UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

11 de Septiembre de 2013(3)

Examen de Métodos Matemáticos II – Grupo D

Nombre y Apellidos:

DNI y firma:

Consejo muy útil: Siempre que sea **razonable** es conveniente comprobar los resultados. Respuestas totalmente correctas sin la debida justificación no recibirán consideración alguna.

1. [1.8 puntos] Sea la ecuación
$$y'' + 2xy' + 2y = 0.$$
a) Encontrar los **cuatro** primeros términos no nulos de una solución en forma de serie que cumpla $y(0) = 1, y'(0) = 0$

b) Hallar el término general de la serie e identificarla con una función elemental.

2. [1.8 puntos] El desarrollo en serie de cosenos de período 4 de $f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & 1 < x < 2 \end{cases}$ es
$$\frac{1}{2} + a\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \dots\right)$$
donde a es una constante. a) Hacer un dibujo de $f(x)$ extendida periodicamente (pinte, por favor, al menos tres períodos de la extensión).

b) Calcular el valor de a.

c) Utilizando la *identidad de Parseval* y el valor de a del apartado anterior calcular la suma de la serie
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

3. [2 puntos] Calcular la única solución del plano que satisface
$$\begin{cases} \Delta u = 0, \quad 1 < r < 2, \\ u(1,\theta) = 1 + \sin 2\theta \\ u_r(2,\theta) = 0. \end{cases}$$
Dibujar el recinto de integración.

4. x [2.4 puntos] Resolver por separación de variables el problema no homogéneo
$$\begin{cases} u_t - u_{xx} = e^{-2t}, \quad 0 < x < \pi, \quad t > 0 \\ u(x,0) = 0, \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

5. [2 puntos] Sea la ecuación
$$(3y+3)u_y - xu_x = 2xy.$$
i) Determinar la solución general por el método de las características. ii) Hallar la solución que cumple el dato $u(x,0) = 0$ y **dos** soluciones sencillas que cumplan $u(0,y) = 0$.

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① Sea la ecuación Puntuación: 1.8

$$y'' + 2xy' + 2y = 0$$

with the conditions $y(0)=1$, $y'(0)=0$

The ecuation is studied at $x=0$ because $y(\underline{0})=1$, $y'(\underline{0})=0$. The point $x=0$ is a regular point. Around $x=0$, the solution of [E] is given by $y=\sum_0^\infty a_n x^n$. In our case $a_0=1$, $a_1=0$, according to the initial conditions. The series solution has $R=\infty$ because $P=2x$, $Q=2$ are convergent series for all $x \in \mathbb{R}$ [all polynomials are]

0.5

$$y=\sum_0^\infty a_n x^n, \quad y'=\sum_0^\infty n a_n x^{n-1}, \quad y''=\sum_0^\infty n(n-1)a_n x^{n-2}.$$

$$0 = y'' + 2xy' + 2y$$
$$= \sum_0^\infty n(n-1)a_n x^{n-2} + \sum_0^\infty 2(n+1)a_n x^n$$
$$= x^{-2}[\underline{0}] + x^{-1}[\underline{0}]$$
$$+ x^0[2\cdot\underline{1}a_2 + 2\cdot\underline{1}a_0]$$
$$+ x[3\cdot\underline{2}a_3 + 2\cdot\underline{2}a_1]$$
$$+ x^2[4\cdot\underline{3}a_4 + 2\cdot\underline{3}a_2]$$
$$+ x^3[5\cdot\underline{4}a_5 + 2\cdot\underline{4}a_3]$$
$$+ x^4[6\cdot\underline{5}a_6 + 2\cdot\underline{5}a_4]$$
$$+ x^5[7\cdot\underline{6}a_7 + 2\cdot\underline{6}a_5]$$
$$\vdots$$
$$+ x^n(n+1)[(n+2)a_{n+2} + 2a_n] + \cdots$$

then $a_{n+2} = -\dfrac{2a_n}{n+2}$, $n=0,1,2,\ldots$

0.5

obviously,

$$a_2 = -a_0$$

$$a_3 = -\frac{2}{3} a_1$$

$$a_4 = -\frac{2a_2}{4} = \frac{a_0}{2}$$

$$a_5 = -\frac{2a_3}{5}$$

$$a_6 = -\frac{2a_4}{6} = -\frac{a_0}{3\cdot 2}$$

$$a_7 = \cdots$$

$$a_8 = -\frac{2a_6}{8} = \frac{a_0}{4\cdot 3\cdot 2}$$

0.5 $$\boxed{a_{2n} = (-1)^n \frac{a_0}{n!}}$$

a_{par}...

