Estructura de la Materia, Septiembre 2018

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

3 de Septiembre de 2018(2)

Examen de Septiembre de Estructura de la Materia, grupo A

Nombre y Apellidos:

Firma y DNI:

El examen consta de teoría y problemas. Justifique las respuestas, excepto cuando sean de elección múltiple, o no se tendrán en cuenta. Los problemas sí necesitan respuestas justificadas, como siempre. Hay algunas ayudas al final de los enunciados.

C1 [10 puntos] Poner un círculo rodeando la respuesta correcta. No hace falta justificación.

1) La energía del estado fundamental del Helio determinada experimentalmente es del orden de: a) $-109\,\text{eV}$, b) $-77.5\,\text{eV}$, c) $-79\,\text{eV}$.

2) Se llama *energía de ionización* a la que se requiere para quitar a un átomo neutro en su estado fundamental: a) los electrones de valencia (uno a los alcalinos, dos a los alcalinotérreos, tres a los térreos, etc), b) un electrón, c) todos los electrones.

3) Las famosas líneas D del sodio son dos líneas muy brillantes del espectro que corresponden a las longitudes de onda: a) $5590\mathring{A}$ y $5596\mathring{A}$, b) $5890\mathring{A}$ y $5896\mathring{A}$, c) $6190\mathring{A}$ y $6196\mathring{A}$. Son de color: d) amarillo, e) rojo, f) naranja. Y son una transición del nivel: g) 5p al 3s, h) 4p al 3s, i) 3p al 3s.

C2 [20 puntos] Al analizar un espectro del hidrógeno un científico llega a la conclusión de que ha ocurrido una *transición dipolar eléctrica* con emisión de un fotón entre dos estados del hidrógeno denotados por $|i\rangle$ y $|f\rangle$. Contestar: a) ¿Pueden ser ambos estados $|i\rangle$ y $|f\rangle$ impares bajo paridad? b) ¿Y pares? Nota: Responder simplemente con monosílabos *sí* o *no* contará sólo como 0.01 sobre 10. O sea, nada.

C3 [40 puntos] [Teoría camuflada] En una configuración electrónica dada aparece un triplete. Si la relación entre las separaciones energéticas entre el término de mínina energía y el siguiente, y entre éste y el de más energía es 4 : 6, determinar: a) El momento angular total J del término de menor energía, b) Los posibles valores de L y S.

Suponga ordenación normal en J.

C4 [40 puntos] [Teoría] Los neutrinos. Características generales, carga, masa, reacciones en las que intervienen, etc. Diga, **sobre todo**, por qué Pauli se vio obligado a postularlos cuando eran desconocidos e indetectable su rastro.

P1 [150 puntos] Un pión en reposo se desintegra en un muón y un neutrino. a) Escribir las ecuaciones (cuántas?) de conservación de energía-momento. b) Escribir los invariantes de Lorentz asociados al cuadrimomento de cada partícula. c) Escribir el momento del neutrino, llamémoslo p, en función de las masas del problema. d) Determinar la velocidad del muón también en función de las masas del problema. e) Aplicación numérica: pc en Mev con dos cifras decimales (use la hoja de Particle Data que se le entrega).

P2 [50+50 puntos] a) Suponiendo acoplamiento LS, contar estados y listar (como hicimos en clase calculando L, S, J) los posibles términos espectrales $^{2s+1}L_J$ que resultan en la configuración

np^4. b) Suponiendo acoplamiento jj, listar los posibles términos $[j_1, j_2]_J$ de la configuración electrónica $np^1\,nd^1$

Usted puede necesitar (o no) alguno de los siguientes datos:
En la siguiente lista de armónicos esféricos, θ, φ son los ángulos polar y azimutal, respectivamente, de las coordenadas esféricas. Asterisco significa el complejo conjugado. Las funciones de onda radiales $R_{nl}(r)$ son las del hidrógeno ($Z = 1$), y a_0 es el radio de Bohr.

