

## Estructura de la Materia

Página web de la asignatura: <http://teorica/ft8/problemas.html>

Los libros de donde se toman los ejercicios están indicados entre corchetes, así [Li] es Liboff, [Gri] Griffiths, [Ga] Gasiorowicz, [LlaEs] son los ejercicios que Felipe J. Llanes-Estrada usaba para impartir este curso, [Brans] es Bransden, [Squ] es G.L. Squires, *Problems in Quantum Mechanics*, [Gri-Part] is Griffiths' *Introduction to Elementary Particles*, etc. Los problemas en **rojo** se proponen como ejercicios al alumno para hacerlos en casa. No hay que entregarlos. Los problemas en negro los hago yo en clase. La solución de estos problemas, si no está escrita en estas páginas, que es lo usual que no estén, me la piden y se la llevo a clase. Yo me los hago todos.

$$\begin{aligned}\alpha &\equiv e^2/4\pi\epsilon_0 \hbar c = 1/137.035\,999\,11 = 7.297\,352\,568 \times 10^{-3} \\ a_0 &= \hbar/m_e c \alpha = 4\pi\epsilon_0 \hbar^2/m_e e^2 = 0.529\,177\,211 \times 10^{-10} \text{ m} \\ E_1 &= -m_e c^2 \alpha^2/2 = 13.605\,692 \text{ eV} \\ E_h &= e^2/4\pi\epsilon_0 a_0 = -2E_1 = 27.211\,385, \text{ eV} \\ m_e &= 0.510\,999\,9 \text{ MeV}/c^2 \\ N_A = L &= 6.022 \times 10^{23} \text{ mol}^{-1}, \quad 1 \text{ \AA} = 10^{-10} \text{ m},\end{aligned}$$

### THE ATOM OF HELIUM

Programa: 1) Primer intento de justificar teóricamente la energía del estado fundamental del helio cuyo valor experimental, o sea, el bueno, es  $-79.0051 \text{ eV}$ . En este primer intento se suprime la repulsión electrón-electrón (ee, de ahora en adelante) y sale una aproximación muy mala, lo que indica que no se puede omitir, que hay que considerarla. 2) Paridad del estado fundamental del átomo de helio. Comentario sobre los estados excitados: sólo uno de los electrones puede estar excitado, el otro no; así es el elemento del sol y de los dirigibles. 3) Segundo intento: usar teoría de perturbaciones a primer orden incluyendo la repulsión ee. El resultado se aproxima más al valor experimental, ha mejorado mucho. 4) Tercer intento: lo mismo pero con el principio variacional de Ritz. Casi estupendo. Sacamos tb  $Z^*$  efectiva igual a 1.6875. Indica que los electrones se apantallan el uno al otro, se impiden mutuamente ver la carga nuclear enterita  $Z = 2$  como debía ser. 4) Aunque el hamiltoniano no considera spin, el principio de exclusion de Pauli tiene mucho que decir: energías  $J$  y  $K$ , de Coulomb y de intercambio. Ambas son positivas. En el caso de  $J$  es obvio que es positiva. El caso de  $K$  hay que trabajárselo para verlo.

- [A first year in Quantum Physics] La *constante de estructura fina* se define como  $\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c}$  donde  $1/(4\pi\epsilon_0) = 9.0 \times 10^9 \text{ J m C}^{-2}$  es la constante de la ley de Coulomb ( $\epsilon_0$  se conoce con el nombre de *constante dieléctrica del vacío*). Comprobar que  $\alpha$  (i) es adimensional, (ii) tiene un valor aproximadamente igual a  $1/137$ .
- En el estado fundamental de un átomo hidrogenoideo de número atómico  $Z$ , ¿a qué distancia es más probable encontrar el electrón?
- [Griffiths] Write the ground state wave function of the helium ignoring the electron repulsion term in the Hamiltonian. What is the energy?

