

**Examen de Métodos Matemáticos II – Grupo D**

Nombre y Apellidos:

DNI y firma:

**Consejo muy útil:** Siempre que sea razonable es conveniente comprobar los resultados. Respuestas totalmente correctas sin la debida justificación no recibirán consideración alguna.

1. [2 puntos] Sea la ecuación

$$x^2 y'' - x(x+5)y' + 9y = 0.$$

- a) Escribir el polinomio indicial en  $x = 0$  y calcular sus raíces.
- b) Encontrar los tres primeros términos no nulos de una solución analítica en  $x = 0$  y su regla de recurrencia. ¿Tienen a cero todas las soluciones cuando  $x \rightarrow 0$ ?

2. [2 puntos] Sea
- $f(x) = \begin{cases} x+1 & -1 < x < 0 \\ x & 0 \leq x < 1 \end{cases}$
- .

- a) Hacer un dibujo de  $f(x)$  extendida periódicamente (pinte, por favor, al menos cuatro períodos de la extensión).
- b) Calcular la serie de Fourier del apartado anterior indicando claramente si es de senos, de cosenos o de senos y cosenos.
- c) Suponiendo que la serie obtenida en b) puede integrarse término a término determinar la constante de integración y calcular la suma de la serie

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots,$$

justificando su respuesta con los teoremas de convergencia puntual de series trigonométricas.

*Nota:* No hay que usar identidad de Parseval.

3. [2 puntos] Resolver el problema en el plano

$$\begin{cases} \Delta u = \sin 4\theta, & r < 1, \quad 0 < \theta < \pi/2 \\ u(1, \theta) = \sin 2\theta \\ u(r, 0) = u(r, \pi/2) = 0. \end{cases}$$

Dibujar el recinto de integración.

4. [2 puntos] Resolver por separación de variables y hallar la solución estacionaria cuando
- $t \rightarrow \infty$
- del problema

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ u(x, 0) = 0, \\ u_x(0, t) = 0, \quad u(1, t) = 1. \end{cases}$$

5. [2 puntos] Sea la ecuación

$$xu_x + (2x - y)u_y = yu^2.$$

Determinar la solución general por el método de las características y la solución que cumple el dato  $u(x, 2x) = \frac{1}{x^2 - 1}$

MW2 June - 2013

$$\textcircled{1} \quad x^2 y'' - x(x+5)y' + 9y = 0 \quad [x]$$

Euler at  $x=0$ :  $x^2 y'' - 5x y' + 9y = 0$

Characteristic polynomial:  $r(r-1) - 5r + 9$

$$= r^2 - 6r + 9 \\ = (r-3)^2, \quad r \begin{cases} 3 \\ 3 \end{cases}$$

solutions (of Euler's):  $y = c_1 x^3 + c_2 x^3 \log x$ .

Both solutions,  $x^3$  and  $x^3 \log x$ , tend to zero when  $x \rightarrow 0$ , but only one ( $x^3$ ) is analytic at  $x=0$ . Same happens with the solutions

of  $[*]$

$$y = \sum_0^{\infty} a_n x^{n+3}, \quad a_0 \neq 0 \quad (\text{Frobenius th})$$

$$y' = \sum_0^{\infty} (n+3) a_n x^{n+2}$$

$$y'' = \sum_0^{\infty} (n+3)(n+2) a_n x^{n+1}$$

Then  $[x]$  reduces to

$$0 = \sum_0^{\infty} n^2 a_n x^{n+3} - \sum_0^{\infty} (n+3) a_n x^{n+4}$$

$$= x^3 [\underline{0}]$$

$$+ x^4 [1^2 a_1 - 3 a_0]$$

$$+ x^5 [2^2 a_2 - 4 a_1]$$

$$+ x^6 [3^2 a_3 - 5 a_2]$$

$$+ x^7 [4^2 a_4 - 6 a_3]$$

$$+ x^8 [5^2 a_5 - 7 a_4]$$

$$+ x^{n+3} [n^2 a_n - (n+2) a_{n-1}] + \dots$$

Recurrence law:  $a_n = \frac{n+2}{n^2} a_{n-1}$ .

