Parcialillo 2 de MM2, grupo C – 13/05/2015

Nombre y Apellidos:

Firma y DNI:

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1. [1.25 punto] Considérese la ecuación diferencial

$$x(x-1)\,y'' + 6x^2y' + 3y = 0.$$

donde $x = 0$ es un punto singular regular. a) Escribir el polinomio indicial en $x = 0$ y calcular sus raíces; b) calcular los primeros **tres** términos de cada una de las dos soluciones linealmente independientes en torno a $x = 0$.

2. [0.50 puntos] Los dibujos de la pizarra corresponden a dos extensiones periódicas distintas de $f(x) = x - x^2$ en $(0, 1)$. Para **cada una de ellas** contestar a las siguientes preguntas: 1) ¿Cuál es el período de la extensión? 2) Indicar si el desarrollo correspondiente es en senos, en cosenos o en senos y cosenos, y los argumentos de estos. 3) Calcular $\sum_1^\infty (a_n^2 + b_n^2)$ en cada caso.

① $x(x-1)y'' + 6x^2 y' + 3y = 0$ [*]

The point $x=0$ is regular singular (o singular regular así en España). Euler ~~about~~ $x=0$ is obtained with

$$x^2[x-1]y'' + 6x[x^2]y' + 3[x]y = 0$$

(substitute $x=0$ in the brackets) and results

$$-x^2 y'' + 0 + 0 = 0$$

whose indicial polynomial is (rectas... obviamente)

$$r(r-1) = 0, \quad r \begin{cases} 1 \equiv r_1 \\ 0 \equiv r_2 \end{cases}$$

[tb se ve de resolver $y''=0$, queda $y = C_1 + C_2 x$ (los exponentes) $r_2 = 0$; $r_1 = 1$]

1st solution: always exists = the newer $r_1 = 1$, $y = \sum_0^\infty a_n x^{n+1}$ with $\underline{a_0 \neq 0}$

$$y' = \sum_0^\infty (n+1) a_n x^n$$

$$y'' = \sum_0^\infty (n+1) n a_n x^{n-1}.$$

Insert y, y', y'' in [*] and write

$$0 = \sum_0^\infty 6(n+1) a_n x^{n+2} + \sum_0^\infty [n(n+1)+3] a_n x^{n+1} - \sum_0^\infty n(n+1) a_n x^n$$

$$= x^0 \underline{(0)} + x^1 [\quad + (0 \cdot 1 + 3) a_0 - 1 \cdot 2 \cdot a_1]$$

$$+ x^2 [6 \cdot 1 \cdot a_0 + (1 \cdot 2 + 3) a_1 - 2 \cdot 3 \cdot a_2]$$

$$+ x^3 [6 \cdot 2 \cdot a_1 + (2 \cdot 3 + 3) a_2 - 3 \cdot 4 \, a_3]$$

$$+ x^4 [6 \cdot 3 \cdot a_2 + (3 \cdot 4 + 3) a_3 - 4 \cdot 5 \, a_4]$$

$$+ x^5 [6 \cdot 4 \cdot a_3 + (4 \cdot 5 + 3) a_4 - 5 \cdot 6 \, a_5] + \dots$$

un lien!

that gives

$$a_1 = \frac{3}{2} a_0,$$

$$a_2 = \frac{1}{6}(6a_0 + 5a_1) = a_0 + \frac{5}{6} a_1 = a_0\left(1 + \frac{5}{4}\right) = \frac{9}{4} a_0,$$

$$a_3 = \frac{1}{12}(12a_1 + 9a_2) = a_1 + \frac{3}{4} a_2 = a_0\left(\frac{3}{2} + \frac{3}{4}\cdot\frac{9}{4}\right) = \frac{51}{16} a_0,$$

$$\vdots$$

and if we take $a_0 = 1$ (optional) we write the 1st solution of [*] as

$$y_1 = \underbrace{x + \frac{3}{2}x^2 + \frac{9}{4}x^3}_{\text{three terms requested...}} + \frac{51}{16}x^4 + \cdots$$