4. • [Brans1983, pg 288] [Hecho en clase] Prove that eigenfunctions  $\psi(\mathbf{r}_1, \mathbf{r}_2)$  of the Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m_e}(\nabla_1 + \nabla_2) - \frac{Ze^2}{4\pi\epsilon_0} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) + \frac{e^2}{4\pi\epsilon_0 r_{12}} \right] \psi(\mathbf{r}_1, \mathbf{r}_2) = E\psi(\mathbf{r}_1, \mathbf{r}_2),$$

corresponding to degenerate eigenvalues can always be chosen to be either space-symmetric  $\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_2, \mathbf{r}_1)$  or space anti-symmetric  $\psi(\mathbf{r}_1, \mathbf{r}_2) = -\psi(\mathbf{r}_2, \mathbf{r}_1)$ .

*Note:* Bransden writes the hamiltonian in a.u., *atomic units*. The list of a.u. is in the webpage, but you, UCM students, are not allowed to use them in this academic year, sorry. When you become a grown Physicist, you will do as you want to. Why? Ask me in the classroom.

5. [Griffiths1995, pg 188] [Helium excited states] Suppose you put both electrons in a helium atom into the  $n = 2$  state; what would the energy of the emitted electron be?
6. [ExPar2017] [Helium problem leads to the treatment of identical particles] El estado fundamental de un átomo de helio es no degenerado como bien sabemos. Pero para este ejercicio vamos a considerar un hipotético helio en el que los dos electrones son substituidos por dos partículas idénticas de spin 1 y carga negativa. No hay interacción entre los spines de las hipotéticas partículas. ¿Cuál es la degeneración del estado fundamental de este novedoso helio? *Sol=6.*
7. •• [Brans1983, pg 288] Calculate the average values of (i)  $r_1^2 + r_2^2$ , (ii)  $\delta^{(3)}(\mathbf{r}_1)$  (this is electron one located at the origin with the nucleus), (iii)  $\delta^{(3)}(\mathbf{r}_{12})$  (electron one and two in the same point of space) for the ground state of helium using:
- the zero-order product of hydrogenic wave function given by the product  $\psi_{100}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2)$ , with  $Z = 2$ ;
  - the simple ‘screened’ variational function given by  $\psi_{1s}^{Z^*}(\mathbf{r}_1)\psi_{1s}^{Z^*}(\mathbf{r}_2)$ , and the ‘effective charge’  $Z^* = 27/16$ ;
  - The Hartree-Fock function found by Byron and Joachain,

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = u_{1s}(\mathbf{r}_1)u_{1s}(\mathbf{r}_2)$$

where the 1s orbital  $u_{1s}(\mathbf{r})$  is given by

$$u_{1s}(\mathbf{r}) = \frac{1}{\sqrt{4\pi}} (Ae^{-\alpha r} + Be^{-\beta r})$$

with  $A = 2.60505, B = 2.08144, \alpha = 1.41, \beta = 2.61$ . This normalized function (check it!) is an analytical fit to Hartree-Fock orbital. The corresponding Hartree-Fock ground states energy is  $E_0 = -2.86167$  a.u.  $= 2.86167 \times 2E_1 = -2.86167 \times 2 \times 13.605692$  eV.

Compare your results with the ‘exact’ values  $\langle r_1^2 + r_2^2 \rangle = 2.39$ ,  $\langle \delta^{(3)}(\mathbf{r}_1) \rangle = 1.81$  and  $\langle \delta^{(3)}(\mathbf{r}_{12}) \rangle = 0.106$  (in a.u.)

*Note from MJRP:* Once you have finished the exercise, there is a table in page 657 where you can check your results. And a comment on them added by the author of the book. The respective wave functions needed in the problem were mentioned during the lectures, save for Hartree-Fock, of course, that is new.

Bransden uses a.u., atomic units, you are not allowed to use them. Example:  $a_0$ , Bohr radius is  $a_0$  and not 1. Read exercise 4.