$$\alpha = \frac{e^2}{4\pi\varepsilon\hbar c} \approx \frac{1}{137}, \qquad \frac{e^2}{4\pi\varepsilon a_0} = 2\,|E_1| = m_e c^2 \alpha^2.$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}, \quad Y_1^{-1} = -Y_1^{1*},$$
$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right), \quad Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}, \quad Y_2^{-1} = -Y_2^{1*},$$
$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\varphi}, \quad Y_2^{-2} = Y_2^{2*}.$$

$$L_+ Y_l^m = \sqrt{l(l+1) - m(m+1)}\,\hbar\, Y_l^{m+1}, \quad L_- Y_l^m = \sqrt{l(l+1) + m(-m+1)}\,\hbar\, Y_l^{m-1}.$$

$$R_{10} = 2\,a_0^{-3/2}\exp(-r/a_0),$$
$$R_{20} = \frac{1}{\sqrt{2}}a_0^{-3/2}\Big(1 - \frac{r}{2a_0}\Big)\exp(-r/2a_0), \quad R_{21} = \frac{1}{\sqrt{24}}a_0^{-3/2}\frac{r}{a_0}\exp(-r/2a_0),$$
$$R_{30} = \frac{2}{\sqrt{27}}a_0^{-3/2}\Big(1 - \frac{2r}{3a_0} + \frac{2}{27}\Big(\frac{r}{a_0}\Big)^2\Big)\exp(-r/3a_0)$$

$$\Big\langle\frac{1}{r}\Big\rangle_{nl} = \frac{1}{n^2\,a_0}, \quad \Big\langle\frac{1}{r^2}\Big\rangle_{nl} = \frac{2}{n^3\,(2l+1)\,a_0^2}, \quad \Big\langle\frac{1}{r^3}\Big\rangle_{nl} = \frac{2}{n^3 l(l+1)(2l+1)\,a_0^3}.$$

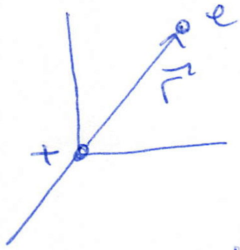
(C1) Answered in the first pep.

(C2) a) NO

b) NO

It is <u>impossible</u> a dipolar transition among states of equal parity because the matrix element $\langle i | e\vec{r} | f \rangle$ is <u>zero</u> in such case.

Parity is to change $\vec{r} \to -\vec{r}$ everywhere. If



$|i\rangle$ and $|f\rangle$ are both odd under parity is because when $\vec{r}$ changes to $-\vec{r}$ the wave functions associated to the states change as

$$\varphi_i(\vec{r}) \to -\varphi_i(\vec{r})$$
$$\varphi_f(\vec{r}) \to -\varphi_f(\vec{r})$$

and

$$\langle i | e\vec{r} | f \rangle = \int_{\mathbb{R}^3} d^3\vec{r}\, \varphi_i^*(\vec{r})\, \vec{r}\, \varphi_f(\vec{r})$$

$$= (-1)^3 \int_{\mathbb{R}^3} d^3\vec{r}\, \varphi_i(\vec{r})\, \vec{r}\, \varphi_f(\vec{r})$$

change $\vec{r} \to -\vec{r}$ everywhere.

$$= -\langle i | e\vec{r} | f \rangle$$

that implies

$$\langle i | e\vec{r} | f \rangle = 0$$

[no dipolar transition whatsoever

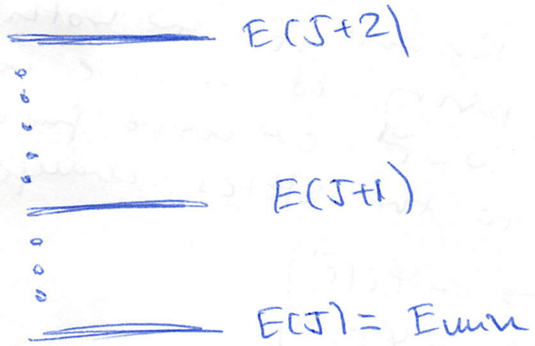
]

Case b) is exactly the same: $(-1)^3$ is now $(-1)^1$: same conclusion.

(C3) A triplet of terms means "same L and S but three different J's". Remember the notation $^{2S+1}L_J$: each level is represented by the state $|L,S,J\rangle$ ($2J+1$ times degenerated in a shell model)

Normal ordering is that the level with smallest J has the smallest energy ($\equiv$ ground state). If this is so, a (see below) is a positive constant.

a) Calculation of the possible values of J



$$E(J) = a\left[J(J+1) - L(L+1) - S(S+1)\right]$$ " a positive const with dimensions of energy"

so

$$E(J+1) - E(J) = a\left[(J+1)(J+2) - J(J+1)\right]$$
$$= a\,(J+1)\left[\cancel{J}+2-\cancel{J}\right]$$
$$= 2a\,(J+1).$$ ← Landé's interval rule

The problem states that
$$\frac{E(J+1)-E(J)}{E(J+2)-E(J+1)} = \frac{4}{6} = \frac{2}{3} = \frac{J+1}{J+2},$$
then $J=1$ corresponds to E_{min}.

