Three-form Cosmology

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What is a three-form?

Totally antisymmetric tensor with three indexes

\[ A_{ijk} = -A_{jik} \]

For example, a three-form defines the cross product

\[ (\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k \]

where \( \epsilon_{ijk} \) is the Levi-Civita symbol.
Three-form action

Action for the three-form $A_{\mu\nu\rho}$

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{48} F^2(A) - V(A^2) \right)$$

where

$$F_{\mu\nu\rho\sigma} = 4\nabla_{[\mu} A_{\nu\rho\sigma]} = \nabla_\mu A_{\nu\rho\sigma} - \nabla_\sigma A_{\mu\nu\rho} + \nabla_\rho A_{\sigma\mu\nu} - \nabla_\nu A_{\rho\sigma\mu}$$

We have the equations of motion:

$$\nabla \cdot F = 12V'(A^2)A$$

and due to antisymmetry we have the additional constraints:

$$\nabla \cdot V'(A^2)A = 0$$
The dual theory

We define duals as:

\[(\star F) = \frac{1}{4!} \epsilon_{\alpha \beta \gamma \delta} F^{\alpha \beta \gamma \delta} \equiv \Phi \]

\[(\star A)_{\alpha} = \frac{1}{3!} \epsilon_{\alpha \beta \gamma \delta} A^{\beta \gamma \delta} \equiv B_{\alpha} \]

which allows us to write the equivalent formulations of the Lagrangians:

\[\mathcal{L}_{IV}(F, \nabla \cdot F) = -\frac{1}{48} F^2 + 2 A^2 (\nabla \cdot F) V' (A^2 (\nabla \cdot F)) - V (A^2 (\nabla \cdot F))\]

\[\mathcal{L}_{III}(A, \nabla A) = -\frac{1}{3} [\nabla A]^2 - V (A^2)\]

\[\mathcal{L}_{I}(B, \nabla \cdot B) = \frac{1}{2} (\nabla \cdot B)^2 - V (-6 B^2)\]

\[\mathcal{L}_{0}(\Phi, \nabla \Phi) = -\frac{1}{2} \Phi^2 - 12 B^2 (\nabla \Phi) V' (-6 B^2 (\nabla \Phi)) - V (-6 B^2 (\nabla \Phi))\]

We recognise \(\mathcal{L}_{0}\) as a \(p(X, \Phi)\) theory with \(X = -\nabla^\mu \Phi \nabla_\mu \Phi\).
Equivalent formulations

Equivalence between Lagrangian descriptions:

\[
\begin{align*}
\mathcal{L}_f &= f \left( F^2(x) \right) - V(x^2), \\
\mathcal{L}_g &= g \left( (\nabla \cdot x)^2 \right) - U(x^2)
\end{align*}
\]
Gauge invariance and stability

\[ \mathcal{L} = -\frac{1}{48} F^2(A) - V(A^2) \]

\(F^2\) is invariant under \(A \rightarrow A + \nabla C\).
\(V(A^2)\) breaks this symmetry resulting in extra degrees of freedom. To see this we can make an expansion in Stückelberg fields s.t. \(A = \tilde{A} + 4[\nabla \Sigma]\)

\[ \mathcal{L}' = -\frac{1}{48} F^2(\tilde{A}) - V((\tilde{A} + F(\Sigma))^2) \]

is now invariant under \(\tilde{A} \rightarrow \tilde{A} + [\nabla C]; \quad \Sigma \rightarrow \Sigma - C/4\).
Expanding the potential around \(\tilde{A}\)

\[ \mathcal{L}' = \mathcal{L} - V'(\tilde{A}^2)F^2(\Sigma) \]

Presence of ghost field for \(V'(\tilde{A}^2) < 0 \Leftrightarrow V,\chi \chi < 0\)
Equations of motion

Consider flat FRW cosmology:

\[ ds^2 = -dt^2 + a^2(t)dx^2 \]

