On the dynamics of some hyperbolic Universe models driven by a spontaneous symmetry breaking scalar field

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Introduction

- Cosmological principle and accelerated expansion imply the existence of a starting point, Big Bang singularity, basis of Big Bang model.
- In the inflationary theory, the Universe had an exponential expansion, solving the flatness problem, the horizon problem and explaining the large-scale structure.
- The line element (metric) that describes the Universe is the Friedmann-Robertson-Walker (FRW) metric, containing a scale factor $S(t)$ and a curvature parameter $k = 0, \pm 1$.
- Cosmological observations suggest a nearly flat Universe, but: dark energy, CMB anisotropies, topology of spacetime.

Hyperbolic FRW metric

Consider the 4D dynamic FRW metric, with negative curvature and scale factor $S(t) = a_0\, e^{f(t)}$:

$$ds_4^2 = a_0^2 e^{2f(t)} \left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] - dt^2$$

We introduce the pseudo-orthonormal frame $e_{a(a=\overline{1,4})}$, whose corresponding dual base $\{\omega^a\}$, is $\omega^1 = a_0 e^f\, d\chi$, $\omega^2 = a_0 e^f\, \sinh\chi\, d\theta$, $\omega^3 = a_0 e^f\, \sinh\chi \sin\theta\, d\varphi$, $\omega^4 = dt$, in order to employ Cartan's first and second equations,

$$d\omega^a = \Gamma^a_{.[bc]}\, \omega^b \wedge \omega^c$$
$$\mathcal{R}_{ab} = d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^c_{.b},$$

where Γ_{abc} are the *connection coefficients*, giving the *connection one-forms* $\Gamma_{ab} = \Gamma_{abc}\, \omega^c$ and $\mathcal{R}_{ab} = R_{abcd}\, \omega^c \wedge \omega^d$ are the *curvature two-forms*.

By using Cartan first eq., we obtain the *connection one-forms*:

$$\Gamma_{12} = -a_0^{-1} e^{-f} \coth\chi\,\omega^2, \qquad \Gamma_{13} = -a_0^{-1} e^{-f} \coth\chi\,\omega^3,$$
$$\Gamma_{14} = f_{|4}\,\omega^1, \qquad \Gamma_{23} = -a_0^{-1} e^{-f} \sinh^{-1}\chi \cot\theta\,\omega^3,$$
$$\Gamma_{24} = f_{|4}\,\omega^2, \qquad \Gamma_{34} = f_{|4}\,\omega^3,$$

while from Cartan second eq., we have the *curvature two-forms*:

$$\mathcal{R}_{12} = -(a_0^{-2} e^{-2f} - f_{|4}^2)\,\omega^1 \wedge \omega^2, \qquad \mathcal{R}_{13} = -(a_0^{-2} e^{-2f} - f_{|4}^2)\,\omega^1 \wedge \omega^3\,,$$
$$\mathcal{R}_{14} = -(f_{|4}^2 + f_{|44})\,\omega^1 \wedge \omega^4, \qquad \mathcal{R}_{23} = -(a_0^{-2} e^{-2f} - f_{|4}^2)\,\omega^2 \wedge \omega^3\,,$$
$$\mathcal{R}_{24} = -(f_{|4}^2 + f_{|44})\,\omega^2 \wedge \omega^4, \qquad \mathcal{R}_{34} = -(f_{|4}^2 + f_{|44})\,\omega^3 \wedge \omega^4.$$

We can compute the *Riemann curvature tensor* and hence the Ricci tensor:

$$R_{\alpha\alpha} = -2a_0^{-2}e^{-2f} + 3f_{|4}^2 + f_{|44};$$
$$R_{44} = -3(f_{|4}^2 + f_{|44}),$$

together with the following Ricci scalar:

$$R = -6a_0^{-2}e^{-2f} + 12f_{|4}^2 + 6f_{|44}.$$

Finally, we obtain the Einstein tensor, $G_{ab} = R_{ab} - (1/2)R\, g_{ab}$, where $g_{ab} = diag(1, 1, 1, -1)$:

$$G_{\alpha\alpha} = -2\ddot{f} - 3\dot{f}^2 + a_0^{-2}e^{-2f}$$
$$G_{44} = 3\dot{f}^2 - 3a_0^{-2}e^{-2f}.$$

