Solutions 2023-2024

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2023-2024

Primer Parcial de Matemáticas: grupo E

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos. No se corregirá nada no incluido en este cuadernillo. Más claramente: si se entregan hojas en sucio ésas no se corrigen.

P1 [2 pt] La función $f(x) = a\left[\left(\frac{b}{x}\right)^{12} - 2\left(\frac{b}{x}\right)^{6}\right]$ se define exclusivamente en $(0,\infty)$, siendo $a, b > 0$ constantes reales.

1) Estudiar el comportamiento de $f(x)$ cuando $x \to 0^+$ y cuando $x \to \infty$. Es f continua en su dominio?

2) Esbozar la gráfica de f indicando si hay simetría en torno a algún punto.

3) Hallar $R\,f$, el recorrido de f.

4) Discutir si f tiene un mínimo absoluto en su dominio.

5) Calcular $f'(x)$, la derivada de f con respecto a x.

P2 [2pt] Sea $f(x) = 2\arctan\left(\frac{3}{x-2}\right)$, $f(2) = -\pi$ definida en toda la recta real.

a) ¿Es $f(x)$ par en torno a $x = 0$? ¿Es impar? ¿Tiene simetria alguna alrededor de $x = 0$?

b) Encontrar los límites de $f(x)$ cuando $x \to \pm\infty$

c) Determimar los extremos absolutos de $f(x)$ si los hubiera.

d) Dibujar $f(x)$ en la recta real y determinar si $f(x) = 0$ en algún punto del intervalo $[1, 3]$

P3 [2 pt] Sea $f(x) = x + \frac{\pi}{4}\tan x$ definida en $(-\pi/2, \pi/2)$.

a) Estudie si f es *uno-uno* en su dominio y pruebe su repuesta tanto si ésta es un *sí* como si es un *no*. b) Encontrar el dominio y recorrido de $g \equiv f^{-1}$. ¿Es g una función? Haga un dibujo aproximado de su gráfica. c) Encontrar $g'(x)$, la derivada de g con respecto a x. d) Determinar la ecuación de la recta tangente a $g(x)$ en $x = \pi/2$.

P4 [1.5pt+1.5pt+1pt]

a) Calcular el límite cuando $n \to \infty$ de $\sqrt{n + \sqrt{n + \sqrt{n}}} - \sqrt{n}$

b) Encontrar los parámetros reales a y b para que la gráfica de $\sqrt[3]{8x^3 + ax^2} - bx$ tenga una asíntota horizontal $y = 1$

c) Determinar si la afirmación

$$(1^2 + 1)\,1! + (2^2 + 1)\,2! + (3^2 + 1)\,3! + \cdots + (n^2 + 1)\,n! = (n + 1)! \cdot n$$

es cierta o falsa usando el *Principio de Inducción.*

(P4) a) $\lim_{n\to\infty} \sqrt{n+\sqrt{n+\sqrt{n}}} - \sqrt{n}$

Need to cancel this with to remove the indetermination $\infty - \infty$. But to cancel we have to get rid of the $\sqrt{} - \sqrt{}$ square roots: how? Multiplying by the conjugate:

$$\frac{\boxed{\left(\sqrt{n+\sqrt{n+\sqrt{n}}} - \sqrt{n}\right)\left(\sqrt{n+\sqrt{n+\sqrt{n}}} + \sqrt{n}\right)}}{\sqrt{n+\sqrt{n+\sqrt{n}}} + \sqrt{n}}$$

$$= \frac{\cancel{n} + \sqrt{n+\sqrt{n}} - \cancel{n}}{\sqrt{n+\sqrt{n+\sqrt{n}}} + \sqrt{n}}$$

behaves as

numerator: $\sqrt{n+\sqrt{n}} = \sqrt{n\left(1+\frac{1}{\sqrt{n}}\right)} \simeq \sqrt{n}$ if n large

In the Denominator: $\sqrt{n+\sqrt{n+\sqrt{n}}} \simeq \sqrt{n+\sqrt{n}}$

we have seen this in the previous line

$\simeq \sqrt{n}$ if n large.

Then

$$\sqrt{n+\sqrt{n+\sqrt{n}}} - \sqrt{n} \simeq \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \frac{\sqrt{n}}{2\sqrt{n}} = \frac{1}{2}$$

if n large.

