Física Cuántica I.

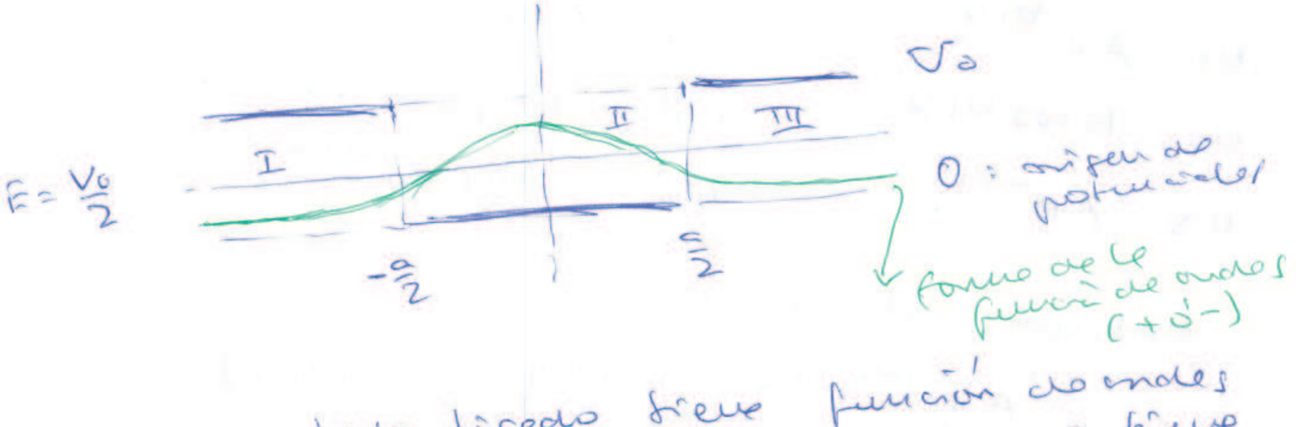
Examen Junio 2013 [2ª parte] [MJSRP070]

1) Una partícula de masa m se halla en el pozo de potencial

$$V(x) = \begin{cases} 0 & -a/2 < x < a/2 \\ V_0 & |x| > a/2, \end{cases}$$

donde V_0 es una constante positiva. La partícula tiene un solo estado ligado de energía $E = V_0/2$. Calcular la probabilidad de encontrar a la partícula en la región i) clásicamente permitida, ii) clásicamente prohibida.

<u>Nota</u>: utilice la condición $E = V_0/2$ desde el principio, pues <u>simplifica considerablemente</u> los cálculos. El pozo va de $-a/2$ a $a/2$ para que usted imponga una paridad definida a la función de onda. Los resultados pedidos son números (0.17, ... o lo que sea).



Un estado ligado tiene función de ondas normalizable, o sea, que en $|x| \to \infty$ tiene que tender a <u>cero</u> con suficiente rapidez. Sabiendo eso y que el estado fundamental es <u>par</u> respecto a $x=0$, se escribe la función de onda. Sin más bla blabla...

Wave function: ground state is <u>even</u> about $x=0$

$u_1 = A e^{k_1 x}$, $k_1 = +\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

region I

$u_2 = B \cos k_2 x$, $= +\sqrt{2m V_0/2\hbar^2}$

$= \sqrt{m V_0/\hbar^2}$.

$u_3 = A e^{-k_1 x}$,

$k_2 = \sqrt{2mE/\hbar^2}$

$= \sqrt{m V_0/\hbar^2}$.

Then $$\boxed{k_1 = k_2 = +\sqrt{\frac{m V_0}{\hbar^2}}}$$

Note: <u>siempre, siempre, siempre</u> las k's se toman <u>reales positivas</u> (ver función de ondas arriba). Si se quiere negativa se pone $-k_1$ ó $-k_2$, si compleja se pone ik_1 ó $-ik_1$. ¡Queda dicho!

