Parcialillo 2 de MM2, grupo C – 27/05/2016

Nombre y Apellidos:

Firma y DNI:

---

1. [0.75 puntos] Sea el desarrollo de $f(x) = x$ en $(0, \pi)$ dado por

$$x \sim b\Big(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\Big).$$

donde b es una constante. ia) Encuentre el período de la serie y dibuje la extensión de la función x que representa (dibuje **tres** períodos al menos, por favor). ib) Calcule b. ic) Diga qué función representa la serie en $(-\pi, 0)$.

iia) Lo mismo para $f(x) = x$ en $(0, 2\pi)$ con

$$x \sim a + c\Big(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\Big),$$

donde a y c son constantes no nulas. iib) Calcule a y c. iic) Diga qué función representa la serie en $(-2\pi, 0)$

iiia) $f(x) = x(1-x)$ en $(0, 1)$ si

$$x(1-x) \sim \frac{8}{\pi^3}\Big(\frac{\sin \pi x}{1^3} + \frac{\sin 3\pi x}{3^3} + \frac{\sin 5\pi x}{5^3} + \cdots\Big)$$

Período y pintar. iiib) ¿Qué función representa la serie en $(-1, 0)$?

Hay que contestar en total a **ocho** apartados.

2. [1 punto] i) Resolver por separación de variables el problema

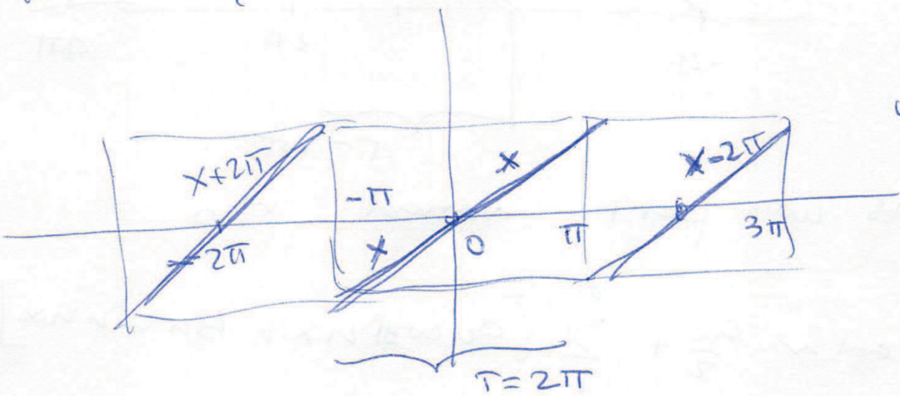
$$\begin{cases} \Delta u = 0, \quad 0 < x, y < \pi, \\ u(x,0) = 0, \quad u(x,\pi) = \sin^2 x = \dfrac{1}{2}(1 - \cos 2x), \quad 0 \le x \le \pi \\ u_x(0,y) = u_x(\pi,y) = 0, \quad 0 \le y \le \pi. \end{cases}$$

ii) Extra: Encontrar la solución si en lugar de $\sin^2 x$ se pone $\sin^4 x$.

Parcial MM2 - 27 Mayo de 2016

(1) i a) $x \sim b\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right)$, $x \in (0,\pi)$

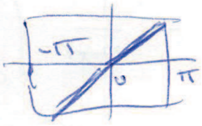
We see from this expansion that $\omega = 1$, and $\boxed{T = 2\pi}$. It is in sines, so it is an odd expansion around $x=0$. The only possibility is



3 periods

i b) $f_{ext} \sim \sum_1^\infty b_n \sin nx$.

Here $b = b_1$. Calculate b_1:

$$\frac{b_1}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}$$  $\cdot \sin x$

$$= \frac{1}{\pi}\int_0^\pi x \sin x = 1 \quad \therefore \quad b_1 = 2$$

or $\boxed{b=2}$

[used: $\int dx\, x \sin ax = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$

i c) From the figure:

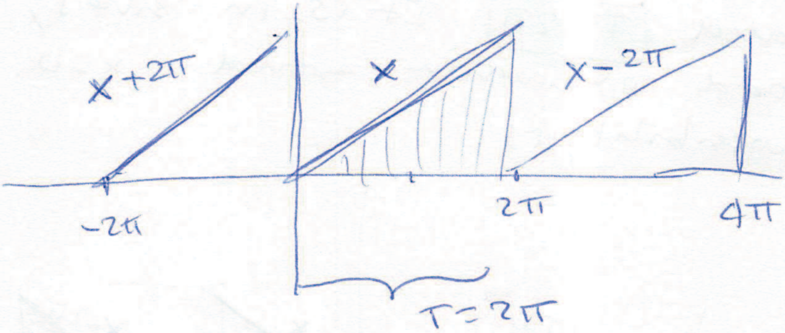
$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right) = \begin{cases} x \text{ when } x \in [0,\pi) \\ x \text{ when } x \in (-\pi, 0] \end{cases}$$

vamos, si es x en todo $(-\pi,\pi)$.

(iia) $x \sim a + c\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$

Picture: a is non-zero. ($\nearrow$ sines and cosines)

~~Ex~~pansion: indicates that $\omega = 1$, $\boxed{T = 2\pi}$, then



Has no parity around $x=0$

(iib) $f_{ext} \sim \frac{a_0}{2} + \sum_1^\infty \left[a_n \cos nx + b_n \sin nx\right]$

Here

$$a = \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\,\frac{(2\pi)^2}{2} = \pi, \qquad \boxed{a = \pi}$$

and c is b_1,

$$\frac{b_1}{2} = \frac{1}{2\pi}\int_0^{2\pi} dx\, x \sin x$$

$$= \frac{1}{2\pi}(-2\pi)\cos 2\pi = -1. \qquad \boxed{c = -2}$$

Then, the expansion is

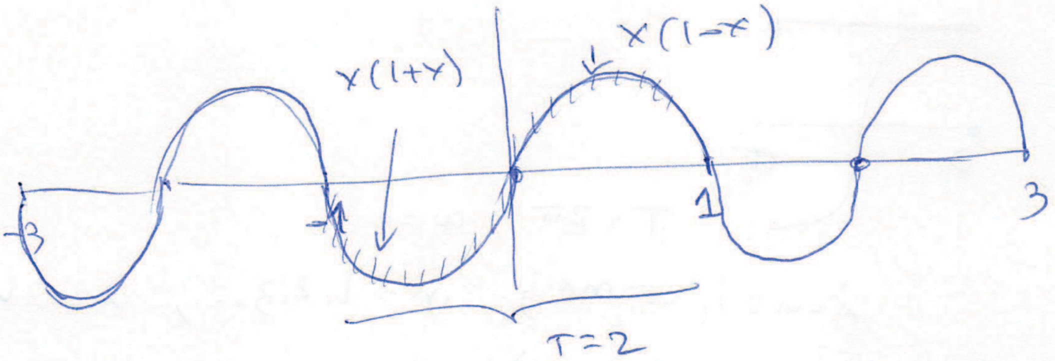
$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$$

(iic) From the picture

$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right) = \begin{cases} x & \text{when } x \in (0, 2\pi) \\ x + 2\pi & \text{when } x \in (-2\pi, 0) \end{cases}$$

UNIVERSIDAD COMPLUTENSE
FACULTAD DE CIENCIAS FISICAS
DEPARTAMENTO DE FISICA TEORICA I
28040 MADRID - ESPAÑA
TELEF.: (34) (1) 394 45 09
FAX.: (34) (1) 394 51 97
TELEX: 47273 FFUC - E

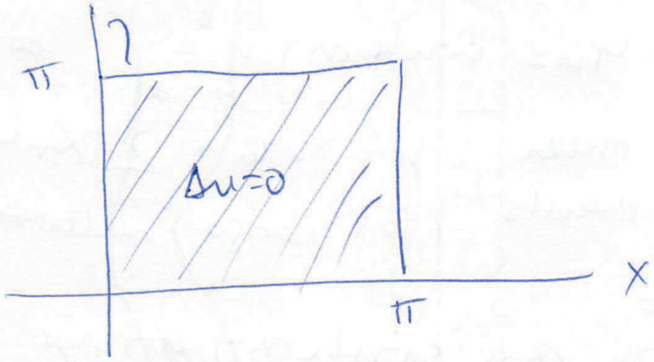
(iia) From the expansion, $\omega=\pi$, $\boxed{T=2}$. The series then odd around $x=0$