Conclusion: $\boxed{\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n+\sqrt{n}}} - \sqrt{n}\right) = \frac{1}{2}}$

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UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2023-2024

First Mid-term Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**

P1 [2 pt] The function $f(x) = a\left[\left(\frac{b}{x}\right)^{12} - 2\left(\frac{b}{x}\right)^{6}\right]$ is defined exclusively on the interval $(0, \infty)$ and $a, b > 0$ are real constants.

1) Indicate how $f(x)$ behaves near $x \to 0^+$ and near $x \to \infty$. Is f continuous in its domain?
2) Sketch the graph of f and indicate point symmetries (if any).
3) Find Rf, the range of f.
4) Discuss whether f has an absolute minimum in its domain.
5) Calculate $f'(x)$, the derivative of f with respect to x.

P2 [2pt] Consider the function $f(x) = 2\arctan\left(\frac{3}{x-2}\right)$, $f(2) = -\pi$ defined for every real number x.

a) Determine if $f(x)$ is even, odd or neither around $x = 0$.

b) Find the limits of $f(x)$ when $x \to \pm\infty$

c) Determine the absolute extrema of $f(x)$

d) Plot $f(x)$ over the whole real axis and determine if $f(x) = 0$ at the interval $[1, 3]$

P3 [2 pt] Consider $f(x) = x + \frac{\pi}{4}\tan x$ defined on $(-\pi/2, \pi/2)$.

a) If f is *one-one* in its domain, prove it, or prove it is not if that were the case. b) Find the domain and range of $g \equiv f^{-1}$. Is g a function? Sketch its graph. c) Calculate $g'(x)$, the derivative of g with respect to x. d) Find the equation of the straight line which is tangent to $g(x)$ at the point $x = \pi/2$.

P4 [1.5pt+1.5pt+1pt]

a) Calculate the limit when $n \to \infty$ of $\sqrt{n + \sqrt{n + \sqrt{n}}} - \sqrt{n}$

b) Find the real parameters a and b so that the graph of $\sqrt[3]{8x^3 + ax^2} - bx$ has the horizontal asymptote $y = 1$

c) Determine whether the statement

$$(1^2 + 1)\,1! + (2^2 + 1)\,2! + (3^2 + 1)\,3! + \cdots + (n^2 + 1)\,n! = (n + 1)! \cdot n$$

is true or false via the *Principle of Induction*.

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b) $\sqrt[3]{8x^3+ax^2}-bx$ has a horizontal asymptote $y=1$ when $x\to\infty$. Find a,b.

We are told that

$$\lim_{x\to\infty} \sqrt[3]{8x^3+ax^2}-bx = 1,$$

or what is equivalent, that

$$\sqrt[3]{8x^3+ax^2} \simeq bx+1, \quad x \text{ large}.$$

Calculating the cube of this relation we have

$$8x^3+ax^2 \simeq (bx+1)^3$$
$$= b^3x^3 + 3b^2x^2 + \dots$$

$$\Rightarrow \quad 8=b^3 \quad \Rightarrow \quad \boxed{b=2}$$

has only one real root.

$$a = 3b^2 = 3\cdot 4 = 12, \quad \boxed{a=12}$$

Conclusion: $\boxed{a=12,\ b=2}$

⌈ I check this result: You do not need it at the exam. But I do check it.

$$8x^3+12x^2 = (2x+1)^2 - 6x - 1$$

$$\sqrt[3]{8x^3+12x^2}-2x = \sqrt[3]{(2x+1)^3\left[1-\frac{6x+1}{(2x+1)^3}\right]} - 2x$$

This term goes to zero when $x\to$ ~~∞ as $\frac{6x}{8x^3}$~~ as $\frac{3}{4x^2}$ when $x\to\infty$

$$\simeq (2x+1)\sqrt[3]{1+\frac{3}{4x^2}} - 2x \qquad \frac{3}{4x^2}\to 0$$

$$\simeq (\cancel{2x}+1) - \cancel{2x}$$
$$= 1.$$

Perfect!!

