SOLUTIONS
P2 . 2023-2024

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UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2023-2024

Segundo Parcial de Matemáticas: grupo E

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos. No se corregirá nada no incluido en este cuadernillo. Más claramente: si se entregan hojas en sucio ésas no se corrigen.

P1 [1pt+1pt] a) Encontrar la serie de potencias de $\dfrac{x}{1+x^3}$ en torno a $x=0$ e indicar su intervalo de convergencia.

b) Integrar término a término la serie anterior para hallar $F(x) = \displaystyle\int_0^x dt\,\frac{t}{1+t^3}$. Indicar el radio de convergencia de $F(x)$. Estudiar si las series $F(-1)$ y $F(1)$ son convergentes o divergentes. Diga **claramente** los test empleados en su respuesta.

P2 [1pt+1pt] a) Determinar si la serie numérica $\sum_{n=1}^{\infty} n^2 \arctan\dfrac{1}{n^2}$ es convergente o divergente. Indicar el test o test utilizados para responder.

b) Escribir la secuencia de las sumas parciales y calcular la suma exacta de

$$\sum_{n=1}^{\infty}\Big(n^2\arctan\frac{1}{n^2} - (n+1)^2\arctan\frac{1}{(n+1)^2}\Big).$$

P3 [0.50pt+0.50pt+1pt] Calcular las siguientes integrales indefinidas

a)$\int dx\,\cosh x\cos(\sinh x)$, b) $\int dx\,\dfrac{\sinh x\cosh x}{1+\cosh^4 x}$, c) $\int dx\,\dfrac{\cosh x}{\sinh^4 x - 2\sinh^3 x}$.

P4 [2pt] a)¿Qué relación hay entre $f(x) = \frac{1}{2}\arcsin(2x-3)$ y $g(x) = \arcsin(\sqrt{x-1})$? b) Escribir el dominio de $f(x)$ y el dominio de $g(x)$.

P5 a) [0.5pt] Use la regla de l'Hôpital (por fin!!) para calcular $\lim_{x\to 2}\dfrac{1}{x-2}\displaystyle\int_2^x dt\,\frac{\sin t}{t}$.

b) [0.5pt] ¿Cúal es el valor exacto de la suma $1-(\frac{1}{2})+\frac{1}{3}(\frac{1}{2})^3-\frac{1}{5}(\frac{1}{2})^5+\frac{1}{7}(\frac{1}{2})^7-\cdots$?

c) [1pt] Encontrar una función tal que $f'(-1)=1/2$, $f'(0)=0$ y $f''(x)>0$ para todo x, o probar que dicha función no puede existir.

(P2) a) Divergent by the "preliminary test":

consider $\sum^{\infty} a_n$

* If $\lim\limits_{n\to\infty} a_n \neq 0 \Rightarrow \sum^{\infty} a_n$ diverges.

Here,

$$\lim_{n\to\infty} \underbrace{n^2 \arctan\frac{1}{n^2}}_{\simeq \frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1 \neq 0$$

$$\left[\begin{array}{l}\arctan x \simeq x \\ \text{or } \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} \dots, |x|<1\end{array}\right]$$

b) $$\sum_{n=1}^{\infty}\left(n^2 \arctan\frac{1}{n^2} - (n+1)^2 \arctan\frac{1}{(n+1)^2}\right)$$

$$= 1 \arctan 1 - \cancel{2^2 \arctan\frac{1}{2^2}}$$
$$+ \cancel{2^2 \arctan\frac{1}{2^2}} - \cancel{3^2 \arctan\frac{1}{3^2}}$$
$$+ \cancel{3^2 \arctan\frac{1}{3^2}} - \cancel{4^2 \arctan\frac{1}{4^2}} + \dots$$

Enough terms to see that it is a "telescoping series". We write $S_1, S_2, \dots S_n, \dots$, sequence of partial sums,

$S_1 = 1 \arctan 1 - 2^2 \arctan\frac{1}{2^2}$

$S_2 = 1 \arctan 1 - 3^2 \arctan\frac{1}{3^2}$

$S_3 = 1 \arctan 1 - 4^2 \arctan\frac{1}{4^2}$

$\vdots$

$S_N = 1 \arctan 1 - (N+1)^2 \arctan\frac{1}{(N+1)^2}$

$\xrightarrow[n\to\infty]{} \arctan 1 - 1$

the sum of the series is $\boxed{\frac{\pi}{4} - 1}$

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Academic Year 2023-2024

2023-2024-Second Mid-term Exam of Mathematics-B

Name and surname:

Signature and DNI:

Important: Books or any other written material are not allowed during the exam. **Calculators are not permitted.** The total score of this script is 10 points. The professor **wont mark anything written on scrap paper**

P1 [1pt+1pt] a) Find the power series expansion for $\dfrac{x}{1+x^3}$ around $x = 0$ and the interval of convergence.

b) Integrate the previous series term by term to find $F(x) = \displaystyle\int_0^x dt\, \frac{t}{1+t^3}$. Indicate the radius of convergence of $F(x)$. Study whether $F(-1)$ and $F(1)$ are convergent or divergent.

