}$ too here.



P3 a) $\displaystyle\int_0^{\log 3} \frac{dx}{\sqrt{e^x+1}}$

a) Use the change

$$u = \sqrt{e^x+1}$$
$$u^2 = e^x + 1 \quad \Rightarrow \quad e^x = u^2 - 1$$
$$2u\,du = e^x\,dx \quad \Rightarrow \quad dx = \frac{2u\,du}{u^2-1}$$

to write

$$\int_0^{\log 3} dx \frac{1}{\sqrt{e^x+1}} = 2\int_{\sqrt{2}}^{2} du\, \frac{1}{u^2-1}$$

$$= \Big[\log(u-1) - \log(u+1)\Big]_{\sqrt{2}}^{2}$$

$\displaystyle\frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]$

$$= \log 1 - \log 3 - \log(\sqrt{2}-1) + \log(\sqrt{2}+1)$$

$$= \log \frac{\sqrt{2}+1}{3(\sqrt{2}-1)}$$

$\displaystyle\frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1} = 3+2\sqrt{2}$

$$= \log\left(1 + \frac{2\sqrt{2}}{3}\right)$$

$$\boxed{\text{Result: } \int_0^{\log 3} \frac{dx}{\sqrt{e^x+1}} = \log\left(1+\frac{2\sqrt{2}}{3}\right)}$$

b) convergent. Three ways to see it, choose any.

1) $\displaystyle\frac{1}{\sqrt{e^x+1}}$ behaves at x large as $\displaystyle\frac{1}{e^{x/2}}$

and $\displaystyle\int^{\infty} dx\, e^{-x/2}$ is finite, i.e. <u>converges</u>.

2) By the "Integral test" $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{e^x+1}}$ and $\displaystyle\sum^{\infty} \frac{1}{\sqrt{e^n+1}}$

$\displaystyle\frac{1}{\sqrt{e^x+1}}$ continuous, positive and decreasing for all $x>0$

have the same character, and $\sum^{\infty} \frac{1}{\sqrt{e^n+1}}$ converges

[By comparison to the limit
$$\sum^{\infty} \frac{1}{\sqrt{e^n+1}} \quad \text{and} \quad \sum^{\infty} \left(\frac{1}{\sqrt{e}}\right)^n$$
have the same character since
$$\lim_{n\to\infty} \frac{1/\sqrt{e^n+1}}{1/\sqrt{e^n}} = \underline{\underline{1}}.$$
But $\sum^{\infty} \left(\frac{1}{\sqrt{e}}\right)^n$ is a geometrical series of ratio $0 < \frac{1}{\sqrt{e}} < 1$, hence convergent

$\Rightarrow \sum^{\infty} \frac{1}{\sqrt{e^n+1}}$ convergent too

]

3) Calculating $\int_0^{\infty} dx \frac{1}{\sqrt{e^x+1}}$ (we know the primitive)

$$= 2\int_{\sqrt{2}}^{\infty} du \frac{1}{u^2-1}$$
$$= \left[\log\left|\frac{u-1}{u+1}\right|\right]_{\sqrt{2}}^{\infty}$$
$$= \log 1 + \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$
$$= 2\log(1+\sqrt{2})$$
$$= \log(3+2\sqrt{2})$$: finite, therefore convergent

$$\frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1} = (\sqrt{2}+1)^2 = 3+2\sqrt{2}$$

(P4) $\frac{1-\sin x}{1-x} = \dots$

This is singular only at $x=1$ (because of $\frac{1}{1-x}$) so $R=1$.

