Examen de Junio de MM2, grupo C – 10/06/2019(10,11)

Nombre y Apellidos:

Firma y DNI:

Consejo muy útil: Siempre que sea **razonable** es conveniente comprobar los resultados.

1. [2 puntos] Sea la función $f(x) = \begin{cases} \cos\dfrac{\pi x}{2} & 0 \le x \le 1 \\ 0 & 1 \le x \le 2 \end{cases}$.

   a) Calcular su serie de Fourier en cosenos de período $T = 4$. b) Dibujar $f_{\text{ext}}(x)$ en $[-6, 6]$ y sobre ella la suma (aproximada) de la serie. c) Utilizando la identidad de Parseval, hallar la suma de
   $$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}.$$

2. [2 puntos] a) Sea la ecuación diferencial de Bessel
   $$x^2 y'' + xy' + \Big(x^2 - \frac{1}{9}\Big)y = 0$$
   considerada en un entorno de $x = 0$. Escribir los cuatro primeros términos **no nulos** de una solución que no esté acotada en el origen y la regla de recurrencia de sus coeficientes.

   b) Estudiar si el punto del infinito es regular, singular regular o irregular para la ecuación
   $$y'' + ay' + by = 0,$$
   donde a, b son constantes siendo por lo menos una de ellas diferente de cero.

3. [2.5 puntos] Resolver por separación de variables
   $$\begin{cases} \Delta u = 0, \quad 0 < x, y < \pi, \\ u(x,0) = x^2,\ u(x,\pi) = 0, \\ u_x(0,y) = u_x(\pi,y) = 0. \end{cases}$$

4. [2 puntos] Calcular por separación de variables la única solución acotada del problema del plano que cumple
   $$\begin{cases} \Delta u = \cos\dfrac{3\theta}{2}, \quad r < 1, \quad 0 < \theta < \pi/3 \\ u(1,\theta) = 0, \\ u_\theta(r,0) = u(r,\pi/3) = 0. \end{cases}$$
   Dibujar el recinto de integración.

5. [1.5 puntos] Calcular por el método de las características la solución general de la ecuación
   $$3x^2 u_y + u_x = x^5$$
   y una solución que satisfaga $u(x,0) = 2x^3$.

Una vez hechos los ejercicios rellene, por favor, los resultados que se piden:

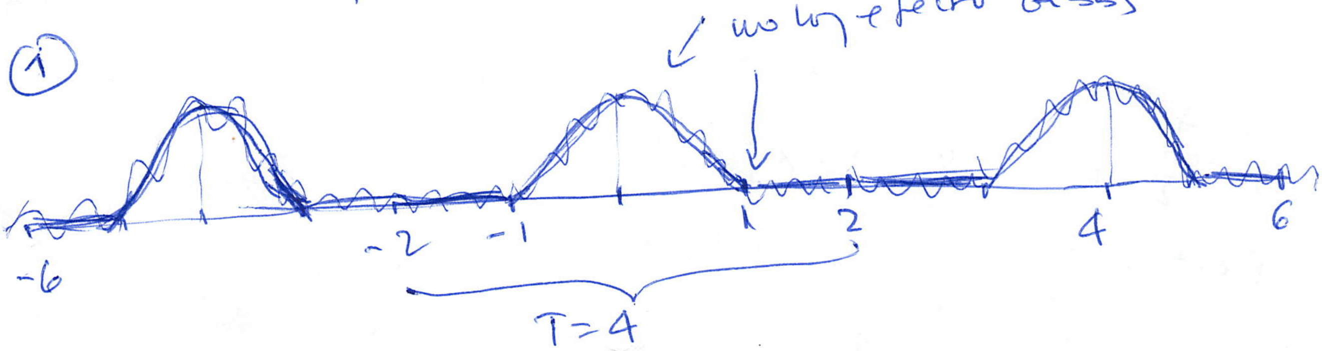
Prob 1:

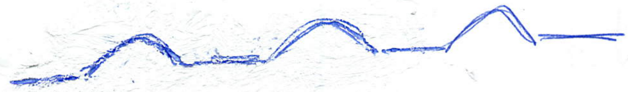
El dibujo del apartado b)