Higgs-like field

Consider, as matter-source, a homogeneous, time-dependent scalar field $\phi(t)$, with *spontaneous Z_2-symmetry breaking* Lagrangian:

$$\mathcal{L}[\phi] = \frac{1}{2}\eta^{ab}\,\phi_{|a}\phi_{|b} - \frac{\mu^2}{2}\,\phi^2 + \frac{\lambda}{4}\,\phi^4,$$

where we define the quartically self-interacting potential:

$$V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{\mu^2}{2}\phi^2 = \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^4}{4\lambda},$$

and leading to the energy-momentum tensor:

$$T_{ab} = \phi_{|a}\phi_{|b} - \eta_{ab}\left[\frac{1}{2}\eta^{cd}\phi_{|c}\phi_{|d} + \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^4}{4\lambda}\right].$$

For a coherent field $\phi_{|\alpha} = 0$, the EM tensor has the components:

$$T_{\alpha\alpha} \equiv P = \frac{1}{2}\phi_{|4}^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 + \frac{\mu^4}{4\lambda},$$

$$T_{44} \equiv \rho = \frac{1}{2}\phi_{|4}^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^4}{4\lambda}$$

The Einstein equations $G_{ab} = k_0 T_{ab}$, where $k_0 = 8\pi/M_P^2$, take the form:

$$2\frac{d^2 f}{dt^2} + 3\left(\frac{df}{dt}\right)^2 - \frac{e^{-2f}}{a_0^2} = -k_0\left[\frac{1}{2}\phi_{|4}^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 + \frac{\mu^4}{4\lambda}\right],$$

$$3\left[\left(\frac{df}{dt}\right)^2 - \frac{e^{-2f}}{a_0^2}\right] = k_0\left[\frac{1}{2}\phi_{|4}^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^4}{4\lambda}\right].$$

Also, we can write down the corresponding Gordon equation:

$$\frac{d^2\phi}{dt^2} + 3\frac{df}{dt}\frac{d\phi}{dt} + \lambda\left(\phi^2 - \frac{\mu^2}{\lambda}\right)\phi = 0.$$

Rescaling

We perform a rescaling of the time coordinate and the field variable, such that they become dimensionless:

$$\tau = \frac{t}{a_0}, \qquad \psi = \sqrt{\frac{k_0}{2}}\phi.$$

Hence, with respect to the new variables τ, ψ, the Einstein-Gordon equations become:

$$2\frac{d^2 f}{d\tau^2} + 3\left(\frac{df}{d\tau}\right)^2 - e^{-2f} = -\left(\frac{d\psi}{d\tau}\right)^2 + \frac{\lambda a_0^2}{k_0}\left(\psi^2 - \frac{k_0\mu^2}{2\lambda}\right)^2 - \frac{\mu^4}{4\lambda}k_0 a_0^2,$$

$$3\left(\frac{df}{d\tau}\right)^2 - 3e^{-2f} = \left(\frac{d\psi}{d\tau}\right)^2 + \frac{\lambda a_0^2}{k_0}\left(\psi^2 - \frac{k_0\mu^2}{2\lambda}\right)^2 - \frac{\mu^4}{4\lambda}k_0 a_0^2,$$

$$\frac{d^2\psi}{d\tau^2} + 3\frac{df}{d\tau}\frac{d\psi}{d\tau} = -2\frac{\lambda a_0^2}{k_0}\left(\psi^2 - \frac{k_0\mu^2}{2\lambda}\right)\psi.$$

By subtracting the first two equations in the system and keeping the third one, we obtain the matter-curvature system:

$$\frac{d^2 f}{d\tau^2} + e^{-2f} = -\left(\frac{d\psi}{d\tau}\right)^2,$$
$$\frac{d^2\psi}{d\tau^2} + 3\frac{df}{d\tau}\frac{d\psi}{d\tau} = -2\frac{\lambda a_0^2}{k_0}\left(\psi^2 - \frac{k_0\mu^2}{2\lambda}\right)\psi.$$

with the rescaled potential:

$$V(\psi) = \frac{\lambda a_0^2}{k_0}\left(\psi^2 - \frac{k_0\mu^2}{2\lambda}\right)^2 - \frac{\mu^4}{4\lambda}k_0 a_0^2.$$