Most general three-form compatible with FRW:

\[ A_{ijk} = a^3(t)\epsilon_{ijk}\chi(t) \]

Equations of motion of the field \( \chi \):

\[ \ddot{\chi} + 3H\dot{\chi} + V_{,\chi} + 3\dot{H}\chi = 0 \]

Equation of motion of background fluid:

\[ \dot{\rho}_B = -3\gamma H\rho_B \]
Equations of motion

Friedmann equation

\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2}(\dot{\chi} + 3H\chi)^2 + V(\chi) + \rho_B \right]
\]

can also write:

\[
H^2 = \frac{\kappa^2}{3} \frac{V + \rho_B}{1 - \kappa^2(\chi' + 3\chi)^2/6}
\]

with \( \dot{} = d/d\ln a \).

Evolution of the Hubble rate and equation of state parameter of \( \chi \)

\[
\dot{H} = -\frac{\kappa^2}{2} (V_{,\chi} \chi + \gamma \rho_B), \quad w_{\chi} = -1 + \frac{V_{,\chi} \chi}{\rho_{\chi}}
\]

Universe de Sitter with \( V = 0 \); Superinflation when \( V_{,\chi} \chi < 0 \).
Rewriting the equations of motion in the form of system of first order differential equations:

\[
\begin{align*}
x' &= 3 \left( \sqrt{\frac{2}{3}} y - x \right) \\
y' &= -\frac{3}{2} \lambda(x) (1 - y^2 - w^2) \left[ xy - \sqrt{\frac{2}{3}} \right] + \frac{3}{2} \gamma w^2 y \\
w' &= -\frac{3}{2} w \left( \gamma + \lambda(x) (1 - y^2 - w^2) x - \gamma w^2 \right)
\end{align*}
\]

\[x \equiv \kappa \chi, \quad y \equiv \frac{\kappa}{\sqrt{6}} (\chi' + 3 \chi), \quad z^2 = \frac{\kappa^2 V}{3H^2}, \quad w^2 \equiv \frac{\kappa^2 \rho_B}{3H^2}, \quad \lambda(x) \equiv -\frac{1}{\kappa} \frac{V,\chi}{V}\]

The Friedmann equation \(\Rightarrow\) \[y^2 + z^2 + w^2 = 1\]
<table>
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<th>$x$</th>
<th>$y$</th>
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The intuitive picture

\[ H^2 = \frac{\kappa^2}{3} \frac{V + \rho_B}{1 - \kappa^2 (\chi' + 3\chi)^2 / 6} \]

\[ \Rightarrow \kappa |\chi' + 3\chi| < \sqrt{6} \]
The intuitive picture

\[ H^2 = \frac{\kappa^2}{3} \frac{V + \rho_B}{1 - \kappa^2 (\chi' + 3\chi)^2 / 6} \]

\[ \Rightarrow \kappa|\chi' + 3\chi| < \sqrt{6} \]
Effective potentials

\[ V_{\text{eff}, \chi} = V_{\chi} + 3 \dot{H} \chi = V_{\chi} \left(1 - \frac{3}{2} (\kappa \chi)^2\right) - \frac{3}{2} \gamma \kappa^2 \rho_B \chi \]

[More in Bruno Barros talk]
Example evolutions: $V = \chi^2$
Example evolutions: $V = \chi^n$

- Epoch of tracking:
  \[ N_s = \frac{1}{3n} \ln \left( \frac{V_i}{3H_i^2 y_i^2} \right) \]

- Point of turn around:
  \[ N_t = \frac{1}{3 (1 + \gamma/2)} \ln \left( \frac{2 B}{\gamma A} \right) \]
  where $A = \sqrt{2/3} y_i/(1 + \gamma/2)$ and $B = \chi_i - A$.