MM2 - Examen Final Ordinario - Junio 2019

He escrito primero las soluciones. Luego los cálculos:

(1)



$f_{ext}(x) =$ 

$$= \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi x}{2} + \frac{2}{\pi}\left(\frac{1}{2^2-1}\cos\pi x - \frac{1}{4^2-1}\cos 2\pi x + \frac{1}{6^2-1}\cos 3\pi x - \frac{1}{8^2-1}\cos 4\pi x + \dots\right)$$

Parseval says that

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

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(2) a) $y = \frac{1}{\sqrt[3]{x}}\left[1 - \frac{3}{8}x^2 + \frac{9}{320}x^4 - \frac{9}{10240}x^6 + \dots\right]$

b) No matter the values of a, b (not zero at the same time): the point $x = \infty$ is irregular.

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(3) $$u = \frac{\pi}{3}(\pi - y) + 4\sum_{n=1}^{\infty}(-1)^n \frac{\sinh n(\pi-y)\cos nx}{n^2 \sinh n\pi}$$

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(4) $$u = \frac{4}{7}\left(r^2 - r^{3/2}\right)\cos\frac{3\theta}{2}$$

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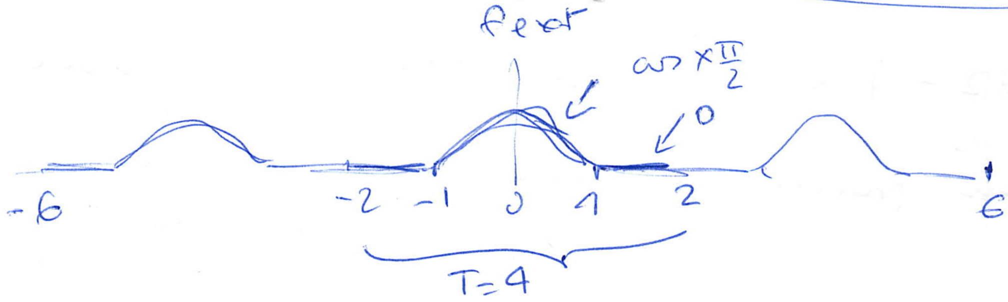
(5) Solución general: $u = \frac{x^6}{6} + f(x^3 - y)$

Solución particular: $u = 2x^3 - 2y - \frac{y^2}{6} + \frac{y x^3}{3}$.

(1)



$T = 4$

$\omega = \frac{\pi}{2}$ " $\{\cos n\frac{\pi}{2}x\}$, n

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos n\frac{\pi}{2}x \quad "$$

$$\frac{a_0}{2} = \frac{1}{4}\int_{-2}^{2} = \frac{1}{2}\int_0^2 = \frac{1}{2}\int_0^1 \cos x\frac{\pi}{2}$$

$$= \frac{1}{2}\,\frac{2}{\pi}\left[\sin\frac{\pi x}{2}\right]_0^1 = \frac{1}{\pi} \quad " \quad \boxed{\frac{a_0}{2} = \frac{1}{\pi}}$$

$$\frac{a_1}{2} = \frac{1}{4}\int_{-2}^{2} \cos\frac{\pi}{2}x = \frac{1}{2}\int_0^2 \cos\frac{\pi}{2}x$$

$$= \frac{1}{2}\int_0^1 \cos^2\frac{\pi x}{2} = \frac{1}{2}\int_0^1 dx\left(\frac{1}{2} + \underbrace{\frac{1}{2}\cos\pi x}_{0 \text{ the integral}}\right)$$

$$= \frac{1}{4} \qquad \boxed{a_1 = \frac{1}{2}}$$

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a_n for $n = 2, 3, 4, 5, \dots$

$$\frac{a_n}{2} = \frac{1}{2}\int_0^1 \cos\frac{\pi x}{2}\cos n\frac{\pi}{2}x$$

$$= \frac{1}{4}\int_0^1 dx\left[\cos\frac{\pi}{2}(n+1)x + \cos\frac{\pi}{2}(n-1)x\right]$$