We calculate the series by long division:

$$1 \cancel{-x} + \cancel{\frac{x^3}{3!}} - \frac{x^5}{5!} + \frac{x^7}{7!} \dots \quad : (1-x) = 1 + \frac{x^3}{3!} + \frac{x^4}{3!} + x^5\frac{19}{5!} + x^6\frac{19}{5!} + \dots$$

$$\cancel{-1} + \cancel{x}$$

$$-\cancel{\frac{x^3}{3!}} + \cancel{\frac{x^4}{3!}}$$

$$-\cancel{\frac{x^4}{3!}} + \frac{x^5}{3!}$$

$$\cancel{x^5\frac{19}{5!}}$$

$$-\cancel{x^5\frac{19}{5!}} + x^6\frac{19}{5!}$$

$$\frac{1}{3!} - \frac{1}{5!} = \frac{5 \cdot 4 - 1}{5!} = \frac{19}{5!}$$

Solution: $1 + \frac{x^3}{3!} + \frac{x^4}{3!} + \frac{19}{5!}x^5 + \frac{19x^6}{5!} + \dots$

$R=1$

That is all!

(P5) a) $\lim_{n\to\infty} n^2\left(\sqrt{n^4+11} - \sqrt{n^4+1}\right) = 5$

[$\infty - \infty$ is an indetermination. We remove it with

$$n^2\left(\sqrt{n^4+11} - \sqrt{n^4+1}\right)\frac{\left(\sqrt{n^4+11} + \sqrt{n^4+1}\right)}{\left(\sqrt{n^4+11} + \sqrt{n^4+1}\right)}$$

removed!

$$= \frac{n^2\left(\cancel{n^4}+11 - (\cancel{n^4}+1)\right)}{\sqrt{n^4+11} + \sqrt{n^4+1}} = \frac{10n^2}{\sqrt{n^4+11}+\sqrt{n^4+1}}$$

$$\underset{n \text{ large}}{\approx} \frac{10n^2}{\sqrt{n^4}+\sqrt{n^4}} = \frac{10n^2}{2n^2} = \underline{\underline{5}}.$$

b) $\frac{x+1}{x-2} \le \frac{x+2}{x+3}$

obviously $x \neq 2, -3$ whatever the solution is.

The easiest way

equivalent

$$\frac{x+1}{x-2} \le \frac{x+2}{x+3} \iff \frac{x+1}{x-2} - \frac{x+2}{x+3} \le 0$$

$$\left[\frac{x+1}{x-2} - \frac{x+2}{x+3} = \frac{(x+1)(x+3) - (x+2)(x-2)}{(x-2)(x+3)} \right.$$

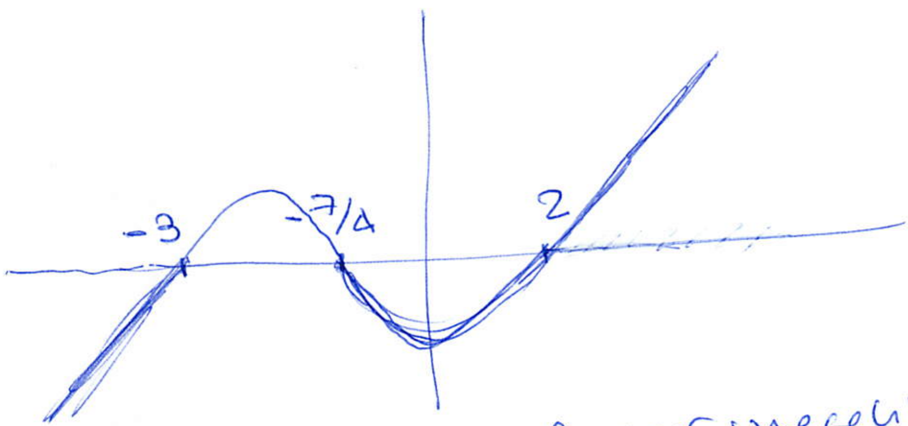
$$= \frac{\cancel{x^2}+4x+3-(\cancel{x^2}-4)}{(x-2)(x+3)}$$

$$\left. = \frac{4x+7}{(x-2)(x+3)} \right.$$

$$\iff \frac{4x+7}{(x-2)(x+3)} \le 0 \iff (4x+7)(x-2)(x+3) \le 0$$

But $x \neq 2, -3$

Plot of the polynomial $(x+3)(4x+7)(x-2)$



Then solutions of our inequality are the x's such that

$$x \in (-\infty, -3) \cup [-7/4, 2),$$

$$a = -3, \quad b = -7/4.$$