- Epoch of inflation:
  \[ \Delta N = \frac{2}{9n} \frac{1}{2/3 - (\kappa \chi_{\text{init}})^2} - \frac{1}{2} \]

- Oscillations:
  \[ \langle w_\chi \rangle = \frac{n - 2}{n + 2} \]

Thus for $n = 2$ the field behaves as dust, $\langle w_\chi \rangle = 0$ and for $n = 4$ it mimics radiation, $\langle w_\chi \rangle = 1/3$. 
Part I: Inflation
Cosmological perturbations

General perturbations about FRW background:

\[ ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt) \]

\[ h_{ij} = a^2 e^{2\zeta} \delta_{ij} \]

\( \zeta \) is the curvature perturbation, and we expand \( N_i \) and \( N \) as:

\[ N_i = \psi, i + \tilde{N}_i \quad N = 1 + \tilde{\alpha} \]

Perturbations of the three-form:

\[ A_{0ij} = a(t) \epsilon_{ijk} \alpha, k \quad A_{ijk} = a^3(t) \epsilon_{ijk} (\chi(t) + \alpha_0) \]

Vector perturbations are decaying and can be ignored.
The second order action

The second order action in scalar perturbations:

$$S_2 = \int dt d^3x \left[ a^3 \frac{\Sigma}{H^2} \dot{\zeta}^2 - a\epsilon(\partial\zeta)^2 \right]$$

where the speed of sound

$$c_s^2 = \frac{V_{,xx}\chi}{V,\chi}$$

and

$$\epsilon = -\frac{\dot{H}}{H^2} \quad \Sigma = \frac{H^2\epsilon}{c_s^2}$$

\(\zeta\) is conserved on the large scales.

$$\dot{\zeta} = c_s^2 \frac{\nabla^2}{\epsilon a^2} \left( \psi + \frac{\zeta}{H} \right)$$
Power spectrum of scalar perturbations

The 2-point correlation function

$$\langle \zeta(k_1)\zeta(k_2) \rangle = (2\pi)^5 \delta^3(k_1 + k_2) \frac{P_\zeta(k_1)}{2k_1^3},$$

The power spectrum

$$P_\zeta \equiv \frac{1}{2\pi^2} k^3 |\zeta_k|^2 = \frac{1}{2(2\pi)^2 \epsilon c_s} \frac{H^2}{M_{Pl}^2} \bigg|_*,$$

* indicates horizon crossing $c_s k = aH$.

The spectral index $n_s$ is

$$1 - n_s = 2\epsilon + \frac{\dot{\epsilon}}{\epsilon H} + \frac{\dot{c}_s}{c_s H}.$$
Power spectrum of tensor perturbations

Since the three-form does not generate tensor perturbations, their evolution equation is as usual,

$$\ddot{h} + 3H\dot{h} - \frac{\nabla^2}{a^2} h = 0$$

Tensor power spectrum:

$$P_T = \frac{2}{\pi^2} \frac{H^2}{M_{\text{Pl}}^2}$$

The tensor spectral index is then

$$n_T = -2\epsilon$$
Consistency relation

Ratio of tensor to scalar perturbations

\[ r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} = 16 c_S |\epsilon| \]

Thus it is in principle possible to distinguish the three-form inflation from scalar field already from the spectra of linear perturbations.
The third order action of scalar perturbations:

\[ S_3 = \int \left\{ dtd^3x \left[ -\epsilon a \zeta (\partial \zeta)^2 - a^3 (\Sigma + 2\lambda) \frac{\dot{\zeta}^3}{H^3} + \frac{3a^3\epsilon}{c_s^2} \dot{\zeta} \zeta^2 \right. \right. \\
+ \left. \left. \frac{1}{2a} \left( 3\zeta - \frac{\dot{\zeta}}{H} \right) \left( \partial_i \partial_j \psi \partial_i \partial_j \psi - \partial^2 \psi \partial^2 \psi \right) - 2a^{-1} \partial_i \psi \partial_i \zeta \partial^2 \psi \right\} \right\} \]

where \( \lambda \) is

\[ \lambda = -\frac{1}{12} \frac{V_{,xx} V_{,xxx}}{V_{,xx}^3} \]