(iib) $\frac{8}{\pi^3}\left(\frac{\sin \pi x}{1^3}+\frac{\sin 3\pi x}{3^3}+\frac{\sin 5\pi x}{5^3}+\dots\right)=\begin{cases} x(1-x) & \text{when } x\in[0,1] \\ x(1+x) & \text{when } x\in[-1,0] \end{cases}$

---

(2)



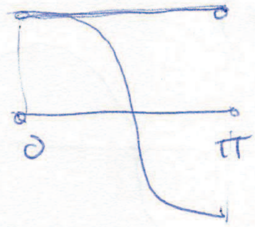
$$\begin{cases} \Delta u=0 & 0<x,y<\pi \\ u(x,0)=0, \; u(x,\pi)=\sin^2 x & 0\le x\le\pi \\ u_x(0,y)=u_x(\pi,y)=0 & 0\le y\le\pi \end{cases}$$

[obviamente el problema de contorno es en las x's: $u_x(0,y)=u_x(\pi,y)=0$. Neumann's conditions].

Separation of variables: $u=XY$

$$\begin{cases} X''+\lambda X=0 \\ X'(0)=X'(\pi)=0 \end{cases} \qquad \begin{cases} Y''-\lambda Y=0 \\ Y(0)=0 \\ \dots\dots \end{cases}$$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$$



$T = 2\pi$, $\omega = 1$

$X_n = \{\cos nx\}$, $n = 1, 2, 3, \dots$, $\lambda_n = n^2$

$X_0 = \{1\}$, $\lambda_0 = 0$

The 'escort' problem:

$$\begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = 0 \end{cases}$$

cojo le representante (sería const·y)

when $\lambda_0 = 0$: $\left.\begin{matrix} Y'' = 0 \\ Y(0) = 0 \end{matrix}\right\} \Rightarrow Y_0 = y$

when $\lambda_n = n^2$ $\quad Y_n = \sinh ny$.

the solution u is then $u = \underbrace{X_0}_{=1}\underbrace{Y_0}_{=y} + \sum_{n=1}^{\infty} \underbrace{X_n}_{=\cos nx}\underbrace{Y_n}_{=\sinh ny}$

with suitable constants (les pongo a mano):

is $u(x,y)$

$$u = \underline{\underline{a}}\, y + \sum_{1}^{\infty} \underline{\underline{a_n}} \sinh ny \cos nx$$

where a, a_n are determined with $u(x,\pi)$.

$$\sin^2 x = a\pi + \sum_{1}^{\infty} a_n \sinh n\pi \cos nx$$

$$= a\pi + a_1 \sinh \pi \cos x + a_2 \sinh 2\pi \cos 2x + \dots$$

mejor escrito como $\frac{1}{2}(1 - \cos 2x)$

$\Rightarrow \boxed{a = \frac{1}{2}}$

$\boxed{a_2 \sinh 2\pi = -\frac{1}{2}}$

Resto de constantes cero !!!

UNIVERSIDAD COMPLUTENSE
FACULTAD DE CIENCIAS FISICAS
DEPARTAMENTO DE FISICA TEORICA I
28040 MADRID - ESPAÑA
TELEF.: (34) (1) 394 45 09
FAX.: (34) (1) 394 51 97
TELEX: 47273 FFUC - E

then,

$$\boxed{u = \frac{y}{2\pi} - \frac{\sinh 2y \cos 2x}{2 \sinh 2\pi}}$$

that it is not a series expansion but a single sum!! (compruebe que cumple todo)

Extra:

$$u(x,\pi) = \sin^4 x = \sin^2 x \sin^2 x = \frac{1}{4}(1-\cos 2x)^2$$

$$= \frac{1}{4}\Big[1 + \underbrace{\cos^2 2x}_{\frac{1}{2}(1+\cos 4x)} - 2\cos 2x\Big]$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x .$$

The solution is then

$$\boxed{u = \frac{3y}{8\pi} - \frac{\sinh 2y \cos 2x}{2\sinh 2\pi} + \frac{\sinh 4y \cos 4x}{8 \sinh 4\pi}}.$$