Wave function	Derivative of the w.f.
$u_1 = A e^{k_1 x}$	$u_1' = A k_1 e^{k_1 x}$
$u_2 = B \cos k_1 x$	$u_2' = -B k_1 \sin k_1 x$
$u_3 = A e^{-k_1 x}$	$u_3' = -A k_1 e^{-k_1 x}$

Matching at $x = -a/2$

$$A e^{-k_1 a/2} = B \cos \frac{k_1 a}{2} \qquad (u\text{'s})$$

$$A \cancel{k_1} e^{-k_1 a/2} = B \cancel{k_1} \sin \frac{k_1 a}{2} \qquad (\text{derivative's}).$$

Note that the matching conditions

$$A e^{-k_1 a/2} = B \cos k_1 \frac{a}{2} = B \sin k_1 \frac{a}{2}$$

mean that

$$\cos \frac{k_1 a}{2} = \sin \frac{k_1 a}{2}$$

or

$$\tan \frac{k_1 a}{2} = 1 \quad , \quad k_1 \frac{a}{2} = \frac{\pi}{4}$$

or

$$\boxed{k_1 a = \frac{\pi}{2}}$$

[This condition gives also the ground state energy

$$E = \frac{V_0}{2} = \frac{\pi^2 \hbar^2}{8 m a^2} .$$

]

The amplitudes A and B are related by

$$A e^{-\pi/4} = B \cos \frac{\pi}{4} .$$

Probability of finding the particle in the forbidden region [classically speaking]:

$$P_{forbidden} = \int_{-\infty}^{-a/2} dx\, u_1^2 + \int_{a/2}^{\infty} dx\, u_3^2$$

wf are real ($|\ |^2$ is $(\ )^2$).

$$= 2 \int_{a/2}^{\infty} dx\, u_3^2$$

$$= 2A^2 \int_{a/2}^{\infty} dx\, e^{-2k_1 x}$$

$$= 2A^2 \frac{1}{2k_1} e^{-k_1 a}$$

$$= \frac{A^2}{k_1} e^{-\pi/2} = \frac{A^2 a}{k_1 a} e^{-\pi/2} = \frac{A^2 a}{\pi/2} e^{-\pi/2}$$

$A e^{-\pi/4} = B \cos \frac{\pi}{4}$

$$= \frac{B^2}{2} \frac{a^2}{\pi/2}$$

$$= \frac{B^2 a}{\pi} .$$

$$P_{allowed}: \int_{-a/2}^{a/2} dx\, u_2^2 = 2\int_0^{a/2} dx\, u_2^2$$

$$= 2B^2 \int_0^{a/2} dx \cos^2 k_1 x$$

$$= 2B^2 \int_0^{a/2} dx \left[\frac{1}{2} + \frac{1}{2}\cos 2k_1 x\right]$$

$$= 2B^2 \left[\frac{a}{4} + \frac{1}{4k_1}\left[\sin 2k_1 x\right]_0^{a/2}\right]$$

$$\sin \underbrace{k_1 a}_{\pi/2} = 1 \Rightarrow \frac{2B^2}{4}\left[a + \frac{1}{k_1}\right] = \frac{2B^2}{4}\left[a + \frac{a}{k_1 a}\right]$$

$$= \frac{B^2 a}{2}\left[1 + \frac{2}{\pi}\right].$$

[Note: $\frac{P_{forb}}{P_{allowed}} = \frac{\pi}{\pi+2}$

But $P_{forbidden} + P_{allowed} = 1$, so

$$\frac{B^2 a}{\pi} + \frac{B^2 a}{2}\left[1 + \frac{2}{\pi}\right] = 1, \qquad B^2 a = \frac{2\pi}{\pi+4}$$

[$B^2 a\left[\frac{1}{\pi} + \frac{1}{2} + \frac{1}{\pi}\right] = B^2 a\left[\frac{1}{2} + \frac{2}{\pi}\right] = B^2 a \frac{\pi+4}{2\pi} = 1$

and

$$\boxed{P_{forbidden} = \frac{2}{4+\pi} \simeq 0.28} \qquad (28\%)$$

$$\boxed{P_{allowed}: \frac{2+\pi}{4+\pi} \simeq 0.72}$$

& $\boxed{E = \frac{\pi^2 \hbar^2}{8ma^2}}$, $\boxed{k_1 a = \frac{\pi}{2}}$

No more to be justified!!

[Un problema bien majo: con introducción, nudo y desenlace].