Now,

$$a_1 = 3a_0$$

$$a_2 = \frac{4a_1}{2^2} = a_1 = 3a_0$$

$$a_3 = \frac{5a_2}{3^2} = \frac{5}{3} a_0$$

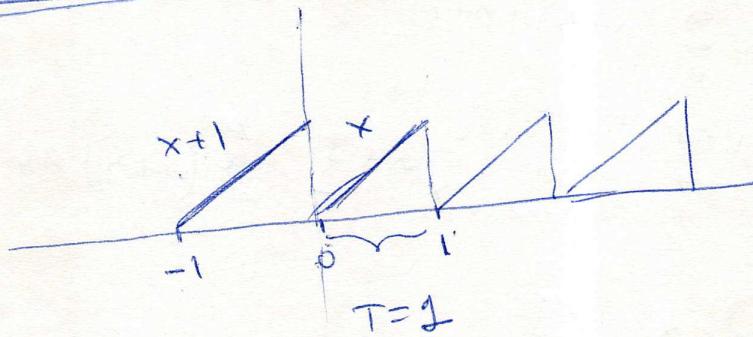
$$a_4 = \frac{6}{4^2} a_3 = \frac{6}{4^2} \cdot \frac{5}{3} a_0 = \frac{5}{8} a_0$$

Take  $a_0 = 1$  and write the first terms of an analytic solution at  $x=0$ :

$$y_1 = x^3 (1 + 3x + 3x^2 + \frac{5}{3}x^3 + \frac{5}{8}x^4 + \dots) \quad -0.4$$

The other soln.  $y_1 \log x + \sum_0^\infty b_n x^{n+4}$  (with  $b_0 = 0$  or not) is non-analytic at  $x=0$ .   
 (Réve de la sueca de Fabr, tres reservado) slip

②



0.25

$T_1$ ,  $\sin \omega x$

(la función es  $f(x) = x$  extendida periódicamente).  $T = 1$ .

$$\omega = 2\pi$$

$$y \sim \sin 2\pi nx, \cos 2\pi nx$$

La serie es en senos pares canónicamente ortogonal a  $x=0$ .

$$\dots = \frac{a_0}{2} + \sum_1^\infty a_n \cos 2\pi nx + b_n \sin 2\pi nx.$$

(2)

$$\frac{a_0}{2} = \frac{1}{T} \int_0^1 x dx = \frac{1}{2} \quad \text{there is } \cancel{\pi} = \frac{1}{2}$$

$$\frac{a_n}{2} = \frac{1}{T} \int_0^1 x \cos 2n\pi x dx \quad (T=1)$$

$$= \left[ \frac{x}{2n\pi} \sin 2n\pi x + \frac{1}{(2n\pi)^2} \cos 2n\pi x \right]_0^1$$

↑  
no contribution  
at  $x=1, x=0$

$$= \frac{1}{(2n\pi)^2} [1 - 1] = 0, \quad n=1, 2, \dots \quad \boxed{a_n = 0}$$

$$\frac{b_n}{2} = \frac{1}{T} \int_0^1 x \sin 2n\pi x dx \quad (T=1)$$

$$= \left[ -\frac{x}{2n\pi} \cos 2n\pi x + \frac{1}{(2n\pi)^2} \sin 2n\pi x \right]_0^1$$

↑  
no contr.  
at  $x=0$

↑  
no contribution  
at  $x=1, x=0$

$$= -\frac{1}{2n\pi}, \quad n=1, 2, 3, \dots$$

$$\boxed{b_n = -\frac{1}{n\pi}}$$

Forced free

$$\int dx \cdot \cos \alpha x = \frac{x}{\alpha} \sin \alpha x + \frac{1}{\alpha^2} \cos \alpha x$$

$a_0/2, a_n, b_n$

$$\int dx \cdot \sin \alpha x = -\frac{x}{\alpha} \cos \alpha x + \frac{1}{\alpha^2} \sin \alpha x$$

$J = 0.2 + 0.4 + 0.4$

L

Result:

$$\begin{aligned} \therefore f(x) &= \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n} \\ &= \frac{1}{2} - \frac{1}{\pi} \left( \sin 2\pi x + \frac{1}{2} \sin 4\pi x + \frac{1}{3} \sin 6\pi x + \dots \right) \end{aligned}$$

series

cosines

Y

sines

0.78 rest

c) Integrate term by term. the previous result:

$$0 \leq x \leq 1, \quad x = \frac{1}{2} - \frac{1}{\pi} \left( \sin 2\pi x + \frac{1}{2} \sin 4\pi x + \frac{1}{3} \sin 6\pi x + \dots \right)$$

and obtain

$$0 \leq x \leq 1, \quad \frac{x^2}{2} = C + \frac{x}{2} + \frac{1}{2\pi^2} \left( \frac{\omega 2\pi x}{12} + \frac{\omega 4\pi x}{22} + \frac{\omega 6\pi x}{32} + \dots \right)$$

or

$$0 \leq x \leq 1, \quad \frac{x^2}{2} - \frac{x}{2} = C + \frac{1}{2\pi^2} \left( \frac{\omega 2\pi x}{12} + \frac{\omega 4\pi x}{22} + \frac{\omega 6\pi x}{32} + \dots \right)$$