$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$

$$= \frac{1}{4}\,\frac{2}{\pi}\left[\frac{1}{n+1}\sin(n+1)\frac{\pi}{2} + \frac{1}{n-1}\sin(n-1)\frac{\pi}{2}\right]$$

no problem because n is not 1

Observe that

$$\begin{cases} \sin(n+1)\frac{\pi}{2} = \sin n\frac{\pi}{2}\underbrace{\cos\frac{\pi}{2}}_{0} + \sin\frac{\pi}{2}\cos n\frac{\pi}{2} = \cos n\frac{\pi}{2} \\ \sin(n-1)\frac{\pi}{2} \quad 0 \quad - \sin\frac{\pi}{2}\cos n\frac{\pi}{2} = -\cos n\frac{\pi}{2} \end{cases}$$

$$= \frac{1}{4}\cdot\frac{2}{\pi}\cos n\frac{\pi}{2}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{4}\frac{2}{\pi}\cos\frac{n\pi}{2}\left[-\frac{2}{n^2-1}\right]$$

$$= -\frac{1}{\pi}\frac{1}{n^2-1}\cos\frac{n\pi}{2} \quad \text{"} \quad \boxed{a_n = -\frac{2}{\pi}\frac{1}{n^2-1}\cos\frac{n\pi}{2}}$$

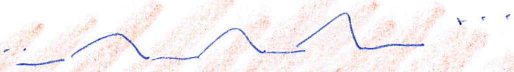
[a_1 can be derived from ↗ too: $a_1 = \frac{0}{0}$ or equal to

$$a_1 = -\frac{2}{\pi}\frac{\left[-\sin n\frac{\pi}{2}\right]}{2n}\cdot\frac{\pi}{2} \underset{n=1}{\nearrow} = \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\cancel{4}}\frac{1}{2} = \frac{1}{2}$$
]

since

$$\cos n\frac{\pi}{2} = \begin{cases} 0 & \text{if } n \text{ odd: } n = 1,3,5,7,\dots \\ -1 & \text{if } n = 2,6,10,14,\dots \\ 1 & \text{if } n = 4,8,12,\dots \end{cases}$$

we have that

$f_{ext} = \dots$  $\dots$

$$= \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi x}{2} + \frac{2}{\pi}\left(\frac{1}{2^2-1}\cos\pi x - \frac{1}{4^2-1}\cos 2\pi x + \frac{1}{6^2-1}\cos 3\pi x - \frac{1}{8^2-1}\cos 4\pi x + \dots\right)$$

Parseval

$$\int_{-2}^{2} f_{ext}^2 = \int_{-2}^{2}\left[\frac{1}{\pi^2} + \frac{1}{4}\cos^2\frac{\pi}{2}x + \frac{4}{\pi^2}\left(\frac{1}{(2^2-1)^2}\cos^2\pi x + \frac{1}{(4^2-1)^2}\cos^2 2\pi x + \frac{1}{(6^2-1)^2}\cos^2 3\pi x + \dots\right)\right.$$

We are integrating even functions around $x=0$, so the previous identity reduces to

$$\int_0^2 f(x)^2 = \int_0^2 dx\left(\frac{1}{\pi^2} + \frac{1}{4}\cos^2\frac{\pi x}{2} + \frac{4}{\pi^2}\left(\left(\frac{1}{2^2-1}\right)\cos^2\pi x + \cdots\right)\right),$$

calculating

$$\int_0^2 dx \frac{1}{\pi^2} = \frac{2}{\pi^2}$$

$$\int_0^2 dx \cos^2\frac{\pi x}{2} = \frac{1}{2}\int_0^2 dx(1+\underbrace{\cos \pi x}_{\text{0 the integral}}) = 1$$

$$\int_0^2 dx \cos^2 \pi x = \frac{1}{2}\int_0^2 dx\,(1+\underbrace{\cos 2\pi x}_{0}) = 1$$

$$\int_0^2 dx \cos^2 2\pi x = \int_0^2 dx \cos^2 3\pi x = \int_0^2 \cos^2 4\pi x = \cdots = 1$$

Besides,

$$\int_0^2 f(x)^2 = \int_0^1 \cos^2\frac{\pi x}{2} = \frac{1}{2},$$

Parseval reduces then to

$$\frac{1}{2} = \frac{2}{\pi^2} + \frac{1}{4} + \frac{4}{\pi^2}\left(\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \cdots\right)$$

this series is also $\sum_1^\infty \frac{1}{(4n^2-1)^2}$.