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c) $\displaystyle\underbrace{\sum_{i=1}^{n}(i^2+1)\,i!}_{} = n\cdot(n+1)!$??

this is also

$(1^2+1)\,1! + (2^2+1)\,2! + \dots + (n^2+1)\,n!$

n	$\sum_{i=1}^{n}(i^2+1)\,i!$	$n(n+1)!$	
1	$(1^2+1)\,1! = 2$	$1\cdot 2! = 2$	✓
2	$2 + (2^2+1)\,2! = 12$	$2\cdot 3! = 12$	✓
3	$12 + (3^2+1)\,3! = 72$	$3\cdot 4! = 3\cdot 24 = 72$	✓

The relation is true for $n=1,2,3$, we suppose (this is the assumption in the Principle of Induction) that it is true for n, and we check whether true (or not) for $n+1$:

$$\sum_{i=1}^{n+1}(i^2+1)\,i! = \sum_{i=1}^{n}(i^2+1)\,i! + \left((n+1)^2+1\right)(n+1)!$$

$$= \underbrace{n\cdot(n+1)! + \left((n+1)^2+1\right)(n+1)!}_{\text{this is...}}$$

compulsory to use the result form

Now we operate to see what this is...

$$n\,\underline{(n+1)!} + \left((n+1)^2+1\right)\underline{(n+1)!}$$

$$= (n+1)!\left[n+1+(n+1)^2\right]$$

$$= (n+1)!\,(n+1)\big[\underbrace{1+n+1}_{n+2}\big]$$

$$= (n+2)!\,(n+1)$$

which is the exact result for the original formula at $n+1$. !!!!

Conclusion:

$$(1^2+1)1!+(2^2+1)2!+\dots+(n^2+1)n! = n(n+1)!.$$

is **true** for all n, and we can use it **safely**.

(P1) f is defined only in $(0,\infty)$. There is nothing in  this part of the Cartesian plane.

f is continuous in every point of $(0,\infty)$ because the only problematic point is $x=0$ and $x=0 \notin (0,\infty)$. Then $\mathrm{Dom}\, f = (0,\infty)$.

You may like to write $f(x)$ as

$$f(x)=a\left(\frac{1}{u^{12}}-\frac{2}{u^{6}}\right), \quad u\equiv\frac{x}{b} = \text{adimensional.}$$

When $x\to 0^+$

$$f(x)\sim\frac{a}{u^{12}}=\frac{a}{(x/b)^{12}} \xrightarrow[x\to 0^+]{} \infty$$

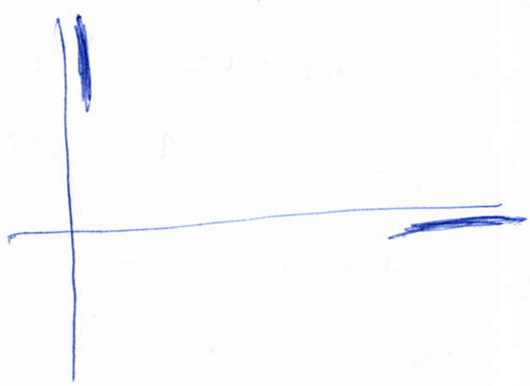
(why?) as $a\dfrac{b^{12}}{x^{12}}$ (that is as $\dfrac{1}{x^{12}}$)

When $x\to\infty$

$$f(x)\sim -\frac{2a}{u^{6}} = \frac{-2a}{(x/b)^{6}} \xrightarrow[x\to\infty]{} 0$$

(why)? as $-2a\left(\dfrac{b}{x}\right)^{6}$

Both behaviors are in this plot



The reason:

$x \to 0$ $\quad x^{12} \ll x^6$ so $\dfrac{1}{x^{12}} \gg \dfrac{1}{x^6}$

and

$$f(x) = a\left(\frac{b}{x}\right)^{12}\left[1 - \cancel{2\left(\frac{x}{b}\right)^6}\right] \xrightarrow[x \to 0]{} \frac{ab^{12}}{x^{12}}.$$

to 0

get outside the dominant term (always)

