UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

27 de Junio de 2014

Examen de Métodos Matemáticos II – Grupo D

Nombre y Apellidos:

DNI y firma:

Consejo muy útil: Siempre que sea **razonable** es conveniente comprobar los resultados.

1. [2 puntos] Sea $f(x) = |x|$ en $-\pi/2 < x < \pi/2$. a) Hacer un dibujo de $f(x)$ extendida periódicamente (pinte, por favor, al menos tres periodos de la extensión). b) Calcular la serie de Fourier del apartado anterior indicando claramente si es de senos, de cosenos o de senos y cosenos. c) Deducir la suma de la serie

   $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots ,$$

   justificando su respuesta con los teoremas de convergencia puntual de series trigonométricas.

   Nota: No hay que usar identidad de Parseval.

2. [1.75 puntos] Sea la ecuación

   $$2xy'' + (2x-1)y' - 5y = 0.$$

   a) Escribir el polinomio indicial en $x = 0$ y calcular sus raíces. b) Encontrar una solución **no analítica** en dicho punto. c) Decir si la afirmación que sigue es cierta o falsa: “Ninguna solución de la ecuación es analítica en $x = 0$”.

3. [2.25 puntos] Resolver por separación de variables

   $$\begin{cases} u_t - 4\,u_{xx} = 0, \quad 0 < x < \pi, \quad t > 0 \\ u(x,0) = 0, \\ u(0,t) = t,\ u_x(\pi,t) = 0. \end{cases}$$

4. [2.25 puntos] Calcular por separación de variables la única constante a para la que tiene solución acotada el problema del plano

   $$\begin{cases} \Delta u = 0, \quad r < 2,\ \ 0 < \theta < \pi/3 \\ u_r(2,\theta) = a + \cos 3\theta, \\ u_\theta(r,0) = u_\theta(r,\pi/3) = 0. \end{cases}$$

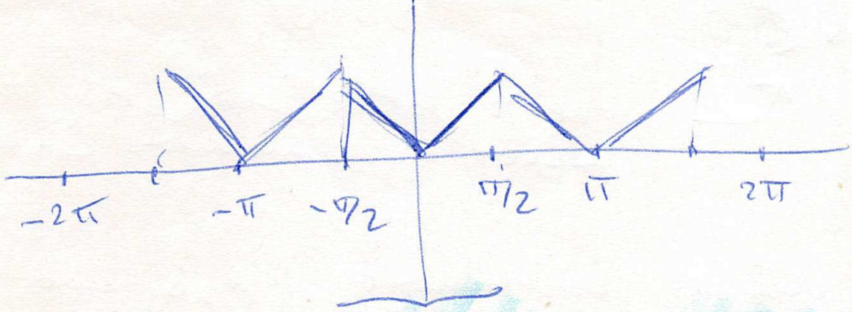
5. [1.75 puntos] Sea la ecuación

   $$u_x + (2x-y)u_y = u^2.$$

   Determinar la solución general por el método de las características y i) la solución que cumple el dato $u(0,y) = 1/y$, ii) ¿cuántas soluciones de la ecuación cumplen $u(x, 2(x-1)) = -\dfrac{1}{x}$?

MM2 - Final - June 2014

(1)



$T = \pi$, $\omega = 2$

$\{\cos 2nx\}$: serie de cosenos.

$$|x| \sim \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos 2nx, \text{ where}$$

$$\frac{a_0}{2} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} = \frac{2}{\pi}\cdot \text{area of} \quad (0,\ \pi/2,\ \pi/2)$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{2}\cdot\frac{\pi}{2}\cdot\frac{1}{\cancel{2}} = \frac{\pi}{4}$$

$$\frac{a_n}{2} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cdot\cos 2nx = \frac{2}{\pi}\int_0^{\pi/2} dx\, x\cos 2nx$$

use $\int x\cos ax\,dx = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax$ // $a = 2n$, $a\cdot\frac{\pi}{2} = n\pi$, $n = 0,1,2,\dots$

$$= \frac{2}{\pi}\left[\frac{1}{a^2}\right]\left[\cos n\pi - 1\right] = \frac{2}{\pi}\frac{1}{4n^2}\left[(-1)^n - 1\right]$$

$$\begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{1}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

Thus

$$a_n \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{2}{\pi n^2} & \text{if } n \text{ odd.} \end{cases}$$

$$|x| = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$$

copying

$$|x| = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right],$$

when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

the function is continuous at $x=\pi/2$.