THE REAL HYDROGEN ATOM. FINE STRUCTURE

Programa: 1) Introducción: la velocidad con la que se mueve el electrón en el átomo de hidrogeno es del orden de un uno por ciento de la velocidad de la luz; es no-relativista, pequeña, sí, pero no *tan pequeña*. Vamos, que va a toda leche comparado con lo que conocemos nosotros. Orden de magnitud de la estructura fina. 2) Sumario de relaciones físicas que serán probadas después: teorema del virial para potenciales centrales, valor medio de  $1/r$  en los estados estacionarios del hidrógeno, Hellmann-Feynman theorem, valor medio de  $1/r^2$ , etc. 3) Energy corrections due to the relativistic speed of the electron in its orbit (makes use of perturbation theory). 4) Darwin term. 5) Spin-orbit coupling; first time along the course where the hamiltonian includes spin. Uses perturbation theory too. 6) The three terms together are called the *fine structure* of the hydrogen atom. 7) Elementary theory of radiation: dipolar selection rules of spontaneous emission. Parity of a atomic state. 8) It all applicable to hydrogen-like atoms as well (átomos hidrogenoideos). 9) Exercises.

8. [LlaEs] [Spin-orbit coupling] ¿Cuántos estados del nivel 4F del átomo de hidrógeno tienen la misma energía de Bohr  $E_4$ ? Sol: 14. Evaluar el desdoblamiento spin-órbita entre los dos valores de  $j$  en el nivel 4F. ¿Qué degeneración persiste tras considerar esta interacción? Sol: 8 y 6.
9. By the name *fine structure* is understood the sum of three terms of the same order of magnitud. They refer to the energy of the electron in a hydrogen atom or H-like atom. The terms are, the energy due to relativistic corrections to the motion of the electron, Darwin term and spin-orbit interaction term. We studied each of them during the lectures. Show that the sum of the three effects gives the following expression for the energy of the electron

$$E_{nj} = E_n \left[ 1 + \frac{Z^2 \alpha^2}{n} \left( \frac{2}{2j+1} - \frac{3}{4n} \right) \right],$$

where  $E_n = Z^2 E_1 / n^2$  are the Bohr energies, and  $E_1 = -m_e c^2 \alpha^2 / 2$ , as above. Note that this expression does not depend on  $l$ . Then  $2S_{1/2}$  and  $2P_{1/2}$  levels of hydronen still have the same energy.

10. [Gri, pg 243] The most prominent feature of the hydrogen spectrum in the visible region is the red Balmer line or Balmer-alpha line, coming from the transition  $n = 3$  to  $n = 2$ . Fine structure splits this line into several closely spaced lines; the question is: *How many*, and *what is their spacing?* First determine how many sublevels the  $n = 2$  level splits into, and find the fine structure energy for each of these, in eV. Then do the same for  $n = 3$ . Draw an energy level diagram showing all possible transitions from  $n = 3$  to  $n = 2$ . The energy released (in the form of a photon) is  $E_3 - E_2 + \Delta E$ , the first part being common to all of them, and the  $\Delta E$  (due to fine structure) varying from one transition to the next. Find  $\Delta E$  (in eV) for each transition.

Este mismo ejercicio lo he hecho en clase, sólo que en clase el enunciado decía: [LlaEs] Identifique todas las transiciones *dipolares eléctricas* que contribuyen a Balmer-alpha en el hidrógeno. Al imponer reglas de selección no sale la línea que el profesor Griffiths escribe de  $j = 5/2$  a  $j = 1/2$ , esa de  $54.33 \times 10^{-6}$  eV. Todo lo demás nos queda igual.

11. [Gri, pg 315] [Selection rules] Calculate the lifetime (in seconds) of each of the four  $n = 2$  states of hydrogen. *Hint:* You will need to evaluate matrix elements of the form  $\langle \psi_{100} | x | \psi_{200} \rangle$ ,  $\langle \psi_{100} | y | \psi_{211} \rangle$ , and so on. Remember that  $x = r \sin \theta \cos \varphi$ ,  $y = r \sin \theta \sin \varphi$  and  $z = r \cos \theta$ . Most of these integrals are zero, so scan them before you start calculating. *Answer:* =  $1.60 \times 10^{-9}$  s for all except  $\psi_{200}$ , which is infinite.