To calculate  $C$  integrate in  $\int_0^1 dx$ :

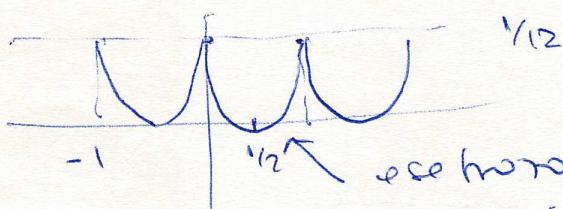
$$\frac{1}{6} - \frac{1}{4} = C + 0 + 0 + 0 + \dots$$

$$\boxed{C = -\frac{1}{12}}.$$

Now

$$\frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} = \frac{1}{2\pi^2} \left( \frac{\omega 2\pi x}{12} + \frac{\omega 4\pi x}{22} + \frac{\omega 6\pi x}{32} + \dots \right)$$

in  $0 \leq x \leq 1$ : este serie de



es decir es  $\frac{x^2 - x + \frac{1}{12}}{2}$  en  $0 \leq x \leq 1$   
en  $x = \frac{1}{12}$

Substitute  $x = \frac{1}{12}$  in this expansion (le  
falta  $\frac{1}{12}$  o combinar así ya los signos lo  
que sume la función deseada para, se dice Dirichlet:

$$\underbrace{\frac{1}{4} - \frac{1}{2} + \frac{1}{12}}_2 = \frac{1}{2\pi^2} \left( -\frac{1}{12} + \frac{1}{22} - \frac{1}{32} + \frac{1}{42} - \dots \right)$$

$-\frac{1}{24}$ ; Conclusion

$$\boxed{\frac{1}{12} - \frac{1}{22} + \frac{1}{32} - \dots = \frac{\pi^2}{12}}$$

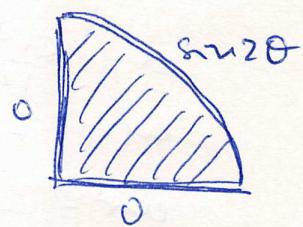
valide:  $\frac{x^2 - x + \frac{1}{6}}{6} = \frac{1}{\pi^2} \left( \frac{\omega 2\pi x}{12} + \frac{\omega 4\pi x}{22} + \frac{\omega 6\pi x}{32} + \dots \right)$   
en  $0 \leq x \leq 1$

(3)

(3)

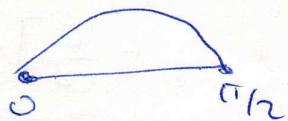
$$\left\{ \begin{array}{l} \Delta u = \sin 4\theta \\ u(\frac{1}{2}, \theta) = \sin 2\theta \\ u(r, 0) = u(r, \frac{\pi}{2}) = 0 \end{array} \right.$$

$$r < 1, \quad \theta \in (0, \frac{\pi}{2})$$



$$u = R\theta$$

$$\left\{ \begin{array}{l} \theta'' + \theta = 0 \\ \theta(0) = \theta(\frac{\pi}{2}) = 0 \end{array} \right.$$



$$T = \pi \\ \omega = 2$$

$$\theta_n = \lambda n \sin 2n\theta, \quad n=1, 2, 3, \dots$$

La solución es

$$u = \sum_{n=1}^{\infty} R_n \sin(n\theta) \quad 0.5$$

con  $R_n(r)$  calculado con

$$\left\{ \begin{array}{l} \Delta u = \sum_{n=1}^{\infty} \left( R_n'' + \frac{R_n'}{r} - 4n^2 \frac{R_n}{r^2} \right) \sin(n\theta) = \sin 4\theta \\ \sin 2\theta = \sum_{n=1}^{\infty} R_n(1) \sin(n\theta). \end{array} \right.$$

We have then:

$$\left\{ \begin{array}{l} R_1'' + \frac{R_1'}{r} - 4 \frac{R_1}{r^2} = 0 \\ R_1(1) = 1 \end{array} \right. \quad 0.5$$

$$\left\{ \begin{array}{l} R_2'' + \frac{R_2'}{r} - 16 \frac{R_2}{r^2} = 1 \\ R_2(1) = 0 \end{array} \right.$$

1.