At tree level in quantum field theory, and in the interaction picture, the In-In (equal time) three-point correlation function is given by the expression

\[ \langle \zeta(t, k_1) \zeta(t, k_2) \zeta(t, k_3) \rangle = -i \int_{t_0}^{t} dt' \langle [\zeta(t, k_1) \zeta(t, k_2) \zeta(t, k_3), H_{\text{int}}(t')] \rangle \]
The bispectrum and non-Gaussianity

The non-Gaussianity of the CMB in the WMAP observations is analyzed by assuming

$$\zeta = \zeta_L - \frac{3}{5} f_{NL} \zeta_L^2$$

where $\zeta_L$ is the linear Gaussian part the perturbations, and $f_{NL}$ is an estimator parameterizing the size of the non-Gaussianity.

The three-point correlation function:

$$\langle \zeta({k_1})\zeta({k_2})\zeta({k_3}) \rangle = (2\pi)^7 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{1}{k_1^3 k_2^3 k_3^3} P_\zeta^2 A(k_1, k_2, k_3)$$

$$f_{NL} = \frac{10}{3} \frac{1}{k_1^3 + k_2^3 + k_3^3} A$$

$$f_{NL}^{\text{equil}} = 30 \frac{1}{K^3 A}$$

where $K/3 = k_1 = k_2 = k_3$.

$$f_{NL}^{\text{equil}} \approx \frac{5}{81} \left( \frac{1}{c_s^2} - 1 - 2 \frac{\lambda}{\Sigma} \right) - \frac{35}{108} \left( \frac{1}{c_s^2} - 1 \right) + ...$$
Example I: Power law potential

In the original three-form theory

\[ \mathcal{L} = -\frac{1}{48} F^2 - V_0 A^{2p} \]

and in the \( p(X, \phi) \) theory

\[ \mathcal{L}_\phi = (2p - 1) \left( \frac{1}{V_0} \right)^{1/(2p-1)} \left( \frac{X}{24p^2} \right)^{p/(2p-1)} - \frac{1}{2} \phi^2 \]

In either approach we obtain:

\( N \) e-folds before the end of inflation:

\[ \chi_N^2 = \frac{2}{3} - \frac{4}{18p} \frac{1}{1 + 2N} \]

The spectral index for \( N = 60 \) gives

\[ n_s \approx -4\epsilon \approx 0.97 \]
Power law potential

Bounds from Planck in blue. Lines are for $N = 50, 60, 70$. 
Power law potential

Amplitude dominant in the equilateral shape.
Example II: Exponential potential

In the original three-form theory

\[ \mathcal{L} = -\frac{1}{48}F^2 - V_0 \exp(\beta A^2) \]

and in the \( p(X, \phi) \) theory:

\[ \mathcal{L}_0 = (W(x) - 1) V_0 \exp \left( \frac{1}{2} W(x) \right) - \frac{1}{2} \phi^2 \]

where \( W(x) \) is the Lambert-W function and \( x = X/12\beta V_0^2 \). 

\( N \) e-folds before the end of inflation

\[ \chi^2_N = \frac{2}{3} - \frac{1}{18\beta} \left( \frac{1}{1 + \sqrt{6N}} \right) \]

and the spectral index gives for \( N = 60 \)

\[ n_s \approx 0.97 \]
Exponential potential
Amplitude dominant in the equilateral shape.
Multifield three-form inflation

[Kumar, Marto, NN, Moniz (2014)]

\[
S = - \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \sum_{n=1}^{N} \left( \frac{1}{48} F_n^2 + V_n(A_n^2) \right) \right]
\]

For two fields:

\[
\begin{align*}
x_1' &= 3 \left( \sqrt{2/3} w_1 - x_1 \right) \\
w_1' &= \frac{3}{2} \left( 1 - (w_1^2 + w_2^2) \right) \left( \lambda_1 \left( x_1 w_1 - \sqrt{2/3} \right) + \lambda_2 x_2 w_1 \right)
\end{align*}
\]

and a similar equation for \( x_2' \), \( w_2' \). \( \lambda_n = V_{(n), x_n}/V \).