(2) El oscilador armónico isótropo en 2 dimensiones tiene hamiltoniano

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{2} m\omega^2 (x^2 + y^2),$$

siendo los estados estacionarios de la forma $X(x)\,Y(y)$ y los niveles de energía igual a $E = (n_x + n_y + 1)\hbar\omega$, con n_x, n_y enteros no negativos.

i) cuántos estados linealmente independientes tienen energía $E = (N+1)\hbar\omega$?

ii) Igual que hice yo en clase, pruebe que el hamiltoniano es invariante bajo rotaciones en torno a un eje perpendicular al plano xy (el eje z) porque después de la rotación se sigue cumpliendo la ecuación de Schrödinger

Alternativa importante: Sólo si el párrafo anterior no le resulta familiar, calcule el conmutador de H con $L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$.

iii) Encuentre una combinación lineal de los estados de energía $E = 2\hbar\omega$ que sea propia de L_z con valor propio $\hbar$ y otra con valor propio $-\hbar$.

Usted puede necesitar (o no) alguno de los siguientes datos:

$$l = \sqrt{\hbar/m\omega}, \quad u_n(x) = C_n H_n\left(\frac{x}{l}\right) e^{-x^2/2l^2}, \quad H_2(x) = 4x^2 - 2,$$

$$H_0(x) = 1, \quad H_1(x) = 2x,$$

donde l es la longitud natural de un oscilador armónico, u_n es el enésimo autoestado en una dimensión, $H_n(x)$ los polinomios de Hermite y C_n constantes de normalización. Muchos libros utilizan la constante $\alpha = 1/l$ en lugar de l.

i) Degeneracy

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{2} m\omega^2 (x^2+y^2)$$

$$|n_x n_y\rangle \equiv X_{n_x}(x)\, Y_{n_y}(y)$$

$$H|n_x, n_y\rangle = E|n_x n_y\rangle \quad , \quad E = \hbar\omega(n_x+n_y+1)$$

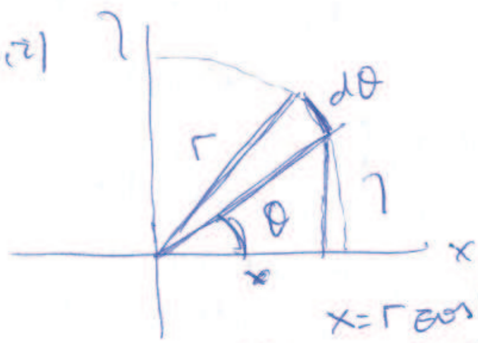
$$n_x, n_y = 0, 1, 2, \dots$$

N	n_x	n_y	#
0	0	0	1
1	1	0	2
	0	1	
2	2	0	3
	1	1	
	0	2	
3	3	0	4
	2	1	
	1	2	
	0	3	

this number is $N+1$

Thus, $N+1$ states have the same energy $E = \hbar\omega(N+1)$. are linearly independent, of course.

ii)



Rotation around z-axis (r: distance from the point (x,y) to the rotation axis does not change)

$$x = r\cos\theta \quad , \quad dx = -r\sin\theta\, d\theta = -y\, d\theta$$

$$y = r\sin\theta \quad , \quad dy = r\cos\theta\, d\theta = x\, d\theta$$

r no lo derivo porque no cambia con el giro.

Under a rotation $(x,y) \to (x',y') = (x,y) + (dx, dy)$,

$$x' = x - y\, d\theta$$
$$y' = y + x\, d\theta$$

φ es $|n_x n_y\rangle$

$H\varphi(x,y) = E\varphi(x,y)$ before the rotation

$H\varphi(x',y') = E\varphi(x',y')$: after the rotation (same E) → lo que importa.