P2 [1pt+1pt] a) Determine whether the numerical series $\sum_{n=1}^{\infty} n^2 \arctan \dfrac{1}{n^2}$ is convergent or divergent. Indicate **clearly** the test or tests that you use to answer.

b) Find the sequence of partial sums and the exact sum of

$$\sum_{n=1}^{\infty} \left(n^2 \arctan \frac{1}{n^2} - (n+1)^2 \arctan \frac{1}{(n+1)^2} \right).$$

P3 [0.50pt+0.50pt+1pt] Calculate the following indefinite integrals:

a) $\int dx\, \cosh x \cos(\sinh x)$, b) $\int dx\, \dfrac{\sinh x \cosh x}{1+\cosh^4 x}$, c) $\int dx\, \dfrac{\cosh x}{\sinh^4 x - 2\sinh^3 x}$.

P4 [2pt] a) What is the relation between $f(x) = \frac{1}{2}\arcsin(2x-3)$ and $g(x) = \arcsin(\sqrt{x-1})$? b) Find the domain of $f(x)$ and the domain of $g(x)$.

P5 a) [0.5pt] Use l'Hôpital rule (finally!!) to evaluate $\lim_{x\to 2} \dfrac{1}{x-2} \displaystyle\int_2^x dt\, \frac{\sin t}{t}$.

b) [0.5pt] Which is the value of the sum $1 - (\frac{1}{2}) + \frac{1}{3}(\frac{1}{2})^3 - \frac{1}{5}(\frac{1}{2})^5 + \frac{1}{7}(\frac{1}{2})^7 - \cdots$?

c) [1pt] Find a function such that $f'(-1) = 1/2$, $f'(0) = 0$ and $f''(x) > 0$ for all x, or prove that such a function cannot exist.

(P4) This is a consequence of MVT: "If $f'(x)=0$ for all $x \in (a,b)$ then $f(x)$ = constant for all x in $I=(a,b)$."

Note that

$$f'(x) = g'(x) = \frac{1}{2}\,\frac{1}{\sqrt{(x-1)(2-x)}}$$

so

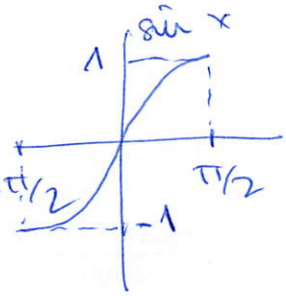
$$\left(f(x)-g(x)\right)' = 0 \quad \forall x \in \operatorname{Dom} f \cap \operatorname{Dom} g$$

intersection $= \min\{\operatorname{Dom} f, \operatorname{Dom} g\}$

and

$$f(x) - g(x) = C_1 = \text{const.}$$

Find C_1. Remember that



$$f(x) = \tfrac{1}{2}\arcsin(2x-3)$$
$$g(x) = \arcsin(\sqrt{x-1})$$

$x=1$ is in Dom $f(x)$ and in Dom $g(x)$

$$f(1) = \tfrac{1}{2}\arcsin(-1) = -\frac{\pi}{4}$$

$$g(1) = \arcsin 0 = 0,$$

Then

$$\boxed{C_1 = -\pi/4}$$

conclusion:

$$\boxed{\arcsin\sqrt{x-1} = \frac{1}{2}\arcsin(2x-3) + \frac{\pi}{4} \quad \forall x \in [1,2].}$$

Why $[1,2] = \operatorname{Dom} f(x) = \operatorname{Dom} g(x)$?

$f(x)$: $-1 \le 2x-3 \le 1 \iff 2 \le 2x \le 4 \iff 1 \le x \le 2$

$g(x)$: $0 \le \sqrt{x-1} \le 1 \iff 0 \le x-1 \le 1 \iff 1 \le x \le 2$

It is a perfect result. Very neat.

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(P3) a) b) are simple direct integrals.

c) It is also simple but need to write $\frac{1}{u^3(u-2)}$ in simple fractions.

a) $\int dx \cosh x \cos(\sinh x) = \int du \cos u$

$u = \sinh x$
$du = \cosh x \, dx$

$= \boxed{\sin u + C_1}$

with $\boxed{u = \sinh x.}$

b) $\int dx \frac{\sinh x \cosh x}{1+\cosh^4 x} = \int du \frac{u}{1+u^4}$

$u = \cosh x$
$du = \sinh x \, dx$

$= \frac{1}{2}\int du \frac{2u}{1+(u^2)^2}$

simple integral

$= \boxed{\frac{1}{2}\arctan u^2 + C_1}$

$\boxed{u = \cosh x}$

c) $\int dx \frac{\cosh x}{\sinh^4 x - 2\sinh^3 x} = \int \frac{du}{u^4 - 2u^3}$

$u = \sinh x$
$du = \cosh x$

$\frac{1}{u^4 - 2u^3} = \frac{1}{u^3(u-2)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C_1}{u^3} + \frac{D}{u-2}$,

A, B, C, D constants to be found.