From here,

$$\sum_1^\infty \frac{1}{(4n^2-1)^2} = \left(\frac{1}{2} - \frac{1}{4} - \frac{2}{\pi^2}\right)\frac{\pi^2}{4}$$

$$= \frac{\pi^2}{16} - \frac{1}{2}.$$

no more to justify.

$$\frac{1}{T}\int_0^T f(x)^2 = \frac{1}{4}$$

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(2) a) $x^2 y'' + x y' + \left(x^2 - \frac{1}{9}\right) = 0$ [*]

Euler around $x=0$ $\quad x^2 y'' + x y' - \frac{1}{9} y = 0$

$$r(r-1) + r - \frac{1}{9} = r^2 - \frac{1}{9} = 0 \qquad r \begin{cases} 1/3 = r_1 \\ -1/3 = r_2 \end{cases}$$

Unbounded at $x=0$ is $r_2 = -1/3$

(wolog because $r_1 - r_2 = 2/3$ is not an integer)

$$y = \sum_0^\infty b_n x^{n-1/3},$$

$$y' = \sum_0^\infty b_n \left(n - \frac{1}{3}\right) x^{n-4/3},$$

$$y'' = \sum_0^\infty b_n \left(n - \frac{1}{3}\right)\left(n - \frac{4}{3}\right) x^{n-7/3}.$$

Inserted y, y', y'' in the equation [*] (Bessel equation),

$$0 = \sum_0^\infty \left(n - \frac{1}{3}\right)\left(n - \frac{4}{3}\right) b_n x^{n-1/3} + \sum_0^\infty \left(n - \frac{1}{3}\right) b_n x^{n-1/3} + \sum_0^\infty b_n x^{n+5/3} - \sum_0^\infty \frac{b_n}{9} x^{n-1/3}$$

$$= \sum_0^\infty n\left(n - \frac{2}{3}\right) b_n x^{n-1/3} + \sum_0^\infty b_n x^{n+5/3}$$

$$\left(n - \frac{1}{3}\right)\left(n - \frac{4}{3}\right) + \left(n - \frac{1}{3}\right) - \frac{1}{9} = \left(n - \frac{1}{3}\right)\left[n - \frac{4}{3} + 1\right] - \frac{1}{9}$$

$$= \left(n - \frac{1}{3}\right)^2 - \frac{1}{9} = n^2 - \frac{2n}{3} = n\left(n - \frac{2}{3}\right)$$

$$= x^{-1/3}\left[\sum_0^\infty \frac{n}{3}(3n-2) b_n x^n + \sum_0^\infty b_n x^{n+2}\right]$$

así quita a este "peso"

$$= x^{-1/3} \begin{cases} 0(-2) b_0 \\ + x\left[\frac{1}{3}(1) b_1\right] \\ + x^2\left[\frac{2}{3}(4) b_2 + b_0\right] \\ + x^3\left[\frac{3}{3}(7) b_3 + b_1\right] \\ + x^4\left[\frac{4}{3}(10) b_4 + b_2\right] \end{cases}$$

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$$+x^5\left[\frac{5}{3}(13)\,b_5 + b_3\right]$$
$$+x^6\left[\frac{6}{3}(16)\,b_6 + b_4\right]$$
$$+x^7\left[7/3\,(19)\,b_7 + b_5\right]$$
$$+x^8\left[\frac{8}{3}(22)\,b_8 + b_6\right] + \dots$$

