Examen de Julio 2018
[Septiembre]

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

13 de Julio de 2018

IET: Examen Final de Julio de Cálculo, Parte 2
Curso 2017/18

Nombre y Apellidos:

Firma y DNI:

El examen son 200 puntos normalizables a 10.

1. Escribir el resultado y **nada más que el resultado**, no hace falta justificación alguna, de los siguientes ejercicios. Use los recuadros correspondientes.

   a) [30pt] Si $xy^m = n$, donde m y n son constantes, encontrar $\frac{dy}{dx}$ y $\frac{d^2y}{dx^2}$.

   $\frac{dy}{dx} = -\frac{y}{mx}$, $\frac{d^2y}{dx^2} = \frac{y(1+m)}{m^2x^2}$

   b) [5pt+10pt] Encontrar las constantes a y b para que $u = x^3 + 3axy^2 - 3bxy$ sea solución de la ecuación de Laplace $u_{xx} + u_{yy} = 0$,

   $a = -1$, $b =$ arbitrario (o sea cualquier valor)

   c) [5pt+20pt] Calcular el vector normal $\mathbf{N}$ a la superficie $x^2 + y^2 - z^2 = 0$ en el punto $(3, 4, c)$ de manera que $\mathbf{N}$ sea paralelo al vector $(1/2, p, q)$. Hallar previamente c sabiendo que es un valor positivo.

   $c = 5$, $\mathbf{N} = \left(\frac{1}{2}, \frac{2}{3}, -\frac{5}{6}\right)$

   d) [30pt] Encontrar la derivada direccional del campo $f(x, y, z) = xy^2 + yz$ en el punto $(1, 1, 2)$ en la dirección del vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. El resultado es el número

   0 (cero)

   e) [30pt] Escribir un vector $\mathbf{t}$ tangente a la superficie de nivel $xy^2 + yz = 3$ del campo del ejercicio anterior en el punto $(1, 1, 2)$.

   $\mathbf{t} = (2, -1, 2)$ o cualquier combinación lineal de $\{(1,0,-1), (0,1,-4)\}$

   o cualquier vector perpendicular $(1, 4, 1)$.

2. Problemas de los de toda la vida. **Con justificación.**

   a) [30pt] Sea el campo vectorial $\mathbf{V} = (2x^2 - yz)\mathbf{i} - 2yz\mathbf{j} + (z^2 - 2xz)\mathbf{k}$. ¿Es posible encontrar un vector $\mathbf{A}$ tal que $\mathbf{V} = \nabla \times \mathbf{A}$? No puntúa decir sí o no a secas, debe dar una razón.

   b) [40pt] Sea $\mathbf{F} = (x + 2y, -2y)$. Calcular la integral

   $$\oint_C \mathbf{F} \cdot \mathbf{dl}$$

   siendo C la línea cerrada $x^2 + y^2 = 1$ orientada positivamente.

①

$\rightarrow -\frac{y^{m+1}}{nm}.$

① a) $xy^m = n$

Differentiating,

$$y^m\,dx + m x y^{m-1}\,dy = 0$$

From here you choose, either

$$\boxed{\frac{dy}{dx} = -\frac{y}{mx}} \; = \quad \text{or} \quad \frac{dx}{dy} = -m\frac{x}{y}.$$

Take $\frac{dy}{dx} = -\frac{y}{mx}$ and derive with respect to x, then

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)\cdot\frac{1}{mx} + \frac{y}{mx^2}$$

$$= \frac{y}{m^2x^2} + \frac{y}{mx^2}$$

subst $\frac{dy}{dx} = -\frac{y}{mx}$

$$= \frac{y}{m^2x^2}\left[1+m\right].$$

Result $\boxed{\frac{d^2y}{dx^2} = \frac{y(1+m)}{m^2x^2}.} = \frac{(1+m)}{n^2m^2}y^{2m+1}$, también

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b) $u = x^3 + 3axy^2 - 3bxy$

$u_x = 3x^2 + 3ay^2 - 3by$ ‖ $u_y = 6axy - 3bx$

$u_{xx} = 6x$ ‖ $u_{yy} = 6ax$

Then $\boxed{a = -1, \quad b \text{ whatever, arbitrary}}$

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c) Surface $x^2 + y^2 - z^2 = 0$

If a point $(3, 4, c)$ is on the surface then $c = 5$ or -5. Here $\boxed{c = 5}$

$F(x,y,z) = x^2 + y^2 - z^2 = 0$

A vector normal to the surface at the point (x,y,z) is $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x, 2y, -2z)$

but it is also parallel to $(x, y, -z)$ or $(3, 4, -5)$ at the point $(3,4,5)$. In our case

$$\vec{N} \parallel (3,4,-5) = 6\left(\tfrac{1}{2}, \tfrac{2}{3}, -\tfrac{5}{6}\right)$$

$$\boxed{\vec{N} = \left(\tfrac{1}{2}, \tfrac{2}{3}, -\tfrac{5}{6}\right)} \quad ; \quad \left[\begin{array}{l} p \text{ is } 2/3 \\ q \text{ is } -5/6 \end{array}\right]$$

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d) $f = xy^2 + yz$

$$\text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y^2, 2xy + z, y)\Big|_{\substack{\text{at the} \\ \text{point} \\ (1,1,2)}} = (1,4,1)$$

$\vec{u}$ = unitary vector in the direction of $(2,-1,2)$ is $\vec{u} = \frac{1}{3}(2,-1,2)$

The directional derivative is then

$$\vec{u} \cdot \vec{\text{grad}}\, f \text{ at the point } (1,1,2) = \vec{u} \cdot (1,4,1)$$
$$= \frac{1}{3}(2,-1,2)\cdot(1,4,1)$$
$$= \frac{1}{3}(2-4+2) = \underline{\underline{0}}$$

scaler vector product

The number is 0

e) This 0 means that along the direction of $\vec{v} = (2,-1,2)$ the function f does not change...