Critical points: - extrema of the potential and
- for \( w_1^2 + w_2^2 = 1 \) and \( x_1^2 + x_2^2 = 2/3 \).
Multifield three-form inflation

Type I

\[ V = x_1^2 + x_2^2 \]

Type II

\[ V = x_1^2 + x_2^4 \]
Entropy perturbations

- Curved trajectories in field space usually lead to the growth of entropy perturbations.
- Curvature perturbations are sourced by entropy perturbations

\[ n_s \equiv \frac{d \ln P_R}{d \ln k} = n_s(t_*) + \frac{1}{H_*} \left( \frac{\partial T_{RS}}{\partial t_*} \right) \sin(2\Delta) \]

\[ r \equiv \frac{P_T}{P_R} = 16\epsilon c_s \bigg|_{*} \cos^2 \Delta \]

\[ \cos \Delta = \frac{1}{\sqrt{1+T_{RS}^2}} \]
Comparison with Planck (2013)

\[ V = V_{10}(x_1^2 + bx_1^4) + V_{20}(x_2^2 + bx_2^4) \]
Three-form couplings with curvature

[Germani, Kehagias (2009)]

\[ S = \int d^4 x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{48} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} - \frac{1}{12} m^2 A_{\mu\nu\rho} A^{\mu\nu\rho} + \frac{1}{8} R A_{\mu\nu\rho} A^{\mu\nu\rho} - \frac{1}{2} A_{\mu\nu\kappa} R^{\kappa\lambda} A^{\mu\nu\lambda} \right) \]

This is equivalent to a scalar field theory with non-minimal kinetic terms coupled to gravity.

\[ \mathcal{L} = \frac{1}{2\kappa^2} R - \frac{1}{2} \Lambda^{\alpha\beta} \partial_{\alpha} \Phi \partial_{\beta} \Phi - \frac{1}{2} m^2 \Phi^2 \]

with

\[ \Lambda^{\alpha\mu} \left[ \left( 1 + \frac{R}{2m^2 g_{\alpha\nu}} \right) - \frac{2}{m^2} R_{\alpha\nu} \right] = \delta^{\mu}_\nu \]

For de Sitter, \( \Lambda^{\alpha\beta} = g^{\alpha\beta} \).
Three-form couplings with curvature

The second order action:

$$\delta S \simeq \int d^4 x \frac{9\dot{\phi}^2}{2m^2} \left[ a^3 \dot{\zeta}^2 - a^3 3H^2 \zeta^2 - \frac{m^2}{9H^2} a(\partial_i \zeta)^2 \right]$$

Curvature perturbation at super-horizon scales

$$\zeta \sim a^{-3/2}$$

- $\zeta$ decays at super-horizon scales;
- Adiabaticity of linear perturbations is lost due to the non-minimal couplings;
- This three-form action cannot be responsible for the CMB temperature fluctuations;
Part II:

Reheating and Preheating
The action

\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{48} F^2(A) - V(A^2) ight. \\
\left. - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \frac{\lambda}{2} \phi F^2 \right) \]

The reheating temperature can be estimated as

\[ T_{rh} \lesssim \lambda \left( \frac{9g_*}{640\pi^4} \right)^{1/4} \left( \frac{m}{M_{Pl}} \right)^{1/2} M_{Pl}, \]

It is \(10^5\) times larger than \(T_{rh}\) for \(f(R) = R + R^2/(6m^2)\)!
Parametric resonance

One can write the equation for $\phi_k$ in the form of a Mathieu equation for $y_k = a^{3/2} \phi_k$

$$\frac{d^2 y_k}{dz^2} + (A_k - 2q \cos(2z)) y_k = 0$$

with

$$A_k = \frac{4k^2}{a^2 m^2} + \frac{4m^2}{m^2}, \quad q = \frac{4\sqrt{8}\lambda M_{Pl}}{\sqrt{3}m^2(t - t_{os})}$$

Instability bands in which the $y_k$ grows exponentially. To guarantee enough efficiency in the production of particles we must have broad-resonance ($A_k \sim 1, 2, 3, \ldots$ and $q \gg 1$).

