UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

8 de Septiembre de 2015

Examen de Métodos Matemáticos II – Grupo C

Nombre y Apellidos:

DNI y firma:

Consejo muy útil: Siempre que sea **razonable** es conveniente comprobar los resultados.

1. [1.5 puntos] El dibujo de la pizarra corresponde a una extensión periódica de $f(x) = x^2$ en $(0, 1)$. a) ¿Cuál es el período de la extensión? b) Indicar si la serie trigonométrica correspondiente es en senos, en cosenos o en senos y cosenos, y los argumentos de estos. c) Calcular mediante la identidad de Parseval $\sum_1^\infty (a_n^2 + b_n^2)$.

2. [2 puntos] Considérese la ecuación diferencial

$$2xy'' + (1 - 2x)y' + 3y = 0$$

en un entorno de $x = 0$. a) Decir de qué tipo es este punto (si regular, singular, singular regular o irregular); b) escribir el polinomio indicial en $x = 0$ y calcular sus raíces; c) encontrar una solución no trivial que satisfaga $y(0) = 0$.

3. [2.5 puntos] Encontrar por separación de variables la función $u(x, t)$ que satisface las condiciones

$$\begin{cases} u_t - u_{xx} + 2tu = 0, \quad 0 < x < 1/2, \ t > 0 \\ u(x, 0) = 1 - 2x, \\ u(0, t) = u_x(1/2, t) = 0. \end{cases}$$

4. [2 puntos] Calcular por separación de variables la solución del problema del plano

$$\begin{cases} \Delta u = 0, \quad 1 < r < 3, \quad 0 < \theta < \pi \\ u(1, \theta) = \sin 3\theta, \ u(3, \theta) = 0, \\ u(r, 0) = u(r, \pi) = 0 \end{cases}$$

Dibujar el recinto de integración.

5. [2 puntos] Calcular por el método de las caraterísticas la solución general de

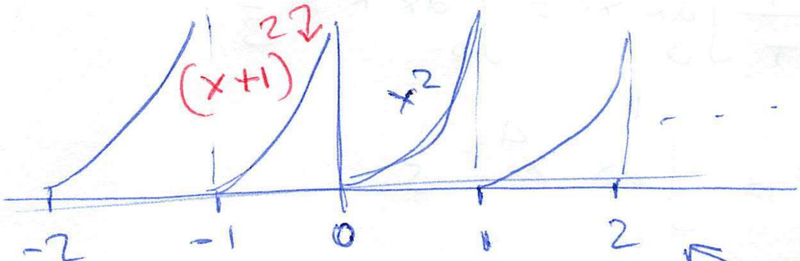
$$xu_x + (xy + e^x)u_y = u$$

y las soluciones particulares que cumplen los datos: i) $u(1, y) = 1/y$, ii) $u(x, -1) = 2x$. No hace falta que discuta unicidad.

(1)

MM2 - septiembre 2015

OJO! Ya no es x^2.



no paridad around $x=0$. Then

$T=1$

$\boxed{T=1}$, $\omega=2\pi$, $\{\cos 2\pi nx, \sin 2\pi nx\}$ $n=1,2,3,\ldots$

Thus $f_{ext}=$  $= \frac{a_0}{2}+\sum_1^\infty \left[a_n\cos 2\pi nx + b_n \sin 2\pi nx\right]$

Nota: He observado a raíz del examen que muchos alumnos (masculino de cortesía) ignoran que argumento quiere decir ángulo, no quiere decir coeficientes. Ex:

$a_n \cos 2\pi nx$

coeficiente. argumento

¿Cómo me iba a imaginar esto?

Identidad de Parseval famoso en el mundo entero. Esto en alguna mecánica.

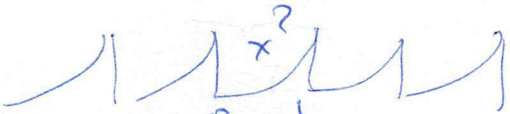
$$\frac{1}{T}\int_{\substack{\text{extendido}\\ \text{a un periodo}}} f_{ext}^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_1^\infty (a_n^2+b_n^2)$$

In our case:

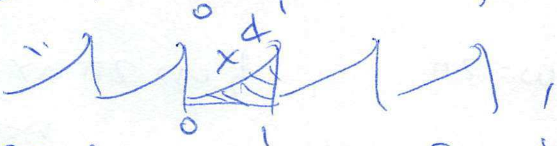
$$\frac{a_0}{2} = \frac{1}{T}\int_0^1 dx\, x^2 = \int_0^1 dx\, x^2 = \frac{1}{3}.$$

$$\int_T f_{ext}^2 = \int_0^1 dx\, x^4 = \frac{1}{5}$$

Γ f_{ext} es



f_{ext}^2 es



en un periodo es lo rayado. Ese área.