All other  $R_n$ 's ( $n=3, 4, 5, \dots$ ) are zero.

$$R_2 = C_3 r^4 + C_4 \frac{r^4}{r^4} - \frac{r^2}{12}$$

$$R_2(1)=0 \text{ is } C_3 = \frac{1}{12} - C_4$$

$$R_1 = C_1 r^2 + \frac{C_2}{r^2}$$

$$R_2 = \frac{1}{12} r^4 + C_4 \left( r^4 - \frac{r^2}{12} \right)$$

0 to avoid singularity

$$R_1(1)=1 \text{ is } C_1 = 1 - C_2$$

$$\text{or } R_1 = r^2 + C_2 \left( \frac{1}{r^2} - r^2 \right)$$

Solución:

$$u = r^2 \sin 2\theta + \frac{1}{12} (r^4 - r^2) \sin 4\theta$$

simplifications

$$\Delta u = \Delta \left[ -\frac{r^2}{12} \sin 4\theta \right]$$

some terms

Comprobación: la homogeneidad es ok. y cumplió el dato. La no homogeneidad:

$$u = -\frac{r^2}{12} \sin 4\theta$$

$$ur = -\frac{r}{6} \sin 4\theta$$

$$ur = -\frac{1}{6} \sin 4\theta$$

$$u_{\theta\theta} = \frac{16}{12} r^2 \sin 4\theta$$

$$\sin 4\theta \left[ -\frac{1}{6} - \frac{1}{6} + \frac{16}{12} \right] = \sin 4\theta \quad \text{OK}$$

④  $\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1, t > 0 \\ u(x, 0) = 0 \\ u_x(0, t) = 0, \quad u(1, t) = 1 \end{cases}$

stationary solution:  $\bar{u} = \bar{u}(x)$

$$\bar{u}_{xx} = 0$$

$$\bar{u}_x(0) = 0, \quad \bar{u}(1) = 1$$

$$\Rightarrow \boxed{\bar{u} = 1}$$

rectas

dependiente de

En las variables

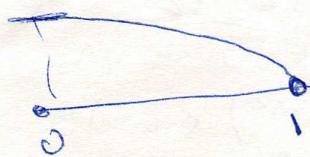
$$w = u - 1 \quad \text{OS}$$

the problem is

$$\begin{cases} w_t - w_{xx} = 0 \\ w(x, 0) = -1 \\ w_x(0, t) = w(1, t) = 0 \end{cases}$$

$$w = T \cdot X$$

$$\begin{cases} X''_x + X_x = 0 \\ X(0) = X(1) = 0 \end{cases} ; \quad \left. \begin{array}{l} T' + \lambda T = 0 \\ \dots \end{array} \right.$$



$$T = 4, \quad \omega = \frac{\pi}{2}$$

$$\text{h. } \omega = n \frac{\pi}{2} x, \quad n = 1, 2, \dots$$

but not all

(4)

$n=0$  desde  $\omega \neq 0$  que  $w_0$  (pero esto no es  
lo que queremos  $n=1, 2, \dots$ ) : la única constante  
se cumple el dato  $x'(0) = x(1) = 0$  es "0".

$$n=1 \quad \omega \frac{\pi}{2} \quad \text{OK}$$

$$n=2 \quad \omega \pi \quad \text{NO}$$

$$n=3 \quad \omega \frac{3\pi}{2} \quad \text{OK}$$

then:

$$x_n = h \omega \frac{2n-1}{2} \pi x, \quad n=1, 2, 3, \dots \quad \text{0.5}$$

$$x_n = \left(\frac{2n-1}{2}\pi\right)^2.$$

$$\int_0^1 dx \cos \frac{2n-1}{2} \pi x \xrightarrow{\omega} \frac{2n-1}{2} \pi x = \frac{1}{2} \delta_{nn}$$

orthogonality relations

the problem is true if

$$T_n' + \left(\frac{2n-1}{2}\right)^2 \pi^2 T_n = 0 \quad \text{0.75}$$

with solution (serve for a multiplying constant)  
 $- (2n-1)^2 \frac{\pi^2}{4} t.$ 

$$T_n = e^{- (2n-1)^2 \frac{\pi^2}{4} t}$$

thus

$$w = \sum_n a_n e^{- (2n-1)^2 \frac{\pi^2}{4} t} \xrightarrow{\omega} \frac{2n-1}{2} \pi x \quad \text{0.75}$$

$$-f = \sum_n a_n \omega \frac{2n-1}{2} \pi x \quad [\text{Corr 1.5 hasta el } f]$$

$$\Gamma - \omega \frac{2n-1}{2} \pi x = \sum_n a_n \omega \frac{2n-1}{2} \pi x \xrightarrow{\omega} \frac{2n-1}{2} \pi x$$

$$-\int_0^1 \omega \frac{2n-1}{2} \pi x = \frac{a_m}{2} \quad \text{or} \quad \frac{a_n}{2} = - \int_0^1 \omega \frac{2n-1}{2} \pi x$$