At $x=0$ the function is continuous (is 0) and the series sums 0 too:

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Hence

$$\boxed{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}}$$

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(2) $$2xy'' + (2x-1)y' - 5y = 0 \qquad [*]$$

Euler at $x=0$ is

$$2x^2y'' - xy' + 0 = 0$$

[why? $2x^2y'' + x\underbrace{[2x-1]}_{-1 \text{ at } x=0}y' - 5xy = 0 \longrightarrow 0 \text{ at } x=0$]

Indicial polynomial associated to Euler equation is [divide [*] per

$$2r(r-1) - r = 0$$

or

$$r(2r-3) = 0 \quad \begin{cases} r_1 = 3/2 & y_1 \sim x^{3/2}[a_0 + \cdots] \\ r_2 = 0 & y_2 \sim C_1 + \cdots \end{cases}$$

$r_1 - r_2$ is not zero nor a positive integer, then the solutions of [*] according to Fröbenius theorem are:

$$\sum_0^\infty a_n x^{n+3/2} \quad \text{with } a_0 \neq 0$$

and

$$\sum_0^\infty b_n x^n \text{ with } b_0 \neq 0.$$

The first one is non-analytic at $x=0$ (via $x^{3/2}$). The second one is analytic at $x=0$. Calculation of

$$y = \sum_0^\infty a_n x^{n+3/2},$$

$$y' = \sum_0^\infty a_n (n+3/2)\, x^{n+1/2},$$

$$y'' = \sum_0^\infty a_n (n+3/2)(n+1/2)\, x^{n-1/2}.$$

Equation (∗) is

$$0 = \sum_0^\infty 2(n+3/2)(n+1/2)\, a_n x^{n+1/2}$$

$$+ \sum_0^\infty 2(n+3/2)\, a_n x^{n+3/2}$$

$$- \sum_0^\infty (n+3/2)\, a_n x^{n+1/2} - \sum_0^\infty 5 a_n x^{n+3/2}$$

$$= \sum_0^\infty (n+3/2)(2n+\cancel{1}-\cancel{1})\, a_n x^{n+1/2} + \sum_0^\infty (2n-2)\, a_n x^{n+3/2}.$$

correction 0.50

Finally,

$$0 = \sum_0^\infty (2n+3)\, n\, a_n x^{n+1/2} + \sum_0^\infty 2(n-1)\, a_n x^{n+3/2}$$

$$= x^{1/2}[\underline{0}] + x^{3/2}[5\cdot 1\, a_1 - 2\cdot 1\cdot a_0]$$

$$+ x^{5/2}[7\cdot 2\, a_2 + 2\cdot \underline{0}\, a_1]$$

coincidence... ok...

$$+ x^{7/2}[9\cdot 3\, a_3 + 2\cdot 1\, a_2]$$

$$+ x^{9/2}[11\cdot 4\, a_4 + 2\cdot 2\, a_3]$$

0.50 desconto

$$+ x^{11/2}[13\cdot 5\, a_5 + 2\cdot 3\, a_4]$$

$$+ x^{13/2}[15\cdot 6\, a_6 + 2\cdot 4\, a_5]$$

$a_2 = a_3 = \dots$ $= 0$

$$+ \dots,$$

$(2n+5)(n+1)\, a_{n+1} + 2(n-1)\, a_n = 0$

$n = 0, 1, 2, \dots$

and then

$$a_1 = \frac{2a_0}{5}$$

$$a_2 = 0$$

$$a_3 = a_4 = a_5 = \dots = 0 \quad \underline{\text{todos}}$$

Solution: $$y = x^{3/2}\left(1+\frac{2}{5}x\right).$$

The other solution is $\underline{\text{analytic}}$ at $x=0$. } 0.25

0.25

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③ Separación de variables

$$\begin{cases} u_t - 4u_{xx} = 0 & , \ 0<x<\pi \ , t>0 \\ u(x,0) = 0 \\ u(0,t) = t \ , \ u_x(\pi,t) = 0 \end{cases}$$

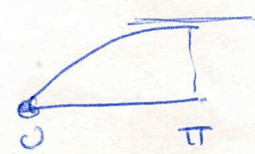
$$w = u-(ax+b) \quad ; \quad w_x = u_x - a$$

$$0 = t - b \quad \boxed{\begin{matrix} b=t \\ a=0 \end{matrix}}$$

change : $\boxed{w = u-t.}$ With this change

$$\begin{cases} w_t - 4w_{xx} = -1 \\ w(x,0) = 0 \\ w(0,t) = w_x(\pi,t) = 0 \end{cases}$$

Solving this boundary condition:



$$T = 4\pi$$

$$\omega = \frac{1}{2}$$

$$\left\{\sin\frac{n}{2}x\right\}, \ n=1,2,3,\dots \quad \text{but not all} \dots$$