Using Parseval:

$$\sum_1^\infty (a_n^2+b_n^2) = 2\left(\frac{1}{5}-\frac{1}{9}\right) = \boxed{\frac{8}{45}}$$

Fin del ejercicio 0 0 0

√ No hace falta, pero cierto. Es pa completar. Ustedes no tienen que hacerlo en el examen

$$\frac{a_n}{2} = \frac{1}{T}\int_0^T dx\, x^2 \cos 2\pi n x = \frac{1}{2n^2\pi^2}$$

$$\frac{b_n}{2} = \frac{1}{T}\int_0^T dx\, x^2 \sin 2\pi n x = -\frac{1}{2n\pi}$$: worse convergence then

$$\sum_1^\infty (a_n^2+b_n^2) = \sum_1^\infty\left(\frac{1}{n^4\pi^4}+\frac{1}{n^2\pi^2}\right) = \frac{1}{\pi^4}\cdot\frac{\pi^4}{90} + \frac{1}{\pi^2}\cdot\frac{\pi^2}{6}$$

sacado de mis notes, libros, apuntes... whatever

$$= \frac{1}{90}+\frac{1}{6}$$

$$= \frac{1+15}{90} = \frac{16}{90} = \frac{8}{45}$$

ok, ok, ok,

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(2) $2xy'' + (1-2x)y' + 3y = 0$ [*]

$x=0$ is a singular point of this equation.

Euler at $x=0$ is

$$2x^2y'' + xy' = 0,$$

i) what makes $x=0$ a regular singular point

$$\left[\begin{array}{l} 2x^2y'' + x[1-2x]y' + 3[x]y = 0 \quad [*] \\ 2x^2y'' + x[1]y' + 0 = 0 \quad \text{Euler.} \end{array}\right.$$

ii) Indicial polynomial is

$$2r(r-1) + r = 0$$

or $2r^2 - r = r(2r-1) = 0$, $r \begin{cases} 1/2 \equiv r_1 \\ 0 \equiv r_2 \end{cases}$

$$y_{\text{Euler}} \sim c_1 x^{1/2} + c_2$$

obvious since $y =$ const was a solution of Euler at $x=0$

iii) The solution asked in the exercise is

$$y = x^{1/2} \sum_0^\infty a_n x^n, \quad \text{with } a_0 \neq 0$$

since it satisfies $y(0)=0$. The other solution

$$y = \sum_0^\infty b_n x^n, \quad \text{with } b_0 \neq 0$$

does not satisfy $y(0)=0$.

$$y = \sum_0^\infty a_n x^{n+1/2},$$

$$y' = \sum_0^\infty (n+1/2) a_n x^{n-1/2},$$

$$y'' = \sum_0^\infty \left(n+\frac{1}{2}\right)(n-1/2) a_n x^{n-3/2},$$

lead to

$$0 = 2xy'' + (1-2x)y' + 3y = \sum_0^\infty -2(n-1)\, a_n x^{n+1/2}$$
$$+ \sum_0^\infty n(2n+1)\, a_n x^{n-1/2}$$

$$= x^{-1/2}[0] + x^{1/2}[-2(-1)a_0 + 1\cdot 3\, a_1]$$
$$+ x^{3/2}[-2(0)\, a_1 + 2\cdot 5\cdot a_2]$$
$$+ x^{5/2}[-2\cdot 1\cdot a_2 + 3\cdot 7\cdot a_3]$$
$$+ x^{7/2}[-2\cdot 2\cdot a_3 + 4\cdot 9\cdot a_4]$$
$$+ x^{9/2}[-2\cdot 3\cdot a_4 + 5\cdot 11\, a_5]$$
$$+ x^{11/2}[-2\cdot 4\cdot a_5 + 6\cdot 13\, a_6]$$
$$+ \dots$$

and later to

$a_1 = -\frac{2}{3}a_0$, a_0 free $\qquad a_n = \frac{2(n-2)\, a_{n-1}}{n(2n+1)}$

$a_2 = 0$
$a_3 = 0$
$\vdots$

all are 0. [Sé que todos son 0 por regla de recurrencia. Es lo que siempre que "no sólo unos pocos" coef son 0, sino todos]

Take $a_0 = 1$ and write the solution that satisfies $y(0) = 0$ as

$$\boxed{y = x^{1/2}\left(1 - \frac{2}{3}x\right)}$$

It is an "elementary" function, not a series.