12. [Gri, pg 318] [Selection rules involving  $l$  and  $l'$ ] [This exercise is redacted in a free manner by myself. If mistakes, they are only mine, MJRPlaza] Prove the commutation relation

$$[\mathbf{L}^2, [\mathbf{L}^2, \mathbf{r}]] = 2\hbar^2 (\mathbf{r}\mathbf{L}^2 + \mathbf{L}^2\mathbf{r}),$$

as follows: 1) First show that

$$[\mathbf{L}^2, z] = 2i\hbar(xL_y - yL_x - i\hbar z)$$

using

$$[L_x, z] = -i\hbar y, \quad [L_y, z] = i\hbar x, \quad [L_z, z] = 0$$

and the Leibniz rule. Observe that the first two commutators follow the right-hand rule. This rule permits to write also that

$$[L_z, x] = i\hbar y, \quad [L_x, y] = i\hbar z, \quad [L_y, z] = i\hbar x$$

and

$$[L_x, z] = -i\hbar y, \quad [L_y, x] = -i\hbar z, \quad [L_z, y] = -i\hbar x$$

without explicit calculation (but I have checked all of them!). 2) Using  $\mathbf{r}\mathbf{L} = \mathbf{r}(\mathbf{r} \times \mathbf{p}) = 0$ , which is  $xL_x + yL_y + zL_z = 0$  and the expression for  $i\hbar(xL_y - yL_x)$  as obtained in the previous point, derive that

$$[\mathbf{L}^2, [\mathbf{L}^2, z]] = 2\hbar^2(z\mathbf{L}^2 + \mathbf{L}^2z).$$

As Griffiths says, the generalization from  $z$  to  $\mathbf{r}$  is trivial.

#### MANY ELECTRON ATOMS

13. [del Bachillerato] [Electronic configuration] With a **periodic table** we easily find that the ground-state electronic configuration of the neon atom is  $1s^22s^22p^6$ . Write the ground-state electronic configuration of Ca ( $Z = 20$ ), Rb ( $Z = 37$ ), Cs ( $Z = 55$ ), and TL ( $Z = 81$ ).
14. [Squires, pg 133] Calculate the degeneracy, and list the possible  $^{2S+1}L_J$  values for each of the following electronic configurations: (a)  $2s^12p^1$ , (b)  $2p^13p^1$ , (c)  $2p^2$ , (d)  $3d^{10}$ , (e)  $3d^9$ .
15. [Squires, pg 134] The electronic configuration of the ground state of nitrogen is  $1s^22s^22p^3$ .
- What is the degeneracy of this electronic configuration?
  - List the permitted products of single-electron states for the electrons in the 2p shell in order of decreasing  $M_L$  value (esto no lo hice en clase, pero dije que lo hicieran).
  - What is the possible  $^{2S+1}L_J$  values for the electronic configurations?
  - Which set has the lowest energy? (Here you have to use Hund's rules).
16. [Squires, pg 134] [ $jj$  coupling] The lead atom obeys  $jj$  coupling, and has two electrons in the 6p shell in the ground state. What are the possible  $j_1, j_2, J$  values for this electronic configuration?
17. [Squires, pg 134] [Nucleus and  $jj$  coupling] In the shell model, the ground nucleus  $^{19}\text{O}$  (an isotope of oxygen) is represented by the three  $d_{5/2}$  neutrons outside closed shells. Given that the nucleons obey  $jj$  coupling, calculate the possible values of the spin  $J$  of the ground state of the nucleus.

18. [Hund's rules. Ground state term] Exercise 13 writes the ground state electronic configuration of some atoms. Now determine the ground state term  $^{2s+1}L_J$  that corresponds to the ground state for H, He,  ${}_3\text{Li}$ ,  ${}_4\text{Be}$ ,  ${}_5\text{B}$ , using Pauli's principle and *Hund's rules*. Remember that fully filled shells or subshells can be excluded from your consideration since they have total  $L = 0, S = 0$ .
19. Same as before but for now for more interesting people  ${}_6\text{C}$ ,  ${}_7\text{N}$ ,  ${}_8\text{O}$ ,  ${}_9\text{F}$ . (Nitrogen is already done in 15).
20. [Thomas-Fermi model] In 1927, L.H. Thomas and E. Fermi, independently proposed a method to characterize the electron distribution in an atom based on the statistics of a noninteracting degenerate gas. The treatment led to the formulation of the well-known nonlinear differential equation