$\sin\frac{x}{2}$: $\underline{\text{yes}}$

$\sin x$: $\underline{\text{no}}$

$\sin\frac{3x}{2}$: $\underline{\text{yes}}$

$$\boxed{X_n = \left\{\sin\frac{2n-1}{2}x\right\}, \ n=1,2,3,\dots}$$

Besides: $$\int_0^\pi dx \, \sin\frac{2n-1}{2}x \, \sin\frac{2m-1}{2}x = \frac{\pi}{2}\delta_{nm}$$

Thus:

$$\boxed{w = \sum_1^\infty T_n \sin\frac{2n-1}{2}x}.$$

T_n → depends on $\underline{t}$ only

Also

$$w_t - 4w_{xx} = \sum_1^\infty \left[T_n' + 4\left(\frac{2n-1}{2}\right)^2 T_n\right]\sin\frac{2n-1}{2}x$$

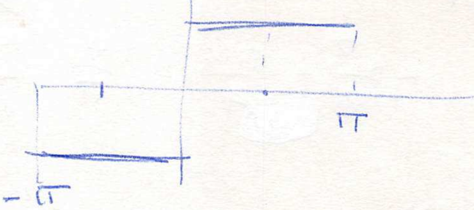
... copying last line...

$$w_t - 4w_{xx} = \sum_1^\infty \left[T_n' + 4\left(\frac{2n-1}{2}\right)^2 T_n\right] \sin \frac{2n-1}{2} x$$

$$= -1$$

$$= \sum_1^\infty -\frac{4}{\pi(2n-1)} \sin\frac{2n-1}{2} x$$

$$1 = \sum_1^\infty b_n \sin \frac{2n-1}{2} x$$



use $\int_0^\pi dx \sin\frac{2n-1}{2}x \sin\frac{2m-1}{2}x = \pi/2\, \delta_{mn}$

to obtain

$$b_m = \frac{2}{\pi}\int_0^\pi dx \sin\frac{2m-1}{2}x$$

$$= -\frac{2}{\pi}\frac{2}{2n-1}\left[\cos\frac{2m-1}{2}\right]_0^\pi$$

$$= \frac{2}{\pi}\frac{2}{2n-1}$$

$$= \frac{4}{\pi(2n-1)},$$

then

$$1 = \sum_1^\infty \frac{4}{\pi(2n-1)} \sin\frac{2n-1}{2}x$$

or

$$\begin{cases} T_n' + (2n-1)^2 T_n = -\dfrac{4}{\pi(2n-1)} \\ T_n(0) = 0 \end{cases}$$

constants are ok solutions of non-homog.

solution:

$$T_n = c_i e^{-(2n-1)^2 t} - \frac{4}{\pi}\frac{1}{(2n-1)^3},$$

$$T_n = \frac{4}{\pi(2n-1)^3}\left[e^{-(2n-1)^2 t} - 1\right]$$

Thus,

$$u = t - \frac{4}{\pi}\sum_{1}^{\infty} \frac{1}{(2n-1)^3}\left[1 - e^{-(2n-1)^2 t}\right] \sin\frac{2n-1}{2}x.$$

Note:

$$-\frac{x^2}{8} + \frac{\pi x}{4} = \frac{4}{\pi}\sum_{1}^{\infty} \frac{1}{(2n-1)^3} \sin\frac{(2n-1)x}{2}; \qquad 0 < x < \pi$$

Then

$$u = t + \underbrace{\frac{x^2}{8} - \frac{\pi x}{4}} + \frac{4}{\pi}\sum_{1}^{\infty} \frac{e^{-(2n-1)^2 t}}{(2n-1)^3} \sin\frac{(2n-1)x}{2}$$

Note that this part $\frac{x^2}{8} - \frac{\pi x}{4}$ could be obtained from

$$\begin{cases} w_t - 4w_{xx} = -1 \\ w(x,0) = 0 \\ w(0,t) = w_x(\pi,0) = 0 \end{cases}$$

with the change $w = \frac{x^2}{8} - \frac{\pi x}{4} + v$ that gives for v:

$$\begin{cases} v_t - v_{xx} = 0 \\ v(x,0) = -\frac{x^2}{8} + \frac{\pi x}{4} \\ v(0,t) = v_x(\pi,0) = 0 \end{cases}$$

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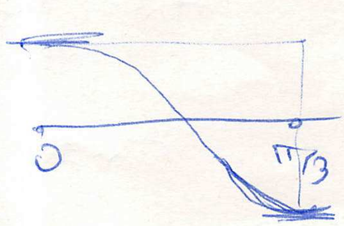
(4) $$\begin{cases} \Delta u = 0 \qquad r < 2, \quad \theta \in \left(0, \frac{\pi}{3}\right) \\ u_r(2,\theta) = a + \cos 3\theta \\ u_\theta(r,0) = u_\theta(r,\pi/3) = 0. \end{cases}$$

If u is solution of this problem, also $u + C_1$ is a solution (C_1 any constant).