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(3)

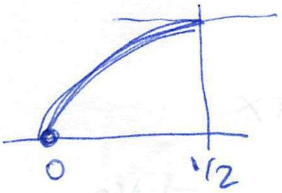
$$\begin{cases} u_t - u_{xx} + 2tu = 0 \\ u(x,0) = 1-2x \\ u(0,t) = u_x(1/2,t) = 0 \end{cases}$$

$u = XT \quad : \quad \dfrac{T'}{T} - \dfrac{X''}{X} + 2t = 0 \quad$ or

$$\frac{T'}{T} + 2t = \frac{X''}{X} = -\lambda ,$$

that reduces to

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(1/2) = 0 \end{cases} \quad \& \quad \begin{cases} T' + (2t+\lambda)T = 0 \\ \ldots\ldots \end{cases}$$



$T = 2$, $\omega = \pi$

$\{\sin n\pi x\}$, $n = 1,2,3,\ldots$ but not all (remember, $X(0) = X'(1/2) = 0$ are mixed boundary conditions):

$n=1$	$\cancel{n=2}$	$n=3$	$n=4$
$\sin \pi x$	$\cancel{\sin 2\pi x}$	$\sin 3\pi x$	$\ldots$
yes	~~no~~	yes	no

$$X_n = \{\sin (2n-1)\pi x\}, \quad n = 1,2,3,\ldots$$
$$\lambda_n = (2n-1)^2\pi^2$$

orthogonality conditions:

$$\int_0^{1/2} dx \, \sin (2n-1)\pi x \ \sin (2m-1)\pi x = \frac{\delta}{4}$$

calculated

$$T' + (2t + (2n-1)^2\pi^2)T = 0$$

solution: $T = C_1 \, e^{-t^2 - (2n-1)^2\pi^2 t}$

constant

The unique solution of the problem is

$$u = \sum_1^\infty b_n e^{-t^2-(2n-1)^2\pi^2 t} \cdot \sin(2n-1)\pi x$$

with b_n the coefficients of

$$1-2x = \sum_1^\infty b_n \sin(2n-1)\pi x$$

or, equivalently,

$$\frac{b_n}{4} = \int_0^{1/2} dx\,(1-2x)\sin(2n-1)\pi x.$$

With

$$\int_0^{1/2} dx\,(1-2x)\sin(2n-1)\pi x$$

$$= \left[-\frac{1-2x}{(2n-1)\pi}\cos(2n-1)\pi x - \frac{2}{(2n-1)^2\pi^2}\sin(2n-1)\pi x\right]_0^{1/2}$$

$$= -\frac{2}{(2n-1)^2\pi^2}\sin(2n-1)\frac{\pi}{2} + \frac{1}{(2n-1)\pi}$$

$$= \frac{2(-1)^n}{(2n-1)^2\pi^2} + \frac{1}{(2n-1)\pi}$$

$\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$, $\sin\frac{5\pi}{2} = 1$, … $\sin(2n-1)\frac{\pi}{2} = -(-1)^n$

Solution: unique:

$$\boxed{u = \sum_{n=1}^\infty \left[\frac{8(-1)^n}{(2n-1)^2\pi^2} + \frac{4}{(2n-1)\pi}\right] e^{-t^2-(2n-1)^2\pi^2 t}\sin(2n-1)\pi x}$$

notice in the above solution the factor e^{-t^2}. It suggests that the change $u=e^{-t^2}w$ in $u_t-u_{xx}+2tu=0$ affords a simpler equation for w than the original one. We have not used this transformation but it is good to know.

notice also: the minus signs in the exponentials. otherwise: danger, danger and danger!!

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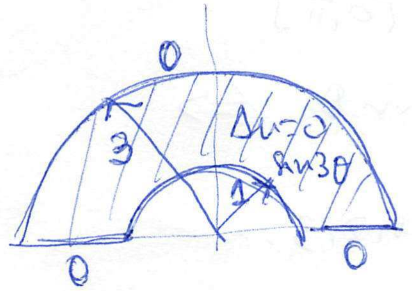
(4)



$$\Delta u = u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}=0$$

$$u=R\Theta$$

$$R''\Theta+\frac{R'\Theta}{r}+\frac{R\Theta''}{r^2}=0 \quad \text{or}$$

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

[1] $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta(\pi)=0\end{array}\right.$, $\left\{\begin{array}{l}r^2R''+rR'-\lambda R=0\\ \dots\end{array}\right.$ [2]

boundary problem here

"escort" problem here (acompañante)

solution [1]:

$T=2\pi$, $\omega=1$

$\Theta_n=\{\sin n\theta\}$, $n=1,2,3,\dots$

$\lambda_n=n^2$

with $\lambda_n=n^2$

solution [2]

$r^2R''+rR'-n^2R=0$, $n=1,2,3,\dots$

$R\sim r^a$

$R'\sim a r^{a-1}$

$R''\sim a(a-1)r^{a-2}$

$(a(a-1)+a-n^2)r^a=0$, $R_n = C_n r^n + \frac{d_n}{r^n}$ "

with c_n, d_n constant to be found with the boundary conditions $u(1,\theta)$ & $u(3,\theta)$.

