Conformal time and radiation

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Motivation

MOTIVATION

- "Tempo Conforme" in Madeira Island.

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- Conformal time in cosmology?

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- "Tempo Conforme" in Madeira Island.
- Conformal time in cosmology?
- Relation between time and radiation.

Goals

GOALS

- Explore the use of conformal time in cosmological models that go beyond the standard FLRW models:
- Consider anisotropic, but spatially homogeneous models and inhomogeneous models.
- Understand the use of conformal time in relation with the general conformal transformations between metrics,
- Connection to the $1 + 3$ threading of spacetimes.

Conformal time in FLRW (I)

Consider the line element of FLRW universes

$$ds^2 = -\mathrm{d}t^2 + \frac{a^2(t)}{\left(1+\frac{k}{4}r^2\right)^2}\left[\mathrm{d}r^2 + R^2(r,t)\mathrm{d}\Omega^2\right] \ . \qquad (1)$$

where $d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric on the 2-sphere.

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Definition of CT Defining the time η through

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)} \quad (2)$$

the metric becomes

$$ds^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{a^2(t)}{\left(1+\frac{k}{4}r^2\right)^2}\left(\mathrm{d}r^2 + R^2(r,t)\mathrm{d}\Omega^2\right)\right] \ . \quad (3)$$

Notice that

$$\eta = \int \frac{\mathrm{d}t}{a(t)} = \int \frac{dr}{\sqrt{1-kr^2}} = \Psi_k(r) \tag{4}$$

where the right side is the comoving distance.

Also

$$\int \mathrm{d}\eta = \int \frac{\mathrm{d}t}{a(t)} = \int \frac{da}{a\dot{a}} = \int \frac{da}{aH_0A_o\sqrt{1-\Omega_0+\Omega_0 a_0/a}} \tag{5}$$

Notice that

$$\eta = \int \frac{\mathrm{d}t}{a(t)} = \int \frac{dr}{\sqrt{1-kr^2}} = \Psi_k(r) \tag{6}$$

where the right side is the comoving distance.

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$$\int \mathrm{d}\eta = \int \frac{\mathrm{d}t}{a(t)} = \int \frac{da}{a\dot{a}} = \int \frac{da}{aH_0A_o\sqrt{1-\Omega_0+\Omega_0 a_0/a}} \tag{7}$$

Redshift

Notice also the underlying role of the the conformal time in the cosmological redshift formula

$$1 + z = \frac{a_{\rm reception}}{a_{\rm emission}} \tag{8}$$

Indeed

$$\frac{\lambda_{\rm reception}}{a_{\rm reception}} = \frac{\lambda_{\rm emission}}{a_{\rm emission}} = \Psi_k(r) = c\eta \ . \tag{9}$$

CT is used in connection to cosmological horizons, perturbations, e.g. conformal longitudinal gauge, gravitational waves.

In conformal time the field equations become

$$3\,(a')^2 + 3\,ka^2 = 8\pi G\;\rho\,a^4 = 8\pi G\;\mu\,a^{4-3\gamma} \tag{10}$$

$$3a'' + 3k\,a = 4\pi G\;(\rho - 3p)\,a^3 = 4\pi G\;(4-3\gamma)\,\mu\,a^{3-3\gamma}\;, \tag{11}$$

where the prime denotes differentiation with respect to the conformal time η.

Matter and Radiation

If we consider combination of pressureless matter (i.e., dust) and radiation, the system then becomes

$$3\,(a')^2 + 3\,ka^2 = \quad 8\pi G\ \mu_M\, a + (\mu_R) \tag{12}$$

$$3\,a'' + 3\,ka = \quad 8\pi G\ \mu_M \tag{13}$$

We easily derive

$$
\begin{aligned}
k = +1\,; \rightarrow \quad a = & \quad b_0 \sin\left(\eta - \eta_0\right) + \tfrac{\kappa^2}{6}\,\mu_M \\
& \text{with } b_0 = \sqrt{(\kappa^2/3)(\mu_M^2/2 + \mu_R)} \\
k = 0 \;\rightarrow \quad a = & \quad b_0\left(\eta - \eta_0\right)^2 + \tfrac{\kappa^2 \mu_M}{6} \\
& \text{with } b_0 = \sqrt