- For 3-forms this is easily achieved as $q \propto \lambda M_{Pl}/m$.
- Also, broad resonance lasts longer than for scalar field inflation.
Still to do...

- Backreaction on the inflaton background field;
- Consider the coupling to another 3-form instead of a scalar field:

\[ \mathcal{L} \sim A_{(2)}^2 F^2 \]
Part III: Dark Energy
Three-form action, couplings to dark matter

Action for the three-form $A_{\mu\nu\rho}$

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{48} F^2(A) - V(A^2) - \sum_a m_a(A^2) \delta(x - x(\lambda)) \sqrt{\dot{x}^2 - g} \right)$$

where

$$F_{\mu\nu\rho\sigma} = 4\nabla_{[\mu} A_{\nu\rho\sigma]} = \nabla_\mu A_{\nu\rho\sigma} - \nabla_\sigma A_{\mu\nu\rho} + \nabla_\rho A_{\sigma\mu\nu} - \nabla_\nu A_{\rho\sigma\mu}$$

We have the equations of motion:

$$\nabla \cdot F = 12 \left( V'(A^2) + 2\rho \frac{m'(A^2)}{m(A^2)} \right) A$$

and due to antisymmetry we have the additional constraints:

$$\nabla \cdot \left( V'(A^2) + 2\rho \frac{m'(A^2)}{m(A^2)} \right) A = 0$$
Equations of motion

Consider flat FRW cosmology:

\[ ds^2 = -dt^2 + a^2(t)dx^2 \]

Most general three-form compatible with FRW:

\[ A_{ijk} = a^3(t)\epsilon_{ijk}\chi(t) \]

Equations of motion of the field \( \chi \) with \( f \equiv 2m_\chi/m \)

\[ \ddot{\chi} + 3H\dot{\chi} + V,\chi + 3\dot{H}\chi = -\kappa \rho_m f \]

Equation of motion of dark matter fluid:

\[ \dot{\rho}_m + 3H\rho_m = \kappa \rho_m f \]
Equations of motion

Friedmann equation

\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \left( \dot{\chi} + 3H\chi \right)^2 + V(\chi) + \rho_m \right) \]

can also write:

\[ H^2 = \frac{\kappa^2}{3} \frac{V + \rho_m}{1 - \kappa^2 (\chi' + 3\chi)^2 / 6} \]

Evolution of the Hubble rate:

\[ \dot{H} = -\frac{\kappa^2}{2} (V,\chi\chi + (1 + \kappa f \chi)\rho_m) \]

Equation of state parameter of \( \chi \):

\[ w_\chi = -1 + \frac{V,\chi + \kappa f \rho_m}{\rho_\chi} \chi \]
Effective potential

\[ \ddot{\chi} + 3H \dot{\chi} + V,\chi + 3\dot{H}\chi = -\kappa \rho_m f \]

\[ V_{\text{eff},\chi} = V,\chi \left( 1 - \frac{3}{2}(\kappa \chi)^2 \right) - \frac{3}{2} \kappa^2 \rho_m \chi + \kappa f \rho_m \left( 1 - \frac{3}{2}(\kappa \chi)^2 \right) \]

We are going to study 4 cases:

(i) \( V = 0, \ f = 0 \);
(ii) \( V = 0, \ f \neq 0 \);
(iii) \( V \neq 0, \ f = 0 \);
(iv) \( V \neq 0, \ f \neq 0 \).
Case: $V = 0, f = 0;$

$$y_i \equiv \chi_i' + 3\chi_i \neq 0 \text{ otherwise } \rho_\chi \equiv 0.$$  

$$w_\chi \equiv -1$$  

It is a cosmological constant!
Case: $V = 0, \ f \neq 0$;

$$w_\chi = -1 + \frac{\kappa f \rho_m}{\rho_\chi}$$
Case: $V \neq 0, f = 0$;