$= \frac{Au^2(u-2) + Bu(u-2) + C_1(u-2) + Du^3}{u^3(u-2)}$

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From
$$1 = Au^2(u-2) + Bu(u-2) + C(u-2) + Du^3,$$
an identity for all u is a sufficient condition to find the constants.

$u=0$	$1 = -2C$	$C = -1/2$
$u=2$	$1 = D\cdot 8$	$D = 1/8$
coeff u^3:	$0 = A + D$	$A = -1/8$
coeff u^2	$0 = -2A + B$	$B = -1/4$

$$\frac{1}{u^4-2u^3} = \frac{-1/8}{u} - \frac{1/4}{u^2} - \frac{1/2}{u^3} + \frac{1/8}{u-2}.$$

Thus
$$\boxed{\int \frac{du}{u^4-2u^3} = -\frac{1}{8}\log|u| + \frac{1}{4}\frac{1}{u} + \frac{1}{4}\frac{1}{u^2} + \frac{1}{8}\log|u-2| + \text{const of integration.}}$$
$u = \sinh x$

(P1) a) $\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots \quad |x|<1$ see below (*)

$$\frac{x}{1+x^3} = x - x^4 + x^7 - x^{10} + \dots + \dots \quad |x|<1$$

$R=1$: It is this 1

$$= \sum_0^\infty (-1)^n x^{3n+1}$$

b) $\int_0^x dt \frac{t}{1+t^3} = \int_0^x dt\,(t - t^4 + t^7 - t^{10} + \dots)$ $R=1$

$$= \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} + \dots \quad |x|<1$$

Integration term by term inherits the same radius of convergence (we saw it in the lectures)

$$= \sum_0^\infty (-1)^n \frac{x^{3n+2}}{3n+2}.$$

$$F(-1) = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \dots + \frac{1}{3n+2} + \dots = \sum_{n=0}^{\infty} \frac{1}{3n+2}$$

divergent as $\sum_{0}^{\infty} \frac{1}{3n}$, which is divergent (comparison by the limit).

$$F(1) = \frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \dots$$

convergent by the AST (Leibnitz test)

1) $a_n = \frac{1}{3n+2}$

2) $a_n > a_{n+1}$

3) $\lim_{n\to\infty} \frac{1}{3n+2} = 0$

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(*) $\frac{x}{1+x^3}$: write the power series around $x=0$

You have many choices.

i) Starting with the geometrical series of ratio $r=-x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad , |x|<1$$

change $x \to x^3$ everywhere

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots \quad |x^3|<1$$

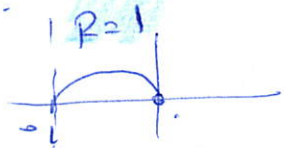
This is $|x|<1$, same set.

$$\frac{x}{1+x^3} = x - x^4 + x^7 - x^{10} + \dots \quad |x|<1$$

ii) By long division:

$$\begin{array}{l|l} x & 1+x^3 \\ \hline \cancel{x} & x - x^4 + x^7 - x^{10} + \dots \\ -\cancel{x} - \cancel{x^4} & \\ \cancel{+x^4} + \cancel{x^7} & \\ \cancel{-x^7} - x^{10} & \\ +x^{10} + x^{13} & \\ \vdots & \end{array}$$

$R=1$: distance from $x=0$ to the nearest singularity of $\frac{x}{1+x^3}$: which is at $x=-1$



iii) By binomial series

$$\frac{1}{1+x^3} = (1+x^3)^{-1} = \binom{-1}{0} + \binom{-1}{1}x^3 + \binom{-1}{2}(x^3)^2 + \binom{-1}{3}(x^3)^3 + \dots$$

$$\binom{m}{0} = 1 \Leftarrow \binom{-1}{0} = 1$$

$$\binom{m}{1} = m \Leftarrow \binom{-1}{1} = -1$$

$$\binom{m}{2} = \frac{m(m-1)}{2!} \Rightarrow \binom{-1}{2} = \frac{(-1)(-2)}{2} = 1$$

$$\binom{m}{3} = \frac{m(m-1)(m-2)}{3!} \Leftarrow \binom{-1}{3} = \frac{(-1)(-2)(-3)}{3!} = -1$$

$\vdots$

because, as you see,

$$\boxed{\binom{-1}{n} = (-1)^n}$$

(no need to prove this, it is)

$$\frac{1}{1+x^3} = 1 - x^3 + (x^3)^2 - (x^3)^3 + \dots \qquad R = 1$$

[How do we calculate R with $a_n, a_{n+1}, \dots$?