[P is the point $(1,1,2)$]. It happens then that the vector $(2,-1,2)$ is tangent to a level surface.

$$\boxed{\vec{t} = (2,-1,2)}$$

at the point $(1,1,2)$.

level surface is $\underline{\underline{f = \text{const}}}$

in our case the constant is 3

The problem can be solved as follows too:
A tangent vector to the surface $xy^2+yz=3$ is $\vec{dl}=(dx,dy,dz)$ with

$$y^2dx+(2xy+z)dy+y\,dz=0$$

from where

$$dz=-y\,dx-\left(2x+\frac{z}{y}\right)dy$$

that gives

$$\vec{dl}=(dx,dy,dz)=\left(dx,dy,-y\,dx-\left(2x+\frac{z}{y}\right)dy\right)$$
$$=dx\,(1,0,-y)$$
$$+dy\left(0,1,-\left(2x+\frac{z}{y}\right)\right).$$

At the point $(1,1,2)$ the vector

$(1,0,-1)$

or the vector

$(0,1,-4)$

are tangent to the surface $xy^2+yz=3$, or a linear combination of $\{(1,0,-1),(0,1,-4)\}$.

In particular

$$2(1,0,-1)-(0,1,-4)=(2,-1,2).$$

as stated. otra manera: un vector perpendicular a $(1,4,1)$ es tangente a la superficie en $P(1,1,2)$

Ya lo había dicho. !!! quizás lo más sencillo

o O o

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② a) If $\vec{V}=\vec{\nabla}\times\vec{A}$, then $\operatorname{div}\vec{V}=0$. [clase]

In our case, with $\vec{V}=(V_1,V_2,V_3)$

$$\operatorname{div}\vec{V}=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}$$
$$=4x-\cancel{2z}+\cancel{2z}-2x$$
$$=2x\quad.\quad \underline{\underline{no}}, \text{ it is not } \equiv 0$$

idénticamente cero.

since

$$\operatorname{div}\vec{V}\neq 0 \Rightarrow \nexists \text{ such vector } \vec{A}$$

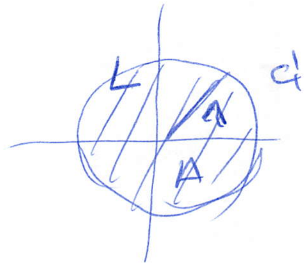
(el que busque, no existe).

b) Quick answer: "Green's theorem in the plane states that for our $\vec{F}$

$$\oint_C \vec{F}\cdot d\vec{l} = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

with $\vec{F} = (P,Q)$ and A the area enclosed by C. C oriented in the anticlock... manner."

In our case



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 2 = -2$$

and

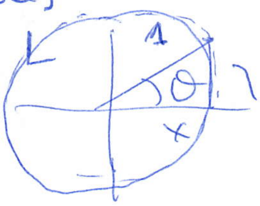
$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -2 \underbrace{\iint_A dx\,dy}_{= \pi \text{: the area of the circle of radius 1}} = -2\pi$$

Conclusion: $\boxed{\oint_C \vec{F}\cdot d\vec{l} = -2\pi}$

Fletxes de vector: obligatòries

——— 0 ——— 0 ———

Calculating $\oint_C \vec{F}\cdot d\vec{l}$ explicitly: Use the change of variables



$r = 1$

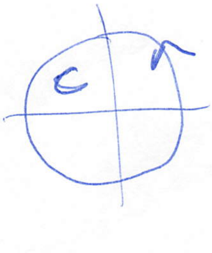
$$x = \cos\theta$$
$$y = \sin\theta$$

parametrització de la corba C (un paràmetre, no dos).

because $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$ is a good choice.

Now

$$d\vec{l} = (dx, dy) = (-\sin\theta, \cos\theta)\, d\theta$$
$$\vec{F} = (\cos\theta + 2\sin\theta, -2\sin\theta)$$
$$\vec{F}\cdot d\vec{l} = -(3\sin\theta\cos\theta + 2\sin^2\theta)\, d\theta$$



$$\oint_C \vec{F}\cdot d\vec{l} = -\int_0^{2\pi} d\theta \left[3\sin\theta\cos\theta + 2\sin^2\theta\right]$$

$$= -\frac{3}{2}\underbrace{\left[\sin^2\theta\right]_0^{2\pi}}_{\text{this is } 0} - \frac{2}{2}\int_0^{2\pi} d\theta \,(1 - \underbrace{\cos 2\theta}_{0 \text{ this integral}})$$

$$= -\int_0^{2\pi} d\theta = -2\pi.$$

as expected!!

$$\boxed{\oint \vec{F}\, d\vec{l} = -2\pi}$$