$u = R \cdot \Theta$

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta'(0) = \Theta'(\pi/3) = 0 \end{cases}$$

Boundary problem solution:

$\Theta_0 = \{1\}$ $\lambda_0 = 0$.

$\Theta_n = \{\cos 3n\theta\}$, $n = 1, 2, 3, \dots$; $\lambda_n = 9n^2$

$$u = R_0 + \sum_1^\infty R_n \cos 3n\theta$$

with

$$R_0 = c_1 + c_2 \log r$$

$$R_n = c_3 r^{3n} + \frac{c_4}{r^{3n}}$$

Also

$$u_r = R_0' + \sum_1^\infty R_n' \cos 3n\theta \Big|_{r=2}$$

$$= a + \cos 3\theta,$$

so then

① $\begin{cases} R_0 = c_1 + c_2 \log r \\ R_0'(2) = a \end{cases}$, ② $\begin{cases} R_1 = c_3 r^3 + \frac{c_4}{r^3} \\ R_1'(2) = 1 \end{cases}$

③ $\begin{cases} \text{other } n\text{'s } (n \neq 0, 1) \\ R_n = c_5 r^{3n} + \frac{c_6}{r^{3n}} . \\ R_n'(2) = 0 \end{cases}$

solving these equations:

① $R_0'(r) = \frac{c_2}{r}$, $\frac{c_2}{2} = a$, $\boxed{c_2 = 2a}$ $\boxed{c_1 \text{ free}}$ $\Rightarrow \boxed{R_0 = c_1 + 2a \log r}$

② $R_1' = 3\left(c_3 r^2 - \frac{c_4}{r^4}\right)$

$R_1'(2) = 3\left(c_3 4 - \frac{c_4}{16}\right) = 1$, $\boxed{c_3 = \frac{1}{12} + \frac{c_4}{64}}$

$\Downarrow$

③ $R_n \equiv 0$ for all $n \geq 2$.

$\boxed{R_1 = \frac{r^3}{12} + c_4\left(\frac{r^3}{64} + \frac{1}{r^3}\right)}$

consequently,

$$u = c_1 + 2a \log r + \frac{r^3}{12}\cos 3\theta + c_4\left(\frac{r^3}{64} + \frac{1}{r^3}\right)\cos 3\theta.$$

For u to be bound at $r=0$, $\boxed{a=0}$

$\boxed{c_4=0}$

Solution:

$$\boxed{u = c_1 + \frac{r^3}{12}\cos 3\theta}$$

unique up to an additive constant.

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(5) $u_x + (2x-y)u_y = u^2$

$(u_x, u_y, -1)\cdot(1, 2x-y, u^2) = 0$

$dx = \frac{dy}{2x-y} = \frac{du}{u^2}$: characteristics equation: (two equations)

$\vdots$

$\left\{\begin{array}{l} \frac{dy}{dx} + y = 2x \quad \text{or} \quad \boxed{y = c_1 e^{-x} + 2x - 2} \\ dx = \frac{du}{u^2} \quad \text{or} \quad \boxed{x + \frac{1}{u} = c_2} \end{array}\right.$ 0.40 0.40

(or $u = \frac{1}{c_2 - x}$ or similar)

$\frac{dy}{dx} + y = 2x$: $y_{homog} = c_1 e^{-x}$

$y = a + bx$ (try this)

$y' = b$

$2x = a + b + bx$, $b = 2$ and $a = -2$:

$y_{part} = 2x - 2$ 0.25

inhomog.

general solution: $\boxed{x + \frac{1}{u} = f\left((y - 2(x-1))e^x\right)}$

$$x + \frac{1}{u} = f\left(\left(\eta - 2(x-1)\right)e^{x}\right)$$

i) $u(0,\eta) = \frac{1}{\eta}$

$\eta = f(\eta + 2)$ or $f(\eta) = \eta - 2$.

then

$$\boxed{x + \frac{1}{u} = \left[\eta - 2(x-1)\right]e^{x} - 2.}$$

0.35

no se simplifica nada, se deja así

ii) $u(x, 2(x-1)) = -\frac{1}{x}$

la sol. pedida se reduce a $0 = f(0)$

there are infinite f s.t. $f(0) = 0$.

0.35 con ejemplo(s)

Ex

$f \equiv 0$ $\qquad u = -\frac{1}{x}$

$f(x) = x$ $\qquad x + \frac{1}{u} = \left(\eta - 2(x-1)\right)e^{x}$

$f(x) = x \log x$, $\qquad x + \frac{1}{u} =$ muy fea ...

Repartido:

1er char	2nd char	general solution	i)	ii)
0.40	0.90	0.25	0.35	0.35