$\boxed{b_0 = \text{free}}$

$b_1 = b_3 = b_5 = b_7 = \dots\ 0$

$\boxed{b_2 = -\frac{3b_0}{8}}$

$b_4 = -\frac{3}{40}b_2 = \frac{9}{320}b_0$; $\boxed{b_2 = \frac{9}{320}b_0}$

$$b_6 = -\frac{3b_4}{6\cdot 16} = -\frac{3\cdot 3\cdot 3}{6\cdot 16\cdot 320} = -\frac{3\cdot 3}{32\cdot 320} = -\frac{9}{10240}b_0$$

$\boxed{b_6 = -\frac{9}{10240}b_0}$

32
$\times 32$
64
96
1024

What gives,

$$\eta = \frac{1}{\sqrt[3]{x}}\left[\overbrace{1 - \frac{3}{8}x^2 + \frac{9}{320}x^4 - \frac{9}{10240}x^6}^{\text{four terms is up to here.}} + \frac{27}{1802240}x^8 \cancel{\dots}\right]$$

(diverges at $x \to 0$ as $\frac{1}{x^{1/3}}$ (expected).

Also

$$\boxed{b_n = -\frac{3\,b_{n-2}}{n(3n-2)}\ ,\quad n = 2,3,4,\dots \qquad b_0 = \text{free} \qquad b_1 = 0}$$

b) We write the equation in the variable s given by $s = \frac{1}{x}$ (point at infinity)

$$\frac{d\eta}{dx} = \frac{d\eta}{ds}\cdot\frac{ds}{dx} = -\frac{1}{x^2}\dot{\eta} = -s^2\dot{\eta}$$

$$\frac{d^2\eta}{dx^2} = \frac{d}{ds}(\quad)\frac{ds}{dx} = \left[-2s\dot{\eta} - s^2\ddot{\eta}\right]\left[-s^2\right]$$

$\dot{} \equiv \frac{d}{ds}$; $' \equiv \frac{d}{dx}$.

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the differential equation with variable s is

$$s^4 \ddot{\eta} + 2s^3 \dot{\eta} + a[-s^2 \dot{\eta}] + b\eta = 0$$

or

$$s^2 \ddot{\eta} + (2s - a)\dot{\eta} + \frac{b}{s^2}\eta = 0$$

or

$$\ddot{\eta} + \left(\frac{2}{s} - \frac{a}{s^2}\right)\dot{\eta} + \frac{b}{s^4}\eta = 0$$

pole of order 2 at $s=0$ pole of order 4

only if $a=b=0$ (not a case of our problem) the point $s=0$ (or $x=\infty$) is a regular point! In our case it is an irregular point

(Esto es un teorema de tiempos de "la Chelito")

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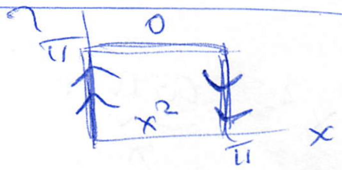
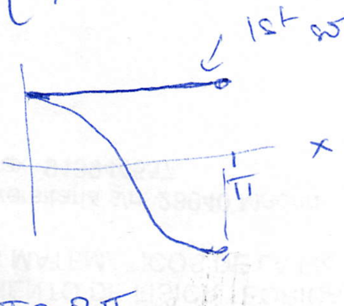
(3) $\begin{cases} \Delta u = 0 \\ u(x,0) = x^2, \; u(x,\pi) = 0 \\ u_x(0,\eta) = u_x(\pi,\eta) = 0 \end{cases}$ ← boundary conditions



$u = XY$

$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$; $\begin{cases} Y'' - \lambda Y = 0 \\ Y(\pi) = 0 \end{cases}$



$T = 2\pi$
$a = 1$
$X_n = \{\cos nx\}$, $n = 1,2,3,\ldots$; $X_0 = \{1\}$
$\lambda_n = n^2$ $\quad$ $\lambda_0 = 0$

Orthogonal relations:

$n = 1,2,3,\ldots$ $\quad \int_0^{\pi} \underset{1}{X_0} \underset{\cos nx}{X_n} = 0$, $\quad \int_0^{\pi} \overset{\cos nx}{X_n} \overset{\cos mx}{X_m} = \frac{\pi}{2}\delta_{nm}$

$$\int_0^{\pi} X_0^2 = \int_0^{\pi} 1\,dx = \pi$$

$$\begin{cases} Y_n'' - \lambda_n Y_n = 0 \\ Y(\pi) = 0 \end{cases} \qquad " \begin{cases} Y'' = 0 \\ Y(\pi) = 0 \end{cases}$$

$$\lambda_n = n^2 \qquad \lambda_0 = 0$$

$$Y_n = \{\sinh n(\pi - x)\} \qquad Y_0 = \pi - y$$

$$n = 1, 2, 3, \dots$$

Structure

$$"U = X_0 Y_0 + \sum_{n=1}^{\infty} X_n Y_n",$$

with the correct constants to be determined

$$U = c_0(\pi - y) + \sum_{n=1}^{\infty} c_n \sinh n(\pi - y) \cos nx$$

where c_0, c_n are found with the condition $U(x,0) = x^2$,

$$x^2 = \underbrace{c_0 \pi}_{\text{call this } \frac{a_0}{2} \text{ (or not, up to you)}} + \sum_{n=1}^{\infty} \underbrace{c_n \sinh n\pi}_{\text{call this } a_n \text{ (as in lectures)}} \cos nx$$

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3} = c_0 \pi, \qquad \boxed{c_0 = \frac{\pi}{3}}$$

$$a_n = \frac{2}{\pi} \underbrace{\int_0^{\pi} x^2 \cos nx\,dx}_{= \frac{2\pi}{n^2}\cos n\pi = (-1)^n \frac{2\pi}{n^2}} = \frac{4(-1)^n}{n^2},$$

$n = 1, 2, 3, \dots$

$$\boxed{c_n = \frac{4(-1)^n}{n^2 \sinh n\pi}}$$

$$\int x^2 \cos bx = \frac{x^2}{b}\sin bx + \frac{2x}{b}\cos bx - \frac{2}{b^3}\sin bx$$

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The solution of the problem is

$$\boxed{u = \frac{\pi}{3}(\pi - y) + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\,\frac{\sinh n(\pi - y)}{\sinh n\pi}\cos nx}$$

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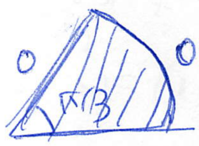
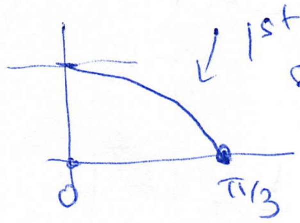
(4)
$$\begin{cases} \Delta u = \cos \frac{3\theta}{2} \\ u(1,\theta) = 0 \\ u_\theta(r,0) = u(r, \frac{\pi}{3}) = 0 \end{cases}$$



$u = R\Theta$

R later...

$$\begin{cases} \Theta'' + \lambda\Theta \\ \Theta'(0) = \Theta(\frac{\pi}{3}) = 0 \end{cases} \quad ''$$



↙ 1st function that satisfies $\Theta'(0) = \Theta(\frac{\pi}{3}) = 0$

$T = \frac{4\pi}{3}$, $\omega = \frac{3}{2}$ '' $\{\cos \frac{3n}{2}\theta\}$ but not all...

$n=1$ $\cos\frac{3\theta}{2}$: yes

$n=2$ $\cos 3\theta$: no ($\cos\pi = -1$ and not 0)

$n=3$ $\cos\frac{9\theta}{2}$ yes

$n=4$ $\cos 6\theta$ no

⋮

Then $\Theta_n = \left\{\cos\frac{3}{2}(2n-1)\theta\right\}$ $n = 1,2,3,\ldots$

$$\lambda_n = \left[\frac{3}{2}(2n-1)\right]^2$$

$\boxed{\lambda = 0 \text{ is not an eigenvalue}}$

$$u = \sum_{1}^{\infty} R_n(r)\cos\frac{3}{2}(2n-1)\theta$$

$$\Delta u = \sum_{1}^{\infty}\left[R_n'' + \frac{R_n'}{r} - \frac{9}{4}(2n-1)^2\frac{R_n}{r^2}\right]\cos\frac{3}{2}(2n-1)\theta$$

