

Parcialillo de MM2, grupo C – 22/03/2019

Nombre y Apellidos:

Firma y DNI:

Consejo muy útil: Siempre que sea razonable es conveniente comprobar los resultados.

1. [0.75 puntos] Considérese la ecuación diferencial

$$x y'' + (1 - 2x)y' + 2y = 0$$

en $x = 0$. a) Indicar si $x = 0$ es un punto regular, singular regular o irregular de la ecuación.
b) Escribir el polinomio indicial y calcular sus raíces. c) Hallar los **cuatro** primeros términos no nulos del desarrollo en serie de las dos soluciones linealmente independientes de la ecuación, indicando claramente si alguna de ellas lleva logaritmos.

2. [0.75 puntos] Sea la ecuación

$$u_{xx} + 2u_{xy} + u_{yy} + u = 0$$

a) Decir si es parabólica, hiperbólica o elíptica. b) Escribir dicha ecuación en su forma canónica y hallar la solución general. c) Hallar la solución particular que cumple $u(x, -x) = 0$, $u_y(x, -x) = 1$.

Una vez hechos los ejercicios rellene, por favor, los resultados que se piden:

Prob 1:

El polinomio indicial de la ecuación de Euler correspondiente y sus soluciones:

$$r^2 = 0, \quad r_1 = r_2 = 0 \text{ (doble)}$$

La primera solución de la ecuación es

$$\gamma_1 = 1 - 2x$$

¿La segunda solución de la ecuación lleva logaritmos? Conteste sí o no

sí

no hay 4 términos
no nulos,
sólo log 2.

La segunda solución es

$$\gamma_2 = (1 - 2x) \ln x + 6x - x^2 - \frac{2}{9}x^3 - \frac{x^4}{18} - \dots$$

Prob 2:

La ecuación es (parabólica, hiperbólica, elíptica). Diga lo que corresponda.

parabólica

Escrita en su forma canónica es

$$4ss + u = 0$$

La solución general es

$$u = f(x-t) \sin x + g(x-t) \cos x$$

f, g funciones arbitrarias de la clase adecuada.

La solución particular que cumple $u(x, -x) = 0$, $u_y(x, -x) = 1$ es

$$u = 2 \sin\left(\frac{x+t}{2}\right)$$

cor

$$u = 2 \cos\left(\frac{x-t}{2}\right) \sin x - 2 \sin\left(\frac{x-t}{2}\right) \cos x.$$

$$\textcircled{1} \quad x\gamma'' + (1-2x)\gamma' + 2\gamma = 0 \quad (\star)$$

The point $x=0$ is a regular singular point of (\star)
because near $x=0$ the equation is well described
by an Euler equation,

$$x^2\gamma'' + x\gamma' = 0. \quad (\gamma = \text{const} \text{ is a solution of Euler's equation})$$

The indicial polynomial of this equation is

$$(r(r-1) + r = r^2 = 0, r_1=0, r_2=0)$$

and the associated solution is

$$\gamma_{\text{Euler}} = C_1 + C_2 \log x$$

already announced: the "constant" is a solution.

Hence, the general solution of (\star) has a logarithm.

Even Euler saw it.

However, one of the two linearly independent solutions, Frobenius theorem states, of (\star) carries no logarithm. This is given by

$$\gamma = \sum_0^\infty a_n x^n \text{ with } a_0 \neq 0.$$

$$\text{with } \gamma' = \sum_{n=0}^\infty n a_n x^{n-1}, \quad \gamma'' = \sum_0^\infty n(n-1)a_n x^{n-2},$$

we write that

$$0 = x\gamma'' + (1-2x)\gamma' + 2\gamma \Rightarrow \sum_0^\infty n^2 a_n x^{n-1} - \sum_0^\infty 2(n-1)a_n x^n$$

$$= x^{-1} [\sum_0^\infty 0 a_n]$$

$$+ x^0 [1^2 a_1 - 2(-1) a_0]$$

$$+ x [2^2 a_2 - 2 \cdot 0 \cdot a_1]$$

$$+ x^2 [3^2 a_3 - 2 \cdot 1 \cdot a_2] +$$

turn the page

$$+ x^3 [4^2 a_4 - 2 \cdot 2 a_3] \\ + x^4 [5^2 a_5 - 2 \cdot 3 a_4] \\ + x^5 [6^2 a_6 - 2 \cdot 4 a_5]$$

+ ...

From here,

$a_0 = \text{free}$

$$a_1 = -2a_0$$

$$a_2 = 0 = a_3 = a_4 = a_5 = \dots \text{ remaining}$$

↓ all coefficients

The solution that comes up logarithms is just the polynomial

$$a_0(1-2x)$$

or just

$$\boxed{y_1 = 1-2x}$$

that has only two terms different from zero, not four.

The second linearly independent solution of (x) is, according to Frobenius theorem,

$$y_2 = y_1 \log x + \sum_0^{\infty} b_n x^{n+1}$$

↑ to avoid recalculating
↑ If you write 0 instead you recalculate
↑ but with

with b_0 equal to zero or not

$\sum b_n x^{n+1}$ solution of

$$x y'' + (1-2x)y' + 2y = 6-4x$$

(6-4x to complete that once $y_1 \log x$ is substituted in

$$x y'' + (1-2x)y' + 2y$$

there is a $-6+4x$ expelled

keep the rhs (check it!!!)

[just a 0!!!]