With $V = V_0 \chi^2$;

$$w_\chi = -1 + \frac{V_0 \chi}{\rho_\chi} \chi$$
Case: $V \neq 0, f \neq 0$;

With $V = V_0 \chi^2$;

$$w_\chi = -1 + \frac{V_\chi + \kappa f \rho_m}{\rho_\chi} \chi$$
Case: $V \neq 0, f \neq 0$;

With $V = V_0 \chi^2$ and $\chi_i = \chi'_i = 0$

$$w_\chi = -1 + \frac{V_\chi + \kappa_f \rho_m}{\rho_\chi} \chi$$
Newtonian limit of linear perturbations

Linear evolution of matter density perturbations

\[ \ddot{\delta}_m + \left( 2H + \kappa f \dot{\chi} - \frac{2 \dot{F}}{1 - F} \right) \dot{\delta}_m = \left( \frac{\kappa_{\text{eff}}^2}{2} \rho_m - \frac{\kappa^2}{a^2 c_{\text{eff}}^2} \right) \delta_m \]

\[ \kappa_{\text{eff}}^2 = \frac{\kappa^2}{1 - F} \left[ 1 + \frac{2}{\kappa^2 \rho_m} \left( \ddot{F} + (2H + \kappa f \dot{\chi}) \dot{F} - \frac{\kappa^2}{2} \frac{V,\chi \kappa f \rho_m}{V,\chi \chi + \kappa f,\chi \rho_m} \right) \right] \]

\[ c_{\text{eff}}^2 = \frac{F}{1 - F}, \quad F \equiv -\frac{\kappa^2 f^2 \rho_m}{V,\chi \chi + \kappa f,\chi \rho_m} \]

\[ F < 0 \Rightarrow c_{\text{eff}}^2 < 0 \Rightarrow \]

for sufficiently large modes there is extra source term for growth of perturbations!
Growth at a given time

\[ \frac{\delta_m(z = 0)}{\delta_m(z = 10^3)} \]

- \( f = 1.7 \times 10^{-21} \)
- \( f = 1.7 \times 10^{-20} \)
Growth for a given mode

\[ \frac{\delta_m(z)}{\delta_m(z = 10^3)} \]

- \( f = 1.7 \times 10^{-21} \)
- \( f = 1.7 \times 10^{-20} \)
Other couplings

[Ngampitipana, Wongjuna (2011)]

\[ \ddot{\chi} + 3H \dot{\chi} + 3\dot{H}\chi + V_{,\chi} = \frac{Q}{\dot{\chi} + 3H\chi} \]

- \( Q_1 = \sqrt{2/3}\kappa\beta \rho_c (\dot{\chi} + 3H\chi) \)
- \( Q_2 = \alpha H \rho_c \)
- \( Q_3 = \Gamma \rho_c \)
- \( Q_4 = \sqrt{6/\kappa^2}\Gamma (\dot{\chi} + 3H\chi) \)

They study of dynamical systems for \( V = V_0e^{-\eta\chi}, \quad V = V_0e^{-\eta\chi^2}, \quad V = V_0\chi^{-n} \).
Part IV:

Magnetic Fields
The issue

- Evidence for magnetic fields from galaxies, clusters, filaments;
- Origin and nature of these magnetic fields is still unclear;
- Possibility I: Magnetogenesis during scalar field inflation;
- Backreaction problem: EM energy density catches up with energy density of inflation and brings inflation to an end;
- Possibility II: Three-Magnetogenesis [Koivisto, Urban (2012)].
The action

\[ \mathcal{L} = -\frac{1}{4} F^2(A) - \frac{1}{48} F^2(A) - V(A^2) - \frac{1}{2} \alpha F_{\mu\nu}(A) F^{\mu\nu}(B) \]

\[ A^\mu = \text{photon vector potential}, \]
\[ A^{\alpha\beta\gamma} = \text{three-form}, \]
\[ B_\alpha = \text{dual of three-form}. \]