$\boxed{\text{The possible values of } J \text{ are } 1,2,3. \text{ The ground state corresponds to } J=1}$

Notice that the problem has no solution if instead of 4/6 we are given, for instance, 4/7 that correspond to $J_{min} = 1/3$.

b) Calculation of L and S

The composition of $\vec{L}$ and $\vec{S}$ to give $\vec{J} = \vec{L} + \vec{S}$ afford the values of J:

$$L+S, \quad L+S-1, \quad \ldots \quad |L-S|.$$

1 by 1 steps

In our case $L+S=3$ is the maximum value of J. Considering that L is an integer and that L, S are positive, the only possibilities are

L	S	J	
0	3	3	no
1	2	3,2,1	yes
2	1	3,2,1	yes
3	0	3	no

First and fourth instances are overruled: they afford a single value of J and we have three.

$\boxed{\text{Possible values of } (L,S) \text{ are } (1,2) \text{ or } (2,1)}$

[not asked in the problem: the energy of each level

$E(3) = 4a$, $d=7$ ← degeneration of the level

$E(2) = -2a$, $d=5$

$E(1) = E_{min} = -6a$, $d=3$

In addition, the "center of gravity" of the levels, considering degeneration is zero, as expected:

$$4a \times 7 - 2a \times 5 - 6a \times 3 = 0$$

7 states with $J=3$

(C4) Why Pauli suggested the neutrino' existence:
In the decay of

$$A \to B + e^-$$

B: another nucleon slightly lighter than A

A: radioactive nucleon

e^-: electron

producing two particles, the energy of the products is determined exclusively by the masses m_A, m_B and m_e. This means that the electron has energy

$$E = \frac{c^2}{2m_A}\left(m_A^2 - m_B^2 + m_e^2\right),$$

a quantity that is fixed ($K = E - m_e c^2$ if you think in terms of kinetic energy w.r.t the reference system in which A is at rest). This E was not observed in experiments. The electron had a kinetic energy lower than $E - m_e c^2$. To preserve conservation of energy-momentum in what we today recall as the reaction

$$n \to p^+ + e^- + \bar{\nu},$$

the neutrino had to be introduced (Pauli se arriesgó temerariamente) by Pauli. Was not a two body decay but a three body decay: quite different!!

Ejercicios del ALUMNO

APELLIDOS ....

NOMBRE | D.N.I. n.º

ASIGNATURA .... GRUPO ....

CURSO | N.º DE MATRICULA | FECHA

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(P4) Answer:

d) $\frac{v}{c} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \simeq 0.244$

e) $pc = 26.61$ MeV

Solution: $\pi \to \nu + \mu$ $\qquad \overset{\nu}{\longleftarrow}\!\!\bullet\!\!\overset{\mu}{\longrightarrow}$

The movement of ν, μ happens along a line in the system of reference where the pion is at rest:

$$\begin{array}{ccccc} \begin{pmatrix} m_\pi c \\ 0 \\ 0 \\ 0 \end{pmatrix} & = & \begin{pmatrix} E_1/c \\ -p \\ 0 \\ 0 \end{pmatrix} & + & \begin{pmatrix} E_2/c \\ p \\ 0 \\ 0 \end{pmatrix} \\ \text{pion} & & \text{neutrino} & & \text{muon} \end{array} , \quad p \text{ positive by choice}$$

energy-momentum conservation

Lorentz invariants are

$$\left(\frac{E_1}{c}\right)^2 - p^2 = 0, \qquad \left[\text{or } \frac{E_1}{c} = p\right]$$

$$\left(\frac{E_2}{c}\right)^2 - p^2 = m_\mu^2 c^2.$$

Considering conservation of energy-momentum and Lorentz invariants we reduce the problem to three equations ($0 = -p + p$: este apartado ya lleva implícito la conservación del momento).