Suppose we had $\frac{1}{1+2x^3}$ instead of $\frac{1}{1+x^3}$, the radius of convergence would be

$$R = \text{dist from } x=0 \text{ to } x = -\frac{1}{\sqrt[3]{2}}$$

$$= \frac{1}{\sqrt[3]{2}}$$

Since

$$\frac{1}{1+2x^3} = 1 - (2x^3) + (2x^3)^2 - (2x^3)^3 + (2x^3)^4 + \dots$$

$$\dots + (-1)^n (2x^3)^n + (-1)^{n+1}(2x^3)^{n+1}$$

$$= \sum_0^\infty (-1)^n 2^n x^{3n}.$$

Since

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_{n+2}}\right| = \lim_{n\to\infty}\left|\frac{a_{n+2}}{a_{n+3}}\right| \text{ "}$$

$$R^3 = \lim_{n\to\infty}\left|\frac{a_n}{\cancel{a_{n+1}}} \cdot \frac{\cancel{a_{n+1}}}{\cancel{a_{n+2}}} \cdot \frac{\cancel{a_{n+2}}}{a_{n+3}}\right| = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+3}}\right|$$

$$= \lim_{n \to \infty} \left| \frac{(-1)^n \, 2^n}{(-1)^{n+1} \, 2^{n+1}} \right| = \frac{1}{2}$$

$\Rightarrow \quad R = 1/\sqrt[3]{2}$ as expected

In our case $R^3 = 1$ so $R = 1$ and you do not see this subtle point. Pity.

(P5) a) $\lim_{x \to 2} \frac{1}{x-2} \int_2^x dt \, \frac{\sin t}{t} = \frac{1}{0} \int_2^2 = \frac{0}{0}$:

Apply l'Hôpital:

$$\lim_{x \to 2} \frac{1}{x-2} \int_2^x dt \, \frac{\sin t}{t} = \lim_{x \to 2} \frac{\frac{\sin x}{x}}{1} = \frac{\sin 2}{2}$$

derivative of $x-2$ is 1

because the derivative wrt x of $\int_0^x dt \, \frac{\sin t}{t}$ is $\frac{\sin x}{x}$.

solution: $\boxed{\frac{\sin 2}{2}}$

b) $1 - \left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots = \boxed{1 - \arctan \frac{1}{2}}$

Remember that

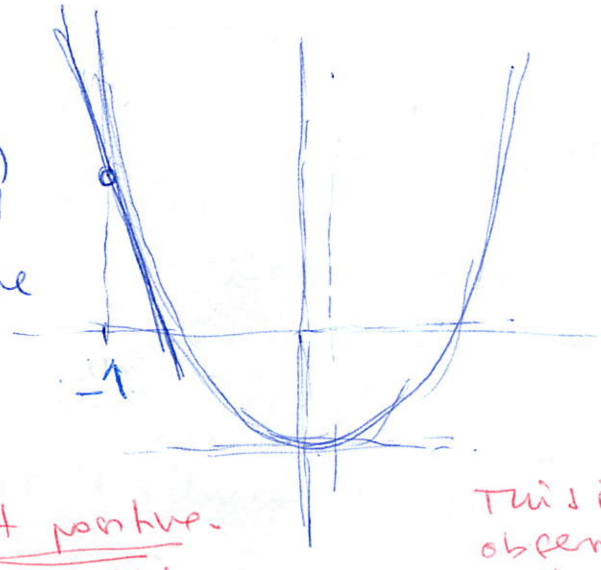
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad |x| < 1$$

or $\arctan \frac{1}{2} = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \dots$, $\frac{1}{2} \in |x| < 1$

c) f'' exits $\Rightarrow$ f' cont $\Rightarrow$ f cont

$f'' > 0$: concave up everywhere.

$f'(-1)$ should be negative



not positive.

!!!!!!

f is concave up $(f''>0)$
Has a minimum at $x=0$ $(f'(0)=0)$

This is an observation: I observe it is incompatible that $f''>0$ everywhere, $f'(0)=0$, a minimum to me with $f'(-1)$ being a positive number.

But there is a theorem that proves observation: The Mean Value theorem applied to f' cont at $[-1,0]$, derivable at $(-1,0)$, $\exists$ c such that

$$\frac{f'(-1)-f'(0)}{-1-0} = f''(c)$$

, look for c in $-1<c<0$

$$\frac{1/2-0}{-1} = -1/2$$

not compatible with $f''(c)>0$

Conclusion: such function DNE. MVT proves it!!
Does Not Exist