But

$$\varphi(x',y') = \varphi(x,y) + \frac{\partial\varphi}{\partial x}\Big|_0 dx + \frac{\partial\varphi}{\partial y}\Big|_0 dy + \cdots\cdots$$

2nd order terms

Taylor expansion

0 is at (x,y)

$$= \varphi(x,y) - d\theta\, y \frac{\partial\varphi}{\partial x}\Big|_0 + d\theta\, x \frac{\partial\varphi}{\partial y}\Big|_0 + \cdots,$$

that substituted in $H\varphi(x',y') = E\varphi(x',y')$ gives

$$H\varphi(x',y') = \cancel{H\varphi(x,y)} + d\theta \cdot H\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\varphi + \cdots$$

$$= E\left[\cancel{\varphi(x,y)} + d\theta\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\varphi + \cdots\right]$$

zero order — 1st order

First order: ($d\theta$ cancels),

$$H\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\varphi = E\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\varphi,$$

that it is interpreted as: "if φ is a proper state of H with energy E, $\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\varphi$ is also a proper state with the same energy E. Then $\left[H, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] = 0$

este método lo "calculé" (dice quien es)

[and $\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ has to be an important operator associated to rotations about the z axis]

Alternatively, show that $\left[H, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] = 0$ but write $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ as $xP_y - yP_x$ instead.

Use more

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$
$$= xP_y - yP_x.$$

to calculate $[H, L_z]$. Divide the hamiltonian in two parts (that do not compensate: one is multiplied by $\frac{1}{2m}$, the other by $\frac{1}{2}m\omega^2$, constants that are not related!)

$P_x^2+P_y^2$

x^2+y^2

$$[P_x^2+P_y^2, xP_y - yP_x] = [P_x^2, xP_y] - [P_y^2, yP_x]$$

Calculate,

$$[P_x^2, xP_y] = P_y[P_x^2, x] = P_yP_x[P_x, x] + P_y[P_x, x]P_x$$
$$= -2i\hbar P_xP_y$$

commutes with x, P_x

$[x, P_x] = i\hbar\cdot\mathbb{1}$.

$$[P_y^2, yP_x] = P_x[P_y^2, y] = \text{same as} = -2i\hbar P_xP_y$$

Obviously,

$$[P_x^2+P_y^2, xP_y - yP_x] = -2i\hbar P_xP_y + 2i\hbar P_xP_y = 0$$

Similarly with $[x^2+y^2, xP_y - yP_x] = [x^2, -yP_x] + [y^2, xP_y] = 0.$

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iii) $E = 2\hbar\omega$ corresponds to $N=1$. Two states $|10\rangle$, $|01\rangle$ (obvious notation!) have the same energy. The states are

$$|10\rangle = u_1(x)u_0(y) = c_0c_1 H_1\left(\frac{x}{l}\right)H_0\left(\frac{y}{l}\right)e^{-r^2/2l^2}$$
$$= c_0c_1\left[\frac{2x}{l}\right]e^{-r^2/2l^2}$$

sabe verlo?

$$|01\rangle = c_0 c_1 \left[\frac{2y}{\ell}\right] e^{-r^2/2\ell^2}$$

calculate $L_z|10\rangle$ and $L_z|01\rangle$. Use $\boxed{L_z f(r) = 0}$

$$L_z|10\rangle = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)|10\rangle$$

$$= -i\hbar\left[0 - \frac{2y}{\ell}\right] c_0 c_1 e^{-r^2/2\ell^2}$$

$$= i\hbar|01\rangle \,,$$

$$L_z|01\rangle = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)|01\rangle$$

$$= -i\hbar\left[\frac{2x}{\ell} - 0\right] c_1 c_0 e^{-r^2/2\ell^2}$$

$$= -i\hbar|10\rangle .$$

From $L_z|10\rangle = i\hbar|01\rangle$ and $L_z|01\rangle = -i\hbar|10\rangle$ is easy to write the combination

$$\varphi_1 = |10\rangle + i|01\rangle$$

that satisfies $L_z\varphi_1 = \hbar\varphi_1$, and

$$\varphi_2 = |10\rangle - i|01\rangle \quad \text{(orthogonal to)}$$

that satisfies $L_z\varphi_2 = -\hbar\varphi_2$.

Note: In 2 dimensions there is no L_x, L_y (no existen). Only L_z. $\vec{L} = (0, 0, L_z)$ and $\vec{L}^2 = L_z^2$.

note too:

N	$n_x\ n_y$	degeneracy	eigenvalue of L_z
0	0 0	1	0
1	1 0; 0 1	2	1, -1
2	2 0; 1 1; 0 2	3	2, 0, -2
3	30, 21, 12, 03	4	3, 1, -1, -3