$$y'' = \frac{y^{3/2}}{\sqrt{x}},$$

with the boundary conditions

$$y(0) = 1, \quad y(\infty) = 0.$$

Find the simplest particular solution of this equation trying the very reasonable ansatz  $ax^m$ . You will observe that your solution satisfies  $y(\infty) = 0$  but it is singular at  $x = 0$ . Don't worry, Sommerfeld had the same problem. Only that he persisted and found after several transformations and an asymptotic saddle point analysis the simple approximate solution (I am not asking you for this one)

$$y = \frac{1}{\left(1 + \left(\frac{x}{\sqrt[3]{144}}\right)^{3/b}\right)^b}$$

where  $b \approx 3.886$ . This solution can be regarded as an asymptotic expansion of the solution that you have worked.

21. [Thomas-Fermi model. Baker's expansion near  $x = 0$ ] The expansion

$$y = 1 + \frac{4}{3}x^{3/2} + \frac{1}{3}x^3 + \frac{2}{27}x^{9/2} + \frac{4}{405}x^6 + \dots$$

is a solution of the Thomas-Fermi equation

$$x^{1/2}y'' = y^{3/2}$$

near  $x = 0$  that satisfies  $y(0) = 1$ . However, this solution has no physical interest because it does not vanish anywhere and diverges for large  $x$ . 1) Obtain it by hand as follows: start with the ansatz  $y = 1 + px^n + \dots$  and determine  $n$  and the constant  $p$  with the TF equation. You will need to use that

$$(1+x)^{3/2} = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \dots,$$

just including not as many terms as I have written but merely those to cancel the dominant contribution. You will end up with  $y = 1 + \frac{4}{3}x^{3/2} + \dots$ . Repeat the algorithm with  $y = 1 + \frac{4}{3}x^{3/2} + px^n + \dots$  and so on. In this manner you will have written the first few non-zero terms of the (nonphysical) expansion. 2) Repeat now what the University of Michigan graduate student Baker did back in 1930. Try the solution

$$y = 1 + sx + \frac{4}{3}x^{3/2} + px^n + \dots,$$

that introduces a slope  $s$  different from zero and determine  $s, p$  and  $n$ , same as before. Remarkably  $s$  remains free, whereas  $n = 5/2$  and  $p = 2s/5$ . Continue (if you are tired of calculating

Table 1:

	<b>H(1)</b>							<b>He(2)</b>
1s	1.00							1.69
	<b>Li(3)</b>	<b>Be(4)</b>	<b>B(5)</b>	<b>C(6)</b>	<b>N(7)</b>	<b>O(8)</b>	<b>F(9)</b>	<b>Ne(10)</b>
1s	2.69	3.68	4.68	5.67	6.66	7.66	8.65	9.64
2s	1.28	1.91	2.58	3.22	3.85	4.49	5.13	5.76
2p			2.42	3.14	3.83	4.45	5.10	5.76

by hand now you may use Maple, Mathematica, MATLAB) and obtain Baker's, now XYZ's (where XYZ is your name) expansion

$$y = 1 + sx + \frac{4}{3}x^{3/2} + \frac{2}{5}sx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}s^2x^{7/2} + \dots$$

Good work.

22. [Shell atomic model] 1) Estimate the total atomic binding energy of the first few atoms whose  $Z_{\text{eff}}$  are given in Table 1. These factors are the Clementi-Raimondi effective factors of nuclear charge, and can be explained by the effect of shielding in different shells and subshells. They are determined in advanced via computer calculations of atomic structure, using *self-consistent field approach* (SCF) based on Hartrees method. With these values one can quickly and easily generate an approximate energy-level diagram and estimate the size of the orbitals for any atom. Notice that the entry corresponding to the Helium, 1.69, is an approximation of  $27/16 = 1.6875$ , number that we already obtained with a variational principle at the beginnig of the semester.

2) Once the energies are obtained (see the solution in a pdf file next to this one) figure if a power of  $Z$  can fit the data. Remember that in Thomas-Fermi model, the ground energy of an atom was of order  $E(Z) \sim Z^{7/3}$ . Perhaps here it is similar although the models are quite different.