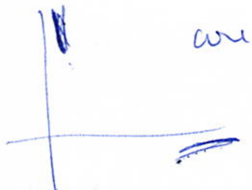
$x \to \infty$ $\quad \dfrac{1}{x^{12}} \ll \dfrac{1}{x^6}$

to 0

and

$$f(x) = a\left(\frac{b}{x}\right)^{6}\left[\cancel{\left(\frac{b}{x}\right)^6} - 2\right] \xrightarrow[x \to \infty]{} -2a\left(\frac{b}{x}\right)^6$$

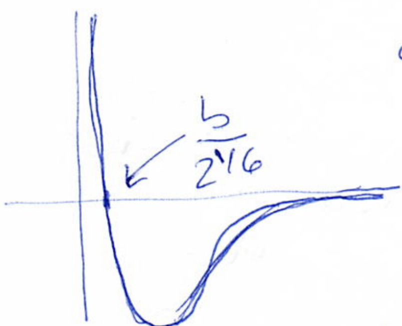
now the dominant term.

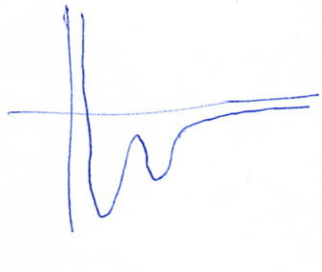
To join  continuously these two pieces we need to cross the x axis once at least: where?

$$f(x) = 0 \iff u^{12} = \frac{u^6}{2}$$

or $u^6 = \frac{1}{2}$, $u = \dfrac{1}{2^{1/6}}$ or $\cancel{-\dfrac{1}{2^{1/6}}} \notin (0,\infty)$

or $x = \dfrac{b}{2^{1/6}}$



We discard plots as  with $f'(x)$.

Calculating $f'(x)$:

$$f(x) = a\left(u^{-12} - 2u^{-6}\right), \quad u = \frac{x}{b}$$

$$f'(x) = \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = \frac{a}{b}\left[-12u^{-13} + 12u^{-7}\right]$$

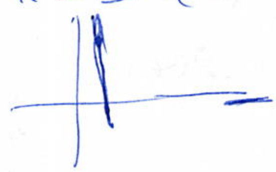
$$= 12\frac{a}{b}u^{-13}\left[-1 + u^6\right]$$

$$f'(x) = 0 \iff u^6 = 1 \iff u \begin{cases} 1 & \text{is } x = b \\ \cancel{-1} & \notin (0,\infty) \end{cases}$$

(only real solutions)

$$\Rightarrow \quad x = b.$$

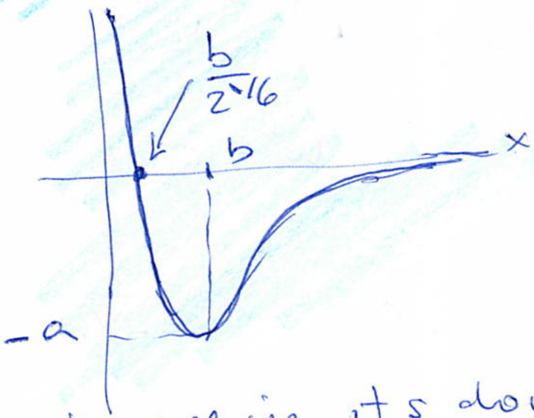
There is a minimum at $x = b$ (no other thing is competing with but a minimum.



$$f(u = 1) = -a$$

All information put together:

- $\mathrm{Dom}\, f = (0,\infty)$
- $R_f = [-a,\infty)$: read from the graph
- graph



vertical asymptote in $x=0$

horizontal asymptote at $x \to \infty$ $(=0^-)$

- f is not one-to-one in its domain
- f has an absolute minimum at $x_{min} = b$, $f_{min} = -a$
- The calculation of $f'(x)$ is $f'(x) = 12\frac{a}{b}u^{-13}\left[-1+u^6\right]$, $u = \frac{x}{b}$