[los impares no interesan]

$n = 1, 2, 3, \ldots$

The solution $[a_0 = 1,\ a_1 = 0]$ is

$$y = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3\cdot 2} + \frac{x^8}{4\cdot 3\cdot 2} - \cdots,$$

[Correcto!!] 0.3

which certainly is: e^{-x^2}. $\boxed{y = e^{-x^2}}$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} = \cdots$$

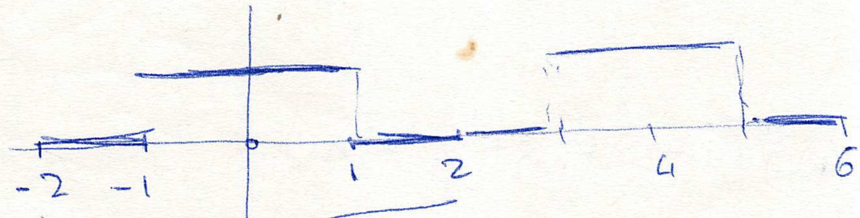
The other solution is $e^{-x^2}\int_0^x ds\, e^{s^2}$

$$= x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots \quad \text{[odd]}$$

[checking: $y = e^{-x^2}$, $y' = e^{-x^2}[-2x]$, $y'' = e^{-x^2}[4x^2-2]$: $y'' + 2xy' + 2y = 0$!! ok

Puntuación: 1.8

(2)

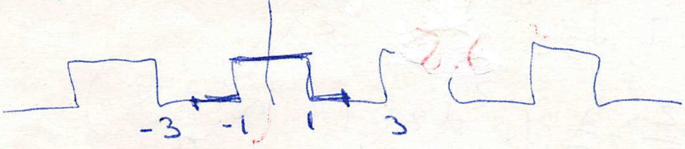


0.3 dibujo

$T = 4$, $\omega = \frac{\pi}{2}$, $\{\cos n\frac{\pi}{2}x\}$.

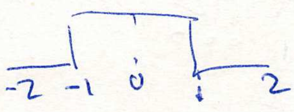
②

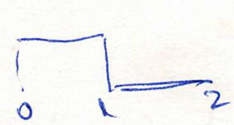
In fact



$$= \frac{1}{2} + a\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \cdots\right)$$

a)

$$\frac{a_n}{2} = \frac{1}{T}\int_{-2}^{2}$$  $$\cos\frac{n\pi}{2}x$$

$$= \frac{2}{T}\int_0^2$$  $$\cos\frac{n\pi x}{2}$$

$$= \frac{2}{T}\int_0^1 \cos\frac{n\pi x}{2}$$

In particular, a is a_1

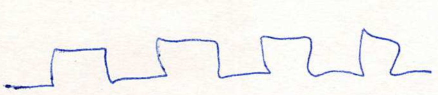
$$a = a_1 = \frac{4}{T}\int_0^1 \cos\frac{\pi x}{2}, \quad T=4$$

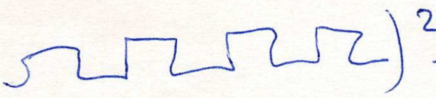
$$= \frac{2}{\pi}\left[\sin\frac{\pi x}{2}\right]_0^1$$

$$= \frac{2}{\pi}.$$

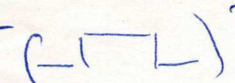
$$\boxed{a = \frac{2}{\pi}}$$ 0.75

b)

 $$= \frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \cdots\right)$$

$\Big($  $\Big)^2$ $$= \frac{1}{4} + \frac{4}{\pi^2}\left(\cos^2\frac{\pi x}{2} + \frac{1}{3^2}\cos^2\frac{3\pi x}{2} + \frac{1}{5^2}\cos^2\frac{5\pi x}{2} + \cdots\right)$$

+ double products.

$$\frac{1}{T}\int_{-2}^{2}\Big($$  $$\Big)^2 = \frac{2}{T}\int_0^2\Big($$  $$\Big)^2$$

$$= \frac{2}{T}\int_0^1 dx$$

$$= \frac{1}{2}.$$

$$\frac{1}{T}\int_{-2}^{2}\frac{1}{4}\,dx = \frac{1}{4}.$$

0.75 valor de la serie.
[menos puntuación si se aplica directamente la fórmula - si se hace -]?