To obtain the second linearly independent out of [*] insert ($r_2 = 0$, remember) $y = \sum_0^\infty b_n x^n$ with $b_0 \neq 0$ in [*] [or change $n \to n-1$ in a previous result]:

$$0 = \sum_0^\infty 6n b_n x^{n+1} + \sum_0^\infty [(n-1)n + 3] b_n x^n - \sum_0^\infty (n-1)n b_n x^{n-1}$$

$$\begin{aligned} &= x^{-1}(0) + x^0[3b_0 - 0\, b_1] \\ &\quad + x[6\cdot 0\cdot b_0 + (0+3)b_1 - 1\cdot 2 b_2] \\ &\quad + x^2[6\cdot 1\cdot b_1 + (1\cdot 2 + 3) b_2 - 2\cdot 3 b_3] \\ &\quad + x^3[6\cdot 2 b_2 + (2\cdot 3 + 3) b_3 - 3\cdot 4 b_4] \\ &\quad + \cdots \end{aligned}$$

and then identify coefficients to obtain that

$b_0 = 0$!!!!!!!!! (coeff of x^0) [Malo!!] malo! malo!

b_1 = free

$b_2 = \frac{3b_1}{2}$ (coeff of x)

$b_3 = \frac{9}{4} b_1$ (coef of x^2)

$\vdots$

Stop here because we are recalculating the series upstairs, and (worse of all) $b_0 = 0$!! This means that problem has

Logarithms enter through $\eta_1 \log x$ with η_1 written previously. But $\eta_1 \log x$, that is not a solution of [*] because inserted in the equation gives a rhs

$$(x-1)\left[2\eta_1' - \frac{\eta_1}{x}\right] + 6x\eta_1 ,$$

$$\begin{aligned} & x(x-1)\left[\eta_1 \log x\right]'' + 6x^2\left[\eta_1 \log x\right]' + 3\eta_1 \log x \\ &= \underbrace{\left[x(x-1)\eta_1'' + 6x^2\eta_1' + 3\eta_1\right]}_{\equiv 0 \text{, since } \eta_1 \text{ is a soln of [*]}} \log x \\ &\quad + (x-1)\left[2\eta_1' - \frac{\eta_1}{x}\right] + 6x\eta_1 . \end{aligned}$$

is the key to correct the unwanted "$b_0=0$".
Calculate with

$$\eta_1 = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \cdots$$

the expansion

$$(x-1)\left[2\eta_1' - \frac{\eta_1}{x}\right] + 6x\eta_1 = -1 - \frac{7}{2}x - \frac{3}{4}x^2 - \frac{33}{16}x^3 + \cdots$$

lo pongo però a fi' per i primi 3 termini cio ho ce fatto...

and with this expansion (but change the sign first!!) as the rhs of

$$\begin{aligned} x^{-1}(0) + x^0 \,[& \qquad\qquad\quad 3b_0 - 0.b_1] \\ + x^1 \,[& \qquad + (0+3)\, b_1 - 1.2\, b_2] \\ + x^2 \,[& 6.1.b_1 + (1.2+3)\, b_2 - 2.3\, b_3] \\ + x^3 \,[& 6.2b_2 + (2.3+3)\, b_3 - 3.4\, b_4] + \cdots \end{aligned}$$

$$= 1 + \frac{7}{2}x + \frac{3}{4}x^2 + \cdots ,$$

deduce that

$b_0 = 1/3$ (¡ya no es cero!!)

$3b_1 - 2b_2 = \frac{7}{2}$ or $b_2 = -\frac{7}{4} + \frac{3}{2} b_1$

$6b_1 + 5b_2 - 6b_3 = \frac{3}{4}$ or $b_3 = -\frac{19}{12} + \frac{9}{4} b_1$

Take $b_1 = 0$ (it was free) and do not recalculate the 1st solution. Collect...

$b_0 = 1/3$, $b_1 = 0$, $b_2 = -\frac{7}{4}$, $b_3 = -\frac{19}{12}, \ldots$

in

$$\frac{1}{3} \quad - \frac{7}{4} x^2 - \frac{19}{12} x^3 - \cdots$$

and write the second solution of [*] as

$$y_1 \log x + \frac{1}{3} - \frac{7}{4} x^2 - \frac{19}{12} x^3 - \cdots$$

Más bonita pe este es (multiplying by 3)

$$3 y_1 \log x + 1 - \frac{21}{4} x^2 - \frac{19}{4} x^3 - \cdots$$

Final solution:

$\boxed{r_1 = 1, \ r_2 = 0}$

$$y_1 = x + \frac{3}{2} x^2 + \frac{9}{4} x^3 + \frac{51}{16} x^4 + \cdots \xrightarrow[x \to 0]{} x$$

no se pedía en el examen tal término

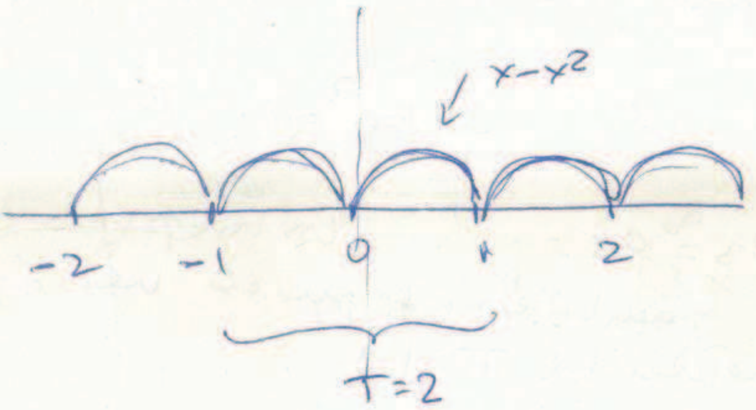
$r_1 = 1$

$$3 y_1 \log x + 1 - \frac{21}{4} x^2 - \frac{19}{4} x^3 - \cdots \xrightarrow[x \to 0]{} 1$$

$r_2 = 0$ [lo decidió el caso de Euler]

La solución general de [*] es una combinación lineal de estas dos soluciones linealmente independientes.

(2)



$T=2$, $\omega=\pi$, $\{\cos n\pi x\}$: since the pattern (graph) is even around $x=0$.

$$\dots = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos n\pi x.$$

Perseval's identity:

$$\frac{1}{T}\int_{-1}^{1} (\quad)^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{1}^{\infty} a_n^2$$ (there are no b_n's ($b_n = 0$))

Calculate each of these terms:

$$\frac{1}{T}\int_{-1}^{1} (\quad)^2 dx = \frac{2}{T}\int_0^1 (\quad)^2 dx$$

even around $x=0$

$$= \int_0^1 (x-x^2)^2 dx = \int_0^1 dx\,(x^2+x^4-2x^3)$$

$$= \frac{1}{3}+\frac{1}{5}-\frac{1}{2}$$

$$= \frac{1}{5}-\frac{1}{6} = \frac{1}{30};$$

$$\frac{a_0}{2} = \frac{1}{T}\int_{-1}^{1} (\quad)\, dx = \frac{2}{T}\int_0^1 (x-x^2)\, dx$$

even around $x=0$

$$= \frac{1}{2}-\frac{1}{3} = \frac{1}{6}.$$

thus, $$\sum_{1}^{\infty} a_n^2 = 2\left(\frac{1}{30}-\frac{1}{36}\right) = \frac{2}{6}\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{90}.$$

Conclusión: primera extensión ;

$$\sum_1^\infty a_n^2 = \frac{1}{90} \quad \text{and} \quad \sum_1^\infty b_n^2 = 0$$

(además $a_1 = a_3 = a_5 = \dots$ = los impares = 0... porque la gráfica también se puede ver como de período fundamental $T=1$).

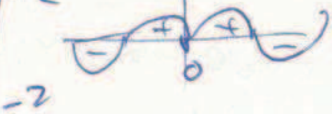
$T = 4$, $\omega = \frac{\pi}{2}$: even around x= too: only cos

$\{\cos n \frac{\pi}{2} x\}$.

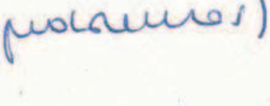
$$= \frac{a_0}{2} + \sum_1^\infty a_n \cos n\frac{\pi}{2} x$$

[pero no los mismos a_0, a_n que en el caso ante-rior. Aunque se usen las mismas letras (usamos otras si tiene problemas) no se debe confundir...

Parseval

$T=4$, $\frac{1}{T}\int_{-2}^{2} \Big[$  $\Big]^2 = \frac{4}{T}\int_0^1 (x-x^2)^2 = \frac{1}{30}$

(igual que antes),

pero $\frac{a_0}{2} = \frac{1}{T}\int_{-2}^{2}$  $= 0$ (se compensan las áreas).

Parseval is now:

$$\frac{1}{30} = 0 + \frac{1}{2}\sum_1^\infty a_n^2, \text{ or } \boxed{\sum_1^\infty a_n^2 = \frac{1}{15}}$$

$+ \sum_1^\infty b_n$ ... (no los ser...