#### MATHEMATICAL IDENTITIES AND PHYSICAL RELATIONS

23. Observación: junto en este apartado las matemáticas. Unas serán para el átomo de helio, otras para el hidrógeno, otras para todo el mundo. Las pongo aquí en un principio. Luego veré si las distribuyo por sus apartados correspondientes, que será lo mejor.
24. [Li, pg 640] [Fourier representation of the Coulomb potential] Often we need to operate with  $\frac{1}{r_{12}}$ , where  $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$  is the distance between two particles. It is useful to know that

$$\frac{1}{r_{12}} = \frac{1}{2\pi^2} \int d^3\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}_{12}}}{k^2},$$

or in general that

$$\frac{1}{r} = \frac{1}{2\pi^2} \int d^3\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}, \quad r = |\mathbf{r}|.$$

Prove any of these relations (are the same!).

25. Show that

$$\int_0^\infty dx \frac{\sin ax}{x} = \frac{\pi}{2}.$$

Notice that the integral is (surprisingly) independent of  $a$ . But not the integral from 0 to  $\pi$ , for instance, or from 0 to a finite number, that depends on  $a$  (something that you can check with Maple). Does not depend on  $a$  when the integration interval runs to infinity.

26. The laplacian of  $1/r$  is

$$\nabla^2 \frac{1}{r} = -4\pi \delta^{(3)}(\mathbf{r}),$$

where  $r$  is the magnitude of vector  $\mathbf{r}$  and  $\delta^{(3)}(\mathbf{r})$  is the 3-dimensional delta function. There are several proofs of this fact. One proof uses the Fourier transform of the Coulomb potential stated in problem 24. Another verification is with the *divergence theorem* that you have studied in previous years (oh yes, I am conscious that this is referred in many papers as an *informal derivation*, dont worry, you are allowed to use informal derivations too. Look at Euler, he did it over and over and some of us thank him for his insight and pemetration). Another well-known manner to demonstrate the above formula is to regularize  $1/r$  in terms of a parameter  $a$  so that the regularized function is well-behaved everywhere for  $a \neq 0$ . For example, Jackson in his book *Classical Electrodynamics*, regularizing  $1/r$  as  $1/\sqrt{r^2 + a^2}$  shows that

$$\nabla^2 \frac{1}{r} = \lim_{a \rightarrow 0} \nabla^2 \frac{1}{\sqrt{r^2 + a^2}} = -4\pi \delta^{(3)}(\mathbf{r}).$$

Try here the first two methods.

27. • The following formula is useful in many atomic calculations. If vectors  $\mathbf{r}_1, \mathbf{r}_2$  denote the position of two particles with respect to the coordinate origin, the inverse of the distance  $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$  is

$$\frac{1}{r_{12}} = \frac{1}{r_1} \sum_{l=0}^{\infty} \left(\frac{r_2}{r_1}\right)^l P_l(\cos \theta), \quad r_1 > r_2,$$

where  $\theta$  is the angle between vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$  and  $P_l(x)$  denote the Legendre polynomials. The first few are, as you well know,  $P_0(x) = 1$ ,  $P_1(x) = x$ ,  $P_2(x) = \frac{1}{2}(3x^2 - 1)$ ,  $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ , etc. Of course if  $r_2 > r_1$  you just change  $r_1$  by  $r_2$  in the previous expression.

#### BRIEF INTRODUCTION TO ELEMENTARY PARTICLES

28. [French] Demostrar que los procesos que se señalan a continuación son imposibles desde un punto de vista dinámico (quiere decir que no se conserva energía-momento: o falla conservación de energía o conservación de momento o ambas):

- (a) Un fotón choca con un electrón en reposo y entrega toda su energía al electrón.
- (b) Un fotón situado en el espacio libre se transforma en un electrón y un positrón.
- (c) Un positrón rápido y un electrón en reposo se destruyen mutuamente dando lugar a un solo fotón.

*Ayuda:* Si una reacción no puede tener lugar en un sistema de referencia no puede tener lugar en ningún otro sistema de referencia pues se pasa de unos a otros mediante transformaciones de Lorentz. Demostrar que en un sistema de referencia dado, el más sencillo en cada caso, la reacción planteada es imposible. Positrón: la antipartícula del electrón: misma masa y carga pero ésta positiva.