The vacuum expectation value v of the field ψ is given by the fixed point of the Hamiltonian:

$$v = 2\sqrt{\frac{\pi}{\lambda}}\frac{\mu}{M_P}.$$

Parametrization condition

We used a Weyl-type shift, as parametrization for the Einstein-Gordon system:

$$s = \sqrt{\frac{\lambda a_0^2}{k_0}}\,\tau, \qquad e^{-2F} = \frac{e^{-2f}}{\frac{\lambda a_0^2}{k_0}},$$

such that the scale factor is given by $S(t) = a_0\, e^{f(t)}$, where t is the *cosmic time*.

Then, using the fact that $k_0 = 8\pi/M_P^2$, the scale factor becomes:

$$S(s) = \sqrt{\frac{8\pi}{\lambda}}\, e^{F(s)}\, M_P^{-1},$$

while the cosmic time can be expressed as:

$$t = \sqrt{\frac{8\pi}{\lambda}}\, M_P^{-1}\, s.$$

Therefore, the system can be rewritten in the following form:

$$2\frac{d^2F}{ds^2} + 3\left(\frac{dF}{ds}\right)^2 - e^{-2F} = -\left(\frac{d\psi}{ds}\right)^2 + \left[(\psi^2 - v^2)^2 - v^4\right],$$

$$3\left(\frac{dF}{ds}\right)^2 - 3e^{-2F} = \left(\frac{d\psi}{ds}\right)^2 + \left[(\psi^2 - v^2)^2 - v^4\right],$$

$$\frac{d^2\psi}{ds^2} + 3\frac{dF}{ds}\,\frac{d\psi}{ds} = -2\left[\psi^2 - v^2\right]\psi,$$

where the vacuum expectation value v of the field ψ is:

$$v = 2\sqrt{\frac{\pi}{\lambda}}\,\frac{\mu}{M_P},$$

such that for $\mu/M_P = 10^{-3}$ and $\lambda = 10^{-6}$, we have $v = 2\sqrt{\pi}$.

Further, we end up with the *matter-curvature* system:

$$\frac{d^2F}{ds^2} + e^{-2F} = -\left(\frac{d\psi}{ds}\right)^2,$$

$$\frac{d^2\psi}{ds^2} + 3\frac{dF}{ds}\frac{d\psi}{ds} = -2\left[\psi^2 - v^2\right]\psi.$$

Slow-roll approximation: neglect the $(d\psi/ds)^2$ term and let $dF/ds \equiv h$, $d\psi/ds \equiv \Pi$. Numerically, we need *initial conditions*:

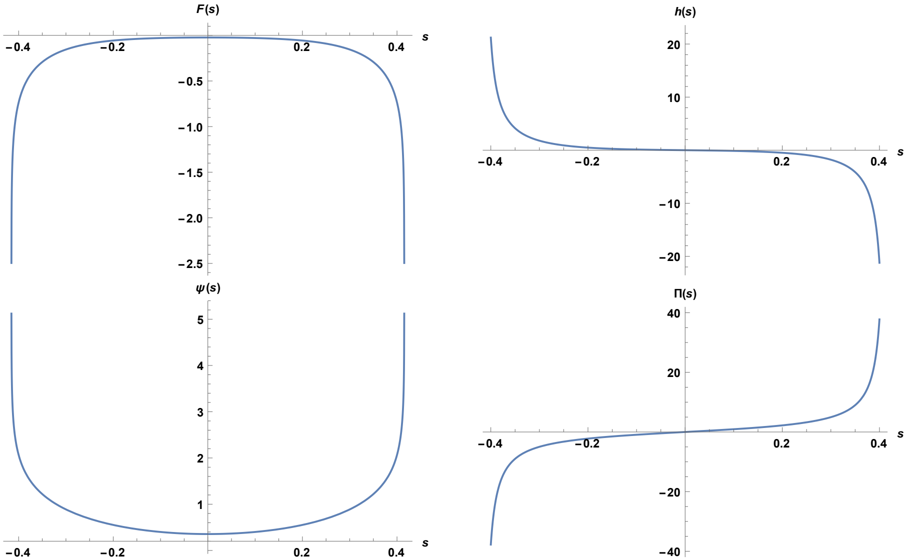
1. for $\psi_0 < v$ (*old inflation*), both the RHS and the LHS in Friedmann equation are negative, so we can take $\frac{dF}{ds}\big|_0 \equiv h_0 = 0$ and,