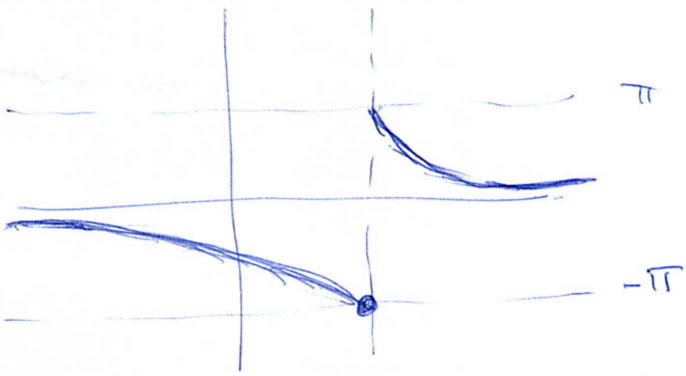
P2 a) no. f(x) has not any symmetry around $x=0$. This can be checked also with the graph.

b)

x	$-\infty$		0	2	∞
$f'(x)$		$-ve$	$-ve$ —	—	$-ve$
$f(x)$	0^-		$-ve$	$-\pi$	0^+

$$\lim_{x\to\infty} 2\arctan\left(\frac{3}{x-2}\right) = 2\arctan 0^+ = 0^+$$

$$\lim_{x\to -\infty} 2\arctan\left(\frac{3}{x-2}\right) = 2\arctan 0^- = 0^-$$

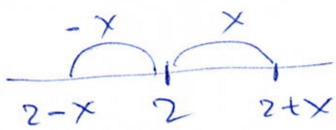


arctan x in the main branch

$$\lim_{x\to 2^+} 2\arctan\left(\frac{3}{x-2}\right) = 2\arctan(\infty) = 2\frac{\pi}{2} = \pi$$

$$\lim_{x\to 2^-} 2\arctan\left(\frac{3}{x-2}\right) = 2\arctan(-\infty) = -2\frac{\pi}{2} = -\pi$$

It is not necessary to calculate $f''(x)$ to plot $f(x)$.
There is no symmetry around $x=0$ but there is a symmetry around $x=2$ because

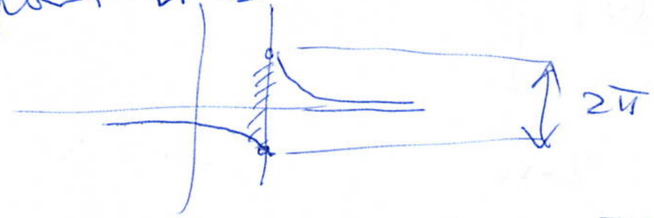


$$f(2-x) = -f(2+x) \quad \forall\, x \neq 2$$

what we check with

$$f(2-x) + f(2+x) = 2\arctan\left(-\frac{3}{x}\right) + 2\arctan\left(\frac{3}{x}\right) = 0 \quad \forall x \neq 2$$

c) The function is discontinuous at $x=2$ because there is a 2π-step at that point $x=2$. Save for this point ~~it~~ it is continuous everywhere.



a) The function is not obliged by the EVT to have ~~nor~~ neither absolute maximum nor absolute minimum in its domain because f is not continuous and the domain is $\mathbb{R} - \{2\}$ (~~neither~~ closed nor bounded), however has a minimum (absolute minimum) at

$$x_{min} = 2, \quad f_{minimum} = -\pi.$$

b) but has not an absolute maximum.

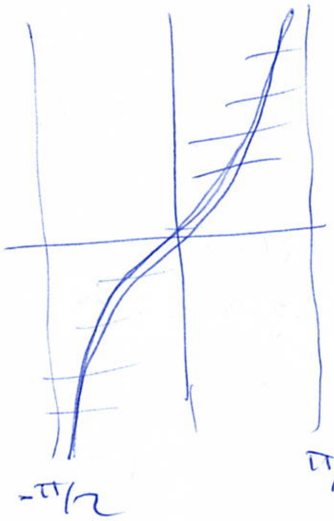
d) The function satisfies $f(1) < 0$ and $f(3) > 0$ but it is not continuous at $[1,3]$, so does not need to take necessarily the value $f=0$ in this interval (and does not take it)

Just to finish:

$$f'(x) = \frac{2\left[-\frac{3}{(x-2)^2}\right]}{1+\left(\frac{3}{x-2}\right)^2} = -ve = \frac{-6}{(x-2)^2+3^2}$$: not needed, as I say.