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(3)

In fact $-6 + 4x$ is $\dots + 0 + 0 + 0 + \dots$ and so on

$$-6 + 4x + 0 + 0 + 0 + 0 + \dots$$

↗
 coeff of x^2
 ↗
 coeff of x^3
 ↗
 coeff of x^4

thus

$$\sum_0^{\infty} (n+1)^2 b_n x^n - \sum_0^{\infty} 2^n b_n x^{n+1} = 6 - 4x + 0 + 0 + 0 \dots$$

$$= x^0 [b_0]$$

$$+ x^1 [2^2 b_1 - 2 \cdot 0 \cdot b_0]$$

$$+ x^2 [3^2 b_2 - 2 \cdot 1 \cdot b_1]$$

$$+ x^3 [4^2 b_3 - 2 \cdot 2 \cdot b_2]$$

$$+ x^4 [5^2 b_4 - 2 \cdot 3 \cdot b_3]$$

$$+ \dots$$

and

$$b_0 = 6$$

$$b_1 = -1$$

$$b_2 = \frac{2b_1}{3^2} = -\frac{2}{9}$$

$$b_3 = \frac{b_2}{4} = -\frac{1}{18}$$

$$b_4 = \frac{6b_3}{25} = -\frac{6}{18 \cdot 25} = -\frac{1}{75}$$

⋮

The second solution predicted by Frobenius theorem

is

$$y_2 = (1-2x) \log x + 6x - x^2 = \frac{2}{9}x^3 - \frac{1}{18}x^4 - \frac{x^5}{75} \dots$$

These are five terms.
Problem asks for four only.

SOURCE: CLASS NOTES
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$$② \quad u_{xx} + 2u_{xy} + u_{yy} + u = 0$$

$\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$. The equation is parabolic

Its canonical form will be

$$uss + \dots = 0$$

([↑] 1st derivatives and the function u only.)

Normal coordinates
(different measures obtain them...)

$$0 = r_x^2 + 2r_x r_y + r_y^2 = (r_x + r_y)^2 \quad (\text{inner})$$

$$\text{is } r_x + r_y = 0 \quad \text{with solution: } r = x - y.$$

Choose

or

$r = x - y$
$s = y$



With this choice

$$u_x = \frac{\partial u}{\partial x} = ur + us$$

$$u_y = -ur$$

$$u_{xx} = urt + 2urs + uss$$

$$u_{xy} = -ur - urs$$

$$u_{yy} = us$$

$$u_{xx} + 2u_{xy} + u_{yy} + u = \cancel{urt + 2urs + uss} - \cancel{2urs} - \cancel{ur} + u = 0 = us + u = 0$$

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The equation $U_{xx} + U = 0$ is similar to the ODE equation

$$U'' + U = 0 \quad \stackrel{*}{=} \frac{d^2}{dx^2}$$

with solution

$$U = C_1 \sin x + C_2 \cos x$$

except that but arbitrary coordinate is the "other" in $U_{xx} + U = 0$) coordinate, that is

$$U = f(r) \sin s + g(r) \cos s$$

with fig arbitrary functions $\in C^2(\mathbb{R})$.

The solution of $U_{xx} + 2U_x + U_{\gamma\gamma} + U = 0$ is then

$$U = f(x-\gamma) \sin x + g(x-\gamma) \cos x$$

Fig arbitrary functions of two variables

Particular solution that satisfies $U(x, -x) = 0$,
 $U_{\gamma}(x, -x) = 1$. We need to obtain U_{γ} from

$$U = f(x-\gamma) \sin x + g(x-\gamma) \cos x,$$

say,

$$U_{\gamma} = -f'(x-\gamma) \sin x + g'(x-\gamma) \cos x.$$

where $\stackrel{*}{=}$ derivative with respect to the argument.
 Here the argument is $x-\gamma$ (call it z if you wish, $z = x-\gamma$, or leave it as it is). What are found of that satisfy

$$\left[\begin{array}{l} 0 = f(2x) \sin x + g(2x) \cos x \\ 1 = -f'(2x) \sin x + g'(2x) \cos x \end{array} \right]$$

The resolution of Ex 7 can be done in many ways (I suppose), for example, rewrite it as?

$$\begin{cases} 0 = f(x) \sin \frac{x}{2} + g(x) \cos \frac{x}{2} \\ -1 = f'(x) \sin \frac{x}{2} + g'(x) \cos \frac{x}{2} \end{cases}$$

I think that the best way to proceed is deriving the first equation wrt x and using the second afterwards, as in

$$0 = \underbrace{f'(x) \sin \frac{x}{2} + g'(x) \cos \frac{x}{2}}_{\text{this is } -1} + \frac{1}{2} \left[f(x) \cos \frac{x}{2} - g(x) \sin \frac{x}{2} \right].$$

Now we have to solve

$$\begin{cases} 0 = f(x) \sin \frac{x}{2} + g(x) \cos \frac{x}{2} & [1] \\ 2 = f(x) \cos \frac{x}{2} - g(x) \sin \frac{x}{2} & [2] \end{cases}$$

$\Gamma [1] \cdot \sin \frac{x}{2} + [2] \cos \frac{x}{2}$ is

and later obtain,

$$\boxed{f(x) = 2 \cos \frac{x}{2}}$$

$$\boxed{g(x) = -2 \sin \frac{x}{2}}$$

Particular solution:

$$u_1 = 2 \cos \left(\frac{x-1}{2} \right) \sin x = 2 \sin \left(\frac{x-1}{2} \right) \cos x$$

$$\Rightarrow = 2 \sin \left(\frac{x+1}{2} \right)$$

also equal to

or or or or

$$\boxed{u_1 = 2 \sin \left(\frac{x+1}{2} \right)}$$

Γ I suspected there was a " $x+$ " because the datum was $u(\underline{x_1 - x}) = 0$

L

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