That $\Delta u = \cos\frac{3\theta}{2}$ indicates that

Prob 1 $\begin{cases} R_1'' + \frac{R_1'}{r} - \frac{9}{4}\frac{R_1}{r^2} = 1 \\ R_1(1) = 0 \end{cases}$

Prob 2. $\begin{cases} R_n'' + \frac{R_n'}{r} - \frac{9}{4}(2n-1)^2\frac{R_n}{r^2} = 0 \\ R_n(1) = 0 \end{cases}$ $\quad n = 2, 3, 4, \ldots$

(we have added to each problem the condition $u(1,\theta)=0$,, $R_1(1)=0$ and $R_n(1)=0$, $n=2,3,\ldots$

We solve problem 2 first

$$R_n = C_1 r^{\frac{3}{2}(2n-1)} + \frac{C_2}{r^{\frac{3}{2}(2n-1)}}$$

with $C_1 = -C_2$,, that is

$$R_n = C_1\left[r^{\frac{3}{2}(2n-1)} - \frac{1}{r^{3/2(2n-1)}}\right]$$

or

$$R_2 = C_1\left[r^{9/2} - \frac{1}{r^{9/2}}\right] \to \text{not bounded at } r \to 0$$

$$R_3 = C_1\left[r^{15/2} - \frac{1}{r^{15/2}}\right] \to \text{not bounded at } r \to 0$$

$\vdots$

$R_2 = R_3 = \cdots = R_{700} = 0$ (todas)

so $C_1 = 0$ (a menos) in each solution.

(this reduces u to $f(r)\cos\frac{3\theta}{2}$ simply),

Now solve problem 1

$$R_1 = C_3 r^{3/2} + \frac{C_4}{r^{3/2}} + \underbrace{\frac{4}{7}r^2}_{\text{particular solution of (a tanteo sale)}}$$

The condition $R_1(1)=0$ is

$$R_1 = -\frac{4}{7}r^{3/2} + C_4\left(-r^{3/2} + \frac{1}{r^{3/2}}\right) + \frac{4}{7}r^2$$

that is bounded at $r=0$ only if $C_4 = 0$.

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We end up with the solution

$$\boxed{u = \frac{4}{7}\left(r^2 - r^{3/2}\right)\cos\frac{3\theta}{2}}.$$

It is the unique solution of the problem.

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(5) $3x^2 u_y + u_x = x^5$

$(u_x, u_y, -1)\cdot(1, 3x^2, x^5) = 0$

$$\boxed{\frac{dx}{1}} = \boxed{\frac{dy}{3x^2}} = \boxed{\frac{du}{x^5}}$$: equations of the characteristics (two)

1st characteristic : $\left[\frac{dy}{dx} = 3x^2\right]$:

$$\boxed{x^3 - y = c_1}, \quad c_1 \text{ a constant}$$

second: $\frac{du}{dx} = x^5$,

$$\boxed{u = \frac{x^6}{6} + c_2}, \quad c_2 \text{ a constant}$$

General solution $\boxed{u = \frac{x^6}{6} + f(x^3 - y)}$,

f an arbitrary function $\in C^1(\Omega)$.

$u(x,0) = 2x^3$

$2x^3 = \frac{x^6}{6} + f(x^3)$ is

$2w = \frac{w^2}{6} + f(w)$, $\quad f(w) = 2w - \frac{w^2}{6}$

then,

$$u = \frac{x^6}{6} + 2(x^3 - y) - \frac{1}{6}(x^3 - y)^2$$

$$= \cancel{\frac{x^6}{6}} + 2x^3 - 2y - \frac{1}{6}\left(\cancel{x^6} + y^2 - 2yx^3\right)$$

$$= 2x^3 - 2y - \frac{y^2}{6} + y\frac{x^3}{3}$$

$$\boxed{u = 2x^3 - 2y - \frac{y^2}{6} + y\frac{x^3}{3}}$$

that satisfies $u(x,0) = 2x^3$ and the equation $3x^2 u_y + u_x = x^5$.