(P4) $f(x) = x + \frac{\pi}{4} \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

This function has a very similar graph to $\tan x$, because it is also



($x + \frac{\pi}{4} \tan x$ tends to zero when $x \to 0$ as $\left(1+\frac{\pi}{4}\right)x$ and not x as $\tan x$, ~~but~~ apart from this $x + \frac{\pi}{4}\tan x$ and $\tan x$ are very similar – the plot – I mean.

(yes) a) $f(x)$ is one-to-one because $f(x)$ is strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

f increasing $\Rightarrow$ f one-to-one.

[You can check this with $f'(x)$

$$f'(x) = 1 + \frac{\pi/4}{\cos^2 x} > 1 \underline{\underline{> 0}} \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

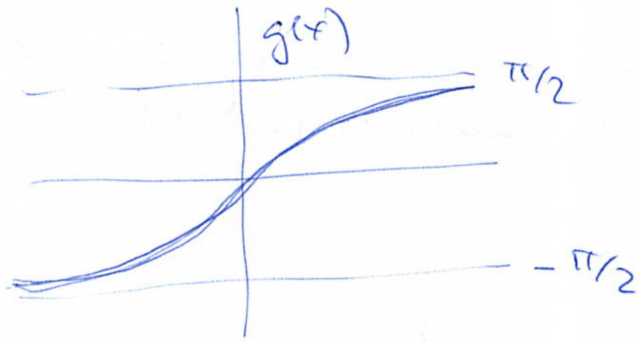
Hence increasing.

b) $\text{Dom} f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\quad \Rightarrow \quad \text{Dom} f^{-1} = (-\infty, \infty)$

$Rf = (-\infty, \infty)$ $\qquad\qquad Rf^{-1} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

and yes, f^{-1} is a function because (theorem) f is one-one.

The graph of $f^{-1} \equiv g$ is



$f(0)=0 \iff g(0)=0$

$f(a)=\pi/2 \iff g(\pi/2)=a$

what is a? obviously if $f(a)=\pi/2$, $\boxed{a=\frac{\pi}{4}}$

c) $\cancel{y = x + \frac{\pi}{4}\tan x}$ $\quad y=f$

$x = g + \frac{\pi}{4}\tan g$, $\quad g=f^{-1}$

Don't try to resolve $x = g + \frac{\pi}{4}\tan g$ in terms of $\tan g$. No one can. But it is not necessary (seen in the lectures) to calculate $g'(x)$

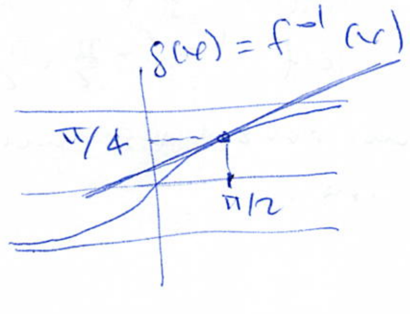
Derive $x = g + \frac{\pi}{4}\tan g$, $\quad g=g(x)$ wrt x.

$$1 = g'\left[1 + \frac{\pi}{4}\frac{1}{\cos^2 g}\right]$$

$$\Rightarrow \boxed{g'(x) = \frac{1}{1+\frac{\pi}{4}\frac{1}{\cos^2 g}}}$$

(other possibilities: $g'(x) = \dfrac{\cos^2 g}{\cos^2 g + \frac{\pi}{4}}, \dots$)

d)



→ we want the equation of this straight line

$$g'(\pi/2) = \frac{1}{1+\frac{\pi}{4}\frac{1}{\cos^2\frac{\pi}{4}}}$$

$g(\pi/2) = \pi/4$

$$= \frac{1}{1+\frac{\pi}{4}\cdot 2}$$

$\cos^2\frac{\pi}{4} = \frac{1}{2}$

$$= \frac{1}{1+\pi/2} \quad " \quad \boxed{g'(\pi/2) = \frac{1}{1+\pi/2}}$$

Straightline is

$$\boxed{\frac{1}{1+\pi/2}\left(x-\frac{\pi}{2}\right)+\frac{\pi}{4}}$$

slope $= \frac{1}{1+\pi/2}$

the point $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ belongs to the line.

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Offending Command: (bin obj seq, type=130, elements=33, size=2675, non-zero unused
Operand Stack:
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