L

$$\begin{aligned}
 \frac{a_n}{2} &= - \int_0^1 dx \cos \left( \frac{(2n-1)\pi}{2} x \right) = - \frac{2}{(2n-1)\pi} \left[ \sin \frac{(2n-1)\pi}{2} x \right]_0^1 \\
 &= - \frac{2}{(2n-1)\pi} \sin \frac{(2n-1)\pi}{2} x \quad \text{since } \frac{(2n-1)\pi}{2} = (-1)^{n+1} \\
 &= - \frac{2(-1)^{n+1}}{(2n-1)\pi} \\
 &= \frac{2(-1)^n}{(2n-1)\pi} \quad " \quad \boxed{a_n = \frac{4(-1)^n}{(2n-1)\pi}}
 \end{aligned}$$

Solução: única

$$w = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} e^{-\frac{(2n-1)^2 \pi^2 t}{4}} \cos \frac{2n-1}{2} \pi x$$

and

$$u = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} e^{-\frac{(2n-1)^2 \pi^2 t}{4}} \cos \frac{2n-1}{2} \pi x. \quad \underline{\text{única}}$$

Sumário estatístico: laje pedreira  
medo  $t \rightarrow$   
(calcular al peso)

$$\begin{cases} x u_x + (2x-7) u_y = \gamma u^2 \\ u(x, 2x) = \frac{1}{x^2-1} \end{cases}$$

$$(u_x, u_{y1}) \cdot (x, 2x-7, \gamma u^2) = 0$$

$$\frac{dx}{x} = \frac{dy}{2x-7} = \frac{du}{\gamma u^2}$$

charakteristische vektor.

(5)

1st characteristic

$\frac{dx}{x} = \frac{dy}{2x-y}$  is also  $\frac{dy}{dx} + \frac{1}{x} = 2$ , whose solution is

$$\left[ y = \frac{C_1}{x} + x \right]$$

particular  
general solution  
of homogeneous  
equation

1st characteristic

0.5

2nd characteristic

$$\frac{dx}{x} = \frac{du}{u^2} \quad \text{or} \quad \gamma \frac{dx}{x} = \frac{du}{u^2} \quad \text{or}$$

$$\left[ \frac{C_1}{x} + x \right] \frac{dx}{x} = \frac{du}{u^2} \quad \text{or} \quad \left( \frac{C_1}{x^2} + 1 \right) dx = \frac{du}{u^2},$$

then it is repeated is

$$-\frac{C_1}{x} + x - C_2 = -\frac{1}{u}$$

or

$$\frac{C_1}{x} - x + C_2 = \frac{1}{u}$$

0.5

or

$$\left[ y - 2x + C_2 = \frac{1}{u} \right] : \quad \text{2nd characteristic}$$

General solution:

$$\left[ \frac{1}{u} = y - 2x + f(x(y-x)) \right]$$

0.5

Particular solution  $u(x, 2x) = \frac{1}{x^2-1}$

$$x^2-1 = f(x^2) \quad \text{or} \quad f(w) = w-1,$$

$$\begin{aligned} \frac{1}{u} &= y - 2x + x(y-x) - 1 = -(x+1)^2 + y(x+1) \\ &= (x+1)(y-x-1). \end{aligned}$$

$$\left[ \frac{1}{u} = (x+1)(y-x-1) \right]$$

0.5

$$\frac{1}{u} = (x+1)(y-x-1)$$

$$-\frac{1}{u^2} ux = y - x - 1 + (x+1) = y - 2x - 2$$

$$-\frac{1}{u^2} uy = x+1$$

$$\begin{aligned} xu + (2x-y)uy - yu^2 &= u^2 [ -x(y-2x-2) - (2x-y)(x+1) - y ] \\ &= u^2 [ -x(\cancel{y}-\cancel{2x}-2) - 2x(\cancel{x}+1) + y(\cancel{x}+1) - y ] \\ &\equiv 0. \quad \underline{\text{ok}} \end{aligned}$$