- $m_\pi c = \frac{E_1}{c} + \frac{E_2}{c}$
- $\frac{E_1}{c} = p$ (because E_1 is positive and p too)
- $\left(\frac{E_2}{c}\right)^2 - p^2 = m_\mu^2 c^2$

A comment about velocity: a relativistic particle of mass m ($m \neq 0$) and momentum p has

$$p = m\gamma v$$
$$E = m\gamma c^2$$

Deduction of $E = m\gamma c^2$:

$$\left(\frac{E}{c}\right)^2 - p^2 = m^2c^2 \quad : \text{ Lorentz invariant}$$

$$E^2 = p^2c^2 + m^2c^4 = m^2\gamma^2 v^2c^2 + m^2c^4$$

$p = m\gamma v$

$$= m^2c^4\left[1 + \gamma^2\frac{v^2}{c^2}\right]$$

$$= m^2c^4\left[1 + \frac{v^2/c^2}{1-v^2/c^2}\right]$$

$$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}} \qquad = m^2c^4\left[\frac{1}{1-v^2/c^2}\right]$$

$$= m^2c^4\gamma^2.$$

Then $E = mc^2\gamma$ as stated.

The comment is precisely that if one knows the momentum of the particle and its energy, the velocity is given by the quotient

$$\frac{v}{c} = \frac{pc}{E} \quad ; \quad \begin{aligned} p &= m\gamma v \\ E &= m\gamma c^2 \end{aligned}$$

this ratio avoids ~~sometimes~~ algebraic calculations.

We will determine the velocity of the muon through p and E_2. Insert [..] into [...],

$$\left(\frac{E_2}{c}\right)^2 - \left(\frac{E_1}{c}\right)^2 = m_\mu^2c^2,$$

that is also

$$\left(\frac{E_2}{c} - \frac{E_1}{c}\right)\underbrace{\left(\frac{E_2}{c} + \frac{E_1}{c}\right)}_{\text{this is } m_\pi c \text{ by [o]}} = m_\mu^2c^2$$

and affords

$$\frac{E_2}{c} - \frac{E_1}{c} = \frac{m_\mu^2 c}{m_\pi}$$

From

$$\left.\begin{aligned} E_2 - E_1 &= \frac{m_\mu^2}{m_\pi} c^2 \\ [\cdot] \quad E_2 + E_1 &= m_\pi c^2 \end{aligned}\right\} \Rightarrow \quad E_1 = \frac{c^2}{2}\left[m_\pi - \frac{m_\mu^2}{m_\pi}\right], \quad E_2 = \frac{c^2}{2}\left[m_\pi + \frac{m_\mu^2}{m_\pi}\right].$$

Since

$$p = \frac{E_1}{c} = \frac{c}{2}\left[m_\pi - \frac{m_\mu^2}{m_\pi}\right]$$

and

$$\frac{v}{c} = \frac{pc}{E_2} = \frac{\frac{c^2}{2}\left[m_\pi - \frac{m_\mu^2}{m_\pi}\right]}{\frac{c^2}{2}\left[m_\pi + \frac{m_\mu^2}{m_\pi}\right]}$$

$$\Rightarrow \boxed{\frac{v}{c} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}} \simeq 0.244$$

Also $\gamma = \dfrac{E_2}{m_\mu c^2} = \dfrac{1}{2}\left[\dfrac{m_\pi}{m_\mu} + \dfrac{m_\mu}{m_\pi}\right]$ " $\boxed{\gamma = \dfrac{1}{2}\left[\dfrac{m_\pi}{m_\mu} + \dfrac{m_\mu}{m_\pi}\right]}$

$\simeq 1.031$

Answers to

c) $\boxed{p = \dfrac{c}{2}\left[m_\pi - \dfrac{m_\mu^2}{m_\pi}\right]}$

e) $m_\pi c^2 = 135.569\,\text{MeV}$
$m_\mu c^2 = 105.659\,\text{MeV}$ [table]

$$pc = \frac{1}{2}\left[\frac{m_\pi^2 c^4 - m_\mu^2 c^4}{m_\pi c^2}\right]$$

$$= \boxed{26.61\,\text{MeV}}$$

this es el momento del muón
[$0 = -p + p$, remember]

$E_1 \simeq 26.61\,\text{MeV}$, $E_2 \simeq 108.96\,\text{MeV}$

P2

a) np^4 : equivalent electrons $\quad m_l \quad m_s$

Each electron can be in one of 6 states: $\left\{\begin{matrix}1\\0\\-1\end{matrix}\right\} \times \left\{\begin{matrix}1/2\\-1/2\end{matrix}\right\}$