General solution of $\Delta u = 0$ with those two zeroes is

$$\boxed{u = \sum_1^\infty \left(c_n r^n + \frac{d_n}{r^n} \right) \sin n\theta}$$

Impose the conditions at $r = 1, 3$.

At $r=3$: for <u>all</u> θ in $[0,\pi]$

$$0 = \sum_1^\infty \left(c_n 3^n + \frac{d_n}{3^n} \right) \sin \theta \iff$$

$$\Rightarrow c_n = -\frac{d_n}{3^{2n}}, \quad n = 1,2,3 \ldots$$

This means that

$$u = \sum_1^\infty d_n \left(-\frac{r^n}{3^{2n}} + \frac{1}{r^n} \right) \sin n\theta .$$

At $r = 1$:

$$\sin 3\theta = \sum_1^\infty d_n \left(-\frac{1}{3^{2n}} + 1 \right) \sin n\theta$$

$$= d_1 \left(1 - \frac{1}{3^2} \right) \sin \theta$$

$$+ d_2 \left(1 - \frac{1}{3^4} \right) \sin 2\theta$$

$$+ d_3 \left(1 - \frac{1}{3^6} \right) \sin 3\theta$$

$$+ \ldots$$

or

$$\begin{cases} c_3 + d_3 = 1 \\ c_3 \cdot 3^3 + \dfrac{d_3}{3^3} = 0 \end{cases}$$

$$c_3 = -1/(3^6 - 1)$$

$$d_3 = \frac{3^6}{(3^6-1)}$$

$\iff$ All d's are zero except $d_3 = \dfrac{1}{1 - \frac{1}{3^6}}$

Solution:

$$\boxed{u(r,\theta) = \frac{\sin 3\theta}{3^6 - 1} \left(\frac{3^6}{r^3} - r^3 \right)}$$

unique

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$$x u_x + (xy + e^x) u_y = u$$

$$0 = (u_x, u_y, -1)(x, xy + e^x, u)$$

Auxiliar equations are : $\dfrac{dx}{x} = \dfrac{dy}{xy+e^x} = \dfrac{du}{u}$

$$\begin{cases} \dfrac{du}{dx} = \dfrac{u}{x} \quad \text{or} \quad \boxed{u = C_1 x} \\ \dfrac{dy}{dx} = y + \dfrac{e^x}{x} \quad \text{or} \quad \dfrac{dy}{dx} - y = \dfrac{e^x}{x}. \end{cases}$$

1st characteristic

linear eqution of 1st order non-homogeneous.

$$\boxed{y = C_2 e^x + e^x \log x}$$

2nd charecteristic

solut homogeneous eqution $\frac{dy}{dx} - y = 0$

[How to obtain $e^x \log x$ here. Use variation of constants (a ojo no lo veo...)

$y = ce^x$ $\qquad {}' = \frac{d}{dx}$

$y' = c'e^x + ce^x$

$y' - y = c'e^x = \dfrac{e^x}{x}$ // $c' = \dfrac{1}{x}$, $c = \log x + C_2$
]

General solution: $\boxed{u = x f(y e^{-x} - \log x)}$

or

$$\boxed{y e^{-x} - \log x = f\left(\frac{u}{x}\right)}$$

f: arbitrary fuction of $C^1(\Omega)$ class.

i) $u(1,\eta)=\frac{1}{\eta}$ in $u=\cancel{x}f(\eta e^{-x}-\log x)$

is

$$\frac{1}{\eta}=f\left(\frac{\eta}{e}\right) \quad \text{or} \quad \eta\to\eta e$$

$$\boxed{\frac{1}{\eta e}=f(\eta)}$$

Solution: $\boxed{u=\dfrac{x}{\eta e^{1-x}-e\log x}}$

ii) $u(x,-1)=2x$ in $u=xf(\eta e^{-x}-\log x)$ is

$$2=f(-e^{-x}-\log x)$$

For all x in Ω (a good domain: $x\neq 0$ por ex) etc...

$f(\ldots)=2$, $f(\ldots)=2$, $f(\ldots)=2$

a given x — another — another different

$\Rightarrow$ $f(x)=2$

and $\boxed{u=2x}$

[no more to justify...]

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