$$\psi_0 = \frac{v}{10}, \;\; \Pi_0 = 0, \;\; F_0 = -\frac{1}{2}\ln\left[\frac{1}{3}\left(v^4 - (\psi_0^2 - v^2)^2\right)\right] \approx -0.02$$

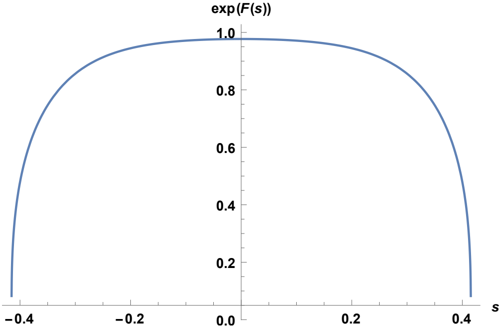
2. for $\psi_0 > v$ (*chaotic inflation*), we take $F_0 = 0$ and,

$$\psi_0 = 2v, \;\; \Pi_0 = 0, \;\; h_0 = \sqrt{\frac{1}{3}\left[\left(\psi_0^2 - v^2\right)^2 - v^4 + 1\right]} \approx 21.$$

Numerical solutions: $\psi_0 < v$ ($\lambda = 10^{-6}$)



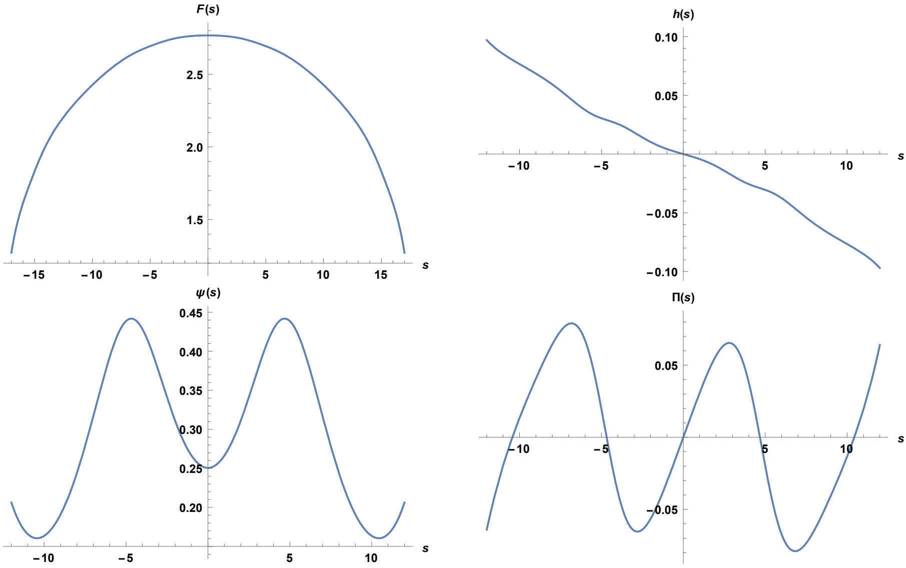
The *scale factor* $S(s)$ behavior is given by:



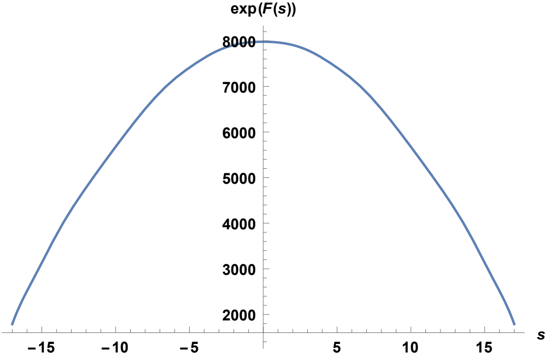
Further, we can consider also the case with $\lambda = 10^{-4}$, such that:

$$v = \frac{\sqrt{\pi}}{5}, \ \ \psi_0 = \frac{v}{\sqrt{2}}, \ \ \Pi_0 = 0, \ \ F_0 \approx 3$$

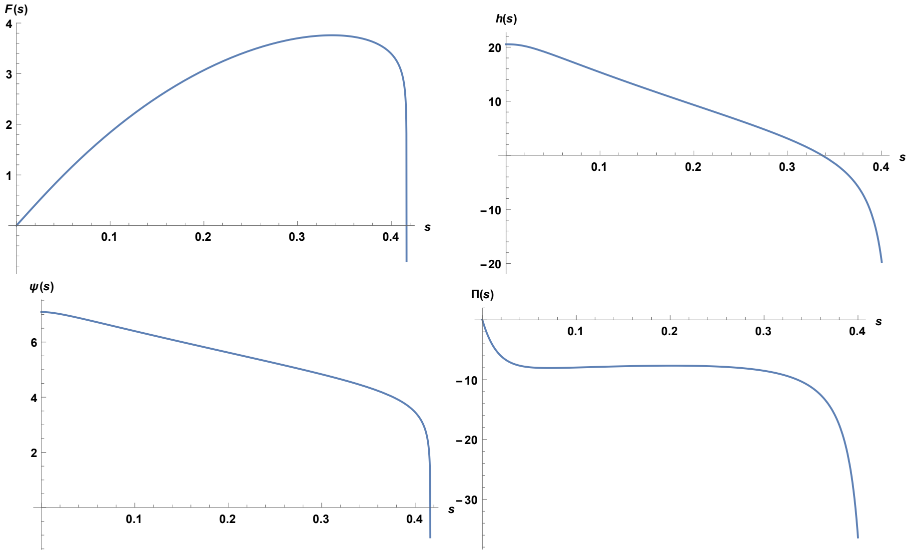
Numerical solutions: $\psi_0 < v$ ($\lambda = 10^{-4}$)



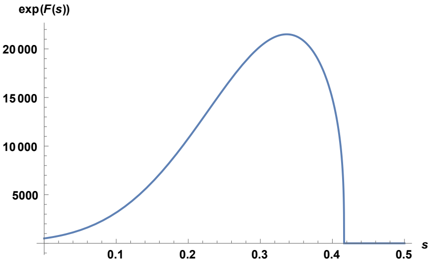
The *scale factor* has the following representation:

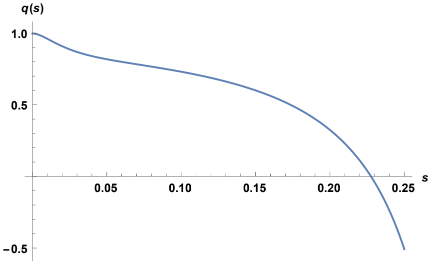


Numerical solutions: $\psi_0 > v$ $(\lambda = 10^{-6})$



The scale factor $S(s)$ and the acceleration parameter $q(s)$ are described by:





where,

$$q = \frac{F''(s)}{F'(s)^2} + 1$$

Conclusions

- Negative curvature FRW metric, sustained by a spontaneous symmetry breaking, dynamic Higgs-like scalar field matter-source.
- Perform a rescaling of the time and the field coordinates, to get dimensionless Einstein-Gordon and matter-curvature equations.
- Use a Weyl-type parametrization to get rid of the scale parameter a_0, and consider the cases $\psi_0 < v$ and $\psi_0 > v$, with the values $\lambda = 10^{-6}$ and $\lambda = 10^{-4}$, for $\mu/M_P = 10^{-3}$.
- We identify an inflationary behavior for $\psi_0 > v$, but also a possible singularity either in the metric or numerical, which need further investigation.

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