de cualquier manera
(la integral es la misma)

$$\frac{1}{T}\int_{-2}^{2} dx \cos^2\frac{\pi x}{2} = \frac{2}{T}\int_0^2 dx \cos^2\frac{\pi x}{2}$$

$$= \frac{2}{T}\cdot\frac{1}{2}\int_0^2 dx\left(1+\cos\pi x\right)$$

0 the integral

$$= \frac{2}{T}\cdot\frac{1}{2}\cdot 2$$

$$= \frac{1}{2}$$

In summary

$$\frac{1}{2} = \frac{1}{4} + \frac{4}{\pi^2}\cdot\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

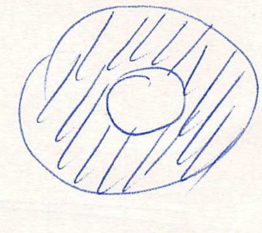
$$\frac{1}{4} = \frac{2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

or

$$\boxed{1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}}$$

Calculate the unique solution that = the plane sehispre:

Puntuación 2



$$\begin{cases} \Delta u = 0, & 1 < r < 2 \\ u(1,\theta) = 1 + \sin 2\theta, & u_r(2,\theta) = 0 \end{cases}$$

La solución más general de la ecuac de Laplace con cond de contorno periódicas en el plano es:

$$u = c_0 + d_0 \log r + \sum_1^\infty \left(a_n r^n + \frac{c_n}{r^n}\right) \cos n\theta + \sum_1^\infty \left(b_n r^n + \frac{d_n}{r^n}\right) \sin n\theta .$$

0.5

Inspired by the boundary conditions we try a solution of the form (we will try something else if it does not work)

0.5

$$\begin{cases} u = c_0 + d_0 \log r + \left(a_2 r^2 + \frac{c_2}{r^2}\right) \sin 2\theta . \\ u_r = \frac{d_0}{r} + 2\left(a_2 r - \frac{c_2}{r^3}\right) \sin 2\theta \end{cases}$$

Impose the conditions

0.5 cond [illegible]

$$\begin{cases} 1 + \sin 2\theta = c_0 + (a_2 + c_2) \sin 2\theta \\ 0 = \frac{d_0}{2} + 2\left(2a_2 - \frac{c_2}{8}\right) \sin 2\theta \end{cases}$$

$$\boxed{c_0 = 1}, \boxed{d_0 = 0}$$

$$\Rightarrow \left.\begin{matrix} a_2 + c_2 = 1 \\ 2a_2 - \frac{c_2}{8} = 0 \end{matrix}\right\} \text{solut: } \boxed{a_2 = \frac{1}{17}}, \boxed{c_2 = \frac{16}{17}}$$

Thus:

$$\boxed{u = 1 + \frac{1}{17}\left(r^2 + \frac{16}{r^2}\right) \sin 2\theta}$$

0.5 resolución.

[This is the solution of the problem. It is unique]

④ Solve the problem

$$\begin{cases} u_t - u_{xx} = e^{-2t}, & x \in (0,\pi),\ t>0 \\ u(x,0)=0 \\ u(0,t)=u(\pi,t)=0 \end{cases}$$

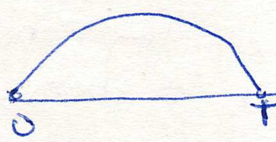
Puntuación 2.4

Take the homogeneous equation to obtain the boundary problem:

$u = XT$, $u_t - u_{xx} = 0$ is $\frac{T'}{T} = \frac{X''}{X}$:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$

0.5



$T = 2\pi$

$\omega = 1$

$\{\sin nx\}$, $n = 1, 2, \dots$

The solution has the structure

$$u = \sum_1^\infty T_n \sin nx$$

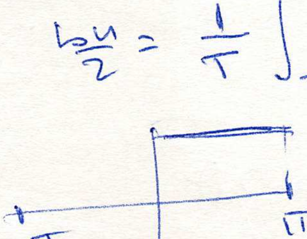
with T_n a function of t to be determined.

$$u_t - u_{xx} = \sum_1^\infty \left(T_n' + n^2 T_n\right) \sin nx$$
$$= e^{-2t} \cdot 1$$

$$1 = \sum_1^\infty b_n \sin nx.$$ Determination of b_n:

1 desarrollo seno

$$\frac{b_n}{2} = \frac{1}{T}\int_{-\pi}^{\pi} \sin nx$$



$T=2\pi$

$$= \frac{2}{T}\int_0^\pi \sin nx$$
$$= -\frac{1}{\pi n}\left[\cos nx\right]_0^\pi$$
$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$$

0 n even; $\frac{2}{n\pi}$, n odd

$$b_n \begin{cases} 0, & n \text{ even} \\ \frac{4}{\pi n}, & n \text{ odd} \end{cases}$$

(cont)

$$u_t - u_{xx} = \sum_1^\infty \left(T_n' + n^2 T_n\right) \sin nx$$

$$= \begin{cases} 0 & , n \text{ even} \\ \dfrac{4e^{-2t}}{\pi n} & , n \text{ odd} \end{cases} ,$$

what implies

0.25

$$\begin{cases} T_n' + n^2 T_n = 0 \\ T_n(0) = 0 \end{cases} \Rightarrow T_n(t) \equiv 0 , \; n \text{ even}$$

$$\begin{cases} T_n' + n^2 T_n = \dfrac{4e^{-2t}}{\pi n} & , n \text{ odd} \\ T_n(0) = 0 \end{cases}$$

$T_n = C_1 e^{-n^2 t}$ (homog)

$T_n = A e^{-bt}$ (non-homog) [trying...]

$T_n' = A e^{-bt} [-b]$

$T_n' + n^2 T_n = A e^{-bt} [-b + n^2] = \dfrac{4}{n\pi} e^{-2t}$

$b = 2$

$A(n^2 - 2) = \dfrac{4}{n\pi}$ // $A = \dfrac{4}{n\pi(n^2-2)}$

$$T_n = C_1 e^{-n^2 t} + \frac{4e^{-2t}}{\pi n(n^2-2)}$$

0.75

with C_1 s.t. $T_n(0) = 0$. Thus

$$T_n(t) = \frac{4}{\pi n(n^2-2)}\left[e^{-2t} - e^{-n^2 t}\right], \; \underline{n \text{ odd}}$$

Converter... añade plis... $n=0$ problems? $\underline{\underline{\text{no}}}$

$n = \sqrt{2}$ problems? $\underline{\underline{\text{no}}}$

Solution:

$$\boxed{u = \frac{4}{\pi} \sum_1^\infty \frac{1}{(2n-1)\left[(2n-1)^2 - 2\right]}\left[e^{-2t} - e^{-(2n-1)^2 t}\right] \sin(2n-1)x}$$

[Distinto es el caso si en lugar de e^{-2t} el enunciado tuviera e^{-t}].

(5) $-xu_x + 3(y+1)u_y = 2xy$

Puntuación 2

i) $u(x,0)=0$

ii) $u(0,y)=0$ [2 solution's]

$(-x, 3(y+1), 2xy)(u_x, u_y, -1) = 0$

$$-\frac{dx}{x} = \frac{dy}{3(y+1)} = \frac{du}{2xy}$$

0.5

• $\frac{dy}{dx} + \frac{3y}{x} = -\frac{3}{x}$ → la const -1 es sol. de la homogénea

solution: $\boxed{y = \frac{c_1}{x^3} - 1}$

1st characteristic

•• $-dx = \frac{du}{2y}$

$-2y\,dx = du$ or $\left(-\frac{2c_1}{x^3} + 2\right)dx = du$

solution: $\boxed{\frac{c_1}{x^2} + 2x + c_2 = u}$ 2nd characteristics

0.5

or $\boxed{yx + 3x + c_2 = u}$

General solution:

$$\boxed{u = yx + 3x + f((y+1)x^3)}$$

0.25

i) $u(x,0)=0$

$0 = 3x + f(x^3)$ // $f(x^3) = -3x$ // $f(x) = -3x^{1/3}$

0.25

$$\boxed{u = x\left[y + 3 - 3(y+1)^{1/3}\right]}$$

ii) $u(0,y)=0$ // $0 = f(0)$.

0.25

Take $f(x) \equiv 0$, $\boxed{u = x(y+3)}$

Take $f(x) = -3x^{1/3}$ $\boxed{u = x\left[y + 3 - 3(y+1)^{1/3}\right]}$, 0.25

that is solution too!!! [curioso...]