In Fourier space, the solution for \( A \) is

\[ A(\eta) = A_1 \cos(k\eta) + A_2 \sin(k\eta) + A_3 e^{\Gamma k\eta} + A_4 e^{-\Gamma k\eta} \]

If \( \Gamma^2 > 0 \), we have exponentially growing/decaying solution.
Stability

\[ V = V_0 \exp(-\beta \chi^2 / M_{Pl}) \]

- \( \beta > 0 \), critical points at \( \chi = \pm \sqrt{2/3} \):

\[ \Gamma^2 = \frac{\kappa \Lambda - \kappa}{\Lambda^2 \kappa^2_\Lambda + \kappa^2} \]

with \( \kappa = k / \mathcal{H}_e \) and \( \Lambda \approx 8 \alpha^2 / 3 \) and \( \kappa^2_\Lambda \approx \beta V / (\alpha^2 M^2_{Pl}) \).

Instability for \( k < k_\Lambda \).

- \( \beta < 0 \), critical points at the minima of the potential:

\[ \Gamma^2 = -\frac{|\beta|V_0}{\alpha^2 M^2_{Pl} k^2} - 1 < 0 \]

There is no instability.
Allowed parameter space

- **green**: magnetic fields observed in intergalactic medium;
- **yellow**: for successful seed to be fed to the magnetohydrodynamic plasma.
\[ \delta^0_B \text{ (Gauss)}, \Lambda = 10^{-2}, k\Lambda/k_{\text{min}} = 10^2. \]

Only a few orders of magnitude in \( k \) are efficiently magnified.
Part V: Screening Mechanisms
Screening with vector fields

[Beltrán-Jiménez, Fróes, Mota (2012)]

Massive vector field with a gauge fixing term

\[ S = \int d^4x \sqrt{-g} \left[ -\frac{R}{2\kappa^2} - \frac{1}{4} F^2 - \frac{1}{2} (\nabla_\mu B^\mu)^2 - \frac{M^2}{2} B^2 \right] + \right. \]

\[ + \int d^4x \mathcal{L}_m[\tilde{g}_{\mu\nu}, \psi] \]

Matter fields couple to gravity via \( \tilde{g}_{\mu\nu} = \Omega^2 (B^2) g_{\mu\nu} \)

Equations of motion give

\[ \Box B_\mu = \left( M^2 + \frac{2\beta \rho}{M_p^2} e^{\beta B^2/M_p^2} \right) B_\mu \]
Screening with vector fields
Screening with vector fields
Screening with three-form fields

[Barreiro, Bertello, NN (in progress)]

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{48} F^2 - V(A^2) \right] + \int d^4x \tilde{\mathcal{L}}_m \]

Conformal coupling \( \tilde{g}^{\mu\nu} = \Omega^2(A^2) g^{\mu\nu} \)

Going to the vector field in the gauge fixing description

\[ \nabla_\alpha (\nabla^\mu B_\mu) = -2 \left( \frac{\partial V}{\partial B^2} + \rho \frac{\partial \Omega}{\partial B^2} \right) B_\alpha \]

(cf. for vectors we had, \( \Box B_\mu = \ldots \))
Screening with three-form fields
Screening with three-form fields
Summary

- Three-forms possess accelerating attractors and saddle points which can describe three-form driven inflation or dark energy;
- Scalar spectral index predicted to be $n_s \approx 0.97$ for $N = 60$.
- Some models are within non-Gaussianity, and ratio of tensor to scalar perturbations bounds;
- Efficient reheating/preheating:
  - Efficient generation of magnetic fields;
  - In the presence of a coupling to dark matter, growth of structure is enhanced for small scales.
- New fifth-force screening solution.