29. [Conservation of energy and momentum] Some conservation laws are absolute whereas some others, parity for instance, are conserved by some of the three fundamental interactions but not all. Three means here that Gravity is excluded. Conservation of energy and momentum is absolute. This exercise is about it.



Un cuerpo de masa  $m_0$  que se encuentra en reposo se desintegra en dos partes cuyas masas son  $m_1$  y  $m_2$ . Demostrar que la energía de cada parte es

$$E_1 = \frac{c^2}{2m_0} (m_0^2 + m_1^2 - m_2^2), \quad E_2 = \frac{c^2}{2m_0} (m_0^2 - m_1^2 + m_2^2).$$

En el libro de Griffiths también viene este ejercicio. Con distintas letras pero el mismo.

30. [DAMTP, University of Cambridge] En el sistema de laboratorio una partícula de masa  $m_1$  y energía  $E_1$  colisiona con otra partícula de masa  $m_2$  supuesta en reposo. Demostrar que la energía combinada en el sistema de centro de masas es

$$\sqrt{(m_1^2 + m_2^2) c^4 + 2E_1 m_2 c^2}.$$

Usando este resultado demostrar que si la colisión tiene lugar entre dos protones (es decir,  $m_1 = m_2 = m_p$ ) moviéndose el primero con energía  $E_1$  y suponiendo el segundo en reposo, se puede formar un par protón-antiprotón (además de los dos protones que había originalmente) si  $E_1 \geq 7m_p c^2$ . A este valor mínimo de la energía se le llama *energía umbral*.

*Nota:* El antiprotón es la antipartícula del protón: misma masa y carga que el protón sólo que la carga es negativa. Un sistema en el que el momento total (el 3-momento total) es cero se llama sistema de *centro de masas* (*centre-of-momentum frame* en inglés).

*Moraleja:* Si un problema parece muy complicado de resolver en el sistema de laboratorio pruébese a analizarlo en el sistema de centro de masas.

31. [DAMTP, University of Cambridge] [A Lorentz transformation] En un sistema inercial  $S$  un fotón con energía  $E$  se mueve en el plano  $xy$  formando un ángulo  $\theta$  con el eje  $x$ . Demostrar que: (1) en un segundo inercial  $S'$  cuya velocidad relativa con respecto a  $S$  es  $v$  dirigida según el eje  $x$ , la energía y el ángulo del fotón están dados por

$$E' = \gamma E (1 - \beta \cos \theta), \quad \cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta},$$

donde  $\beta = v/c$  y  $\gamma = (1 - \beta^2)^{-1/2}$ ; (2) escribir  $E$  y  $\cos \theta$  como funciones de  $E'$  y  $\cos \theta'$ . (3) Demostrar que un fotón moviéndose en la dirección  $x$  experimenta un cambio en la frecuencia dado por el factor  $\sqrt{(1 - \beta)/(1 + \beta)}$ . A este cambio se le llama *efecto Doppler relativista*. (4) A continuación considere una fuente de fotones que está en reposo en  $S'$ . Considere los fotones emitidos en la dirección hacia delante, i.e.,  $\cos \theta' > 0$ . Demuestre que si  $\beta$  es muy próximo a la unidad, esos fotones se verán desde  $S$  concentrados en un estrecho cono alrededor de  $\theta = 0$ . Esto se conoce como *headlight effect*.

32. i) Complete these reactions attending to lepton number conservation: a)  $\pi \rightarrow \mu + \nu$ , b)  $\mu \rightarrow e + \nu + \nu$ . ii) Why is this reaction never observed  $\mu \leftarrow e + \gamma$ ? (look at the piece of paper I gave to you. This decay is not there). iii) Using lepton number, electron number and muon number conservation complete with  $\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu$  the reactions: a) neutron decay,  $n \rightarrow p + e + \nu$ , b) pion decay,  $\pi \rightarrow \mu + \nu$ , c) muon decay,  $\mu \rightarrow e + \nu + \nu$ .
33. a) Why is the proton stable?