= 6 different states. No two electrons are in the same state so there are

$$\binom{6}{4} = \binom{6}{2} = \frac{6!}{2!\,4!} = 15$$

ways of choosing four different states from six

In the case np^6 there is only one. Suppose we use the standard notation α for $m_s = 1/2$, β for $m_s = -1/2$, 1 for $m_l = 1$, $\bar{1}$ for $m_l = -1$,

the unique state would be

$$\underbrace{1\alpha}_{\text{electron 1}}\ \underbrace{0\alpha}_{\text{electron 2}}\ \bar{1}\alpha\ 1\beta\ 0\beta\ \underbrace{\bar{1}\beta}_{\text{electron 6}}.$$

look at this state again as

$$\underbrace{1\alpha 0\alpha \bar{1}\alpha 1\beta}_{np^4}\ \underbrace{0\beta\bar{1}\beta}_{np^2 \equiv \text{gap}}$$

For these electrons $\quad M_L = \text{sum of } ~~ms~~ \; m_l\text{'s} = 1$

$M_S = \text{sum of } m_s\text{'s} = 1 = \alpha + \alpha + \cancel{\alpha} + \cancel{\beta}$

For these electrons

$M_L = -1$

$M_S = -1$

The configuration of np^4 is totally determined by the configuration of np^2 electrons (just negating M_L and M_S).

Conclusion: Compose two electrons (or two gaps, as you wish) and the configuration of four electrons is resolved!

With two electrons np^2:

$l_1 = 1 \qquad l_2 = 1 \qquad L = \overset{s}{2}, \overset{a}{1}, \overset{s}{0}$

$s_1 = 1/2 \qquad s_2 = 1/2 \qquad S = 1^s, 0^a$

[Small s means "symmetric" under the interchange of electron 1 by electron 2, small a means antisymmetric].

In a LS coupling, the possible values of L, S are:

L	S	J	2J+1	sum of 2J+1
2	0	2	5	5
1	1	2, 1, 0	5, 3, 1	9
0	0	0	1	1
				sum: 15 states

remember that the electrons have total antisymmetric wave func. under the interchange of two electrons

Possible terms: $^3S_0$; $^3P_0$, $^3P_1$, $^3P_2$, $^1D_2$

b) jj coupling: np^1nd^1: non equivalent electrons

electron 1: $l_1 = 1$: $m_{l_1} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \times \begin{Bmatrix} 1/2 \\ -1/2 \end{Bmatrix} = m_{s_1}$: 6 possible states

electron 2: $l_2 = 1$ $m_{l_2} = \begin{Bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \end{Bmatrix} \times \begin{Bmatrix} 1/2 \\ -1/2 \end{Bmatrix} = m_{s_2}$: 10 possible states

This makes a total of $6 \times 10 = 60$ states all of them with the same energy under the shell model

jj coupling:

$j_1 = l_1 + s_1, l_1 + s_1 - 1, \dots, |l_1 - s_1| = \frac{3}{2}, \frac{1}{2}$

$j_2 = \frac{5}{2}, \frac{3}{2}$

The possible values of total spin J are:

j_1	j_2	J	$2J+1$	sum of $2J+1$
$\frac{3}{2}$	$\frac{5}{2}$	4, 3, 2, 1	9, 7, 5, 3	24
✱ $\frac{3}{2}$	$\frac{3}{2}$	3, 2, 1, 0	7, 5, 3, 1	16
$\frac{1}{2}$	$\frac{5}{2}$	3, 2	7, 5	12
$\frac{1}{2}$	$\frac{3}{2}$	2, 1	5, 3	8
			sum:	60

of course!!!

✱ Electrons differ in $l_1=1$, $l_2=2$
there are 12 possible terms: notation: $[j_1, j_2]_J$:

$[\frac{3}{2}, \frac{5}{2}]_4$, $[\frac{3}{2}, \frac{5}{2}]_3$, $[\frac{3}{2}, \frac{5}{2}]_2$, $[\frac{3}{2}, \frac{5}{2}]_1$

$[\frac{3}{2}, \frac{3}{2}]_3$, $[\frac{3}{2}, \frac{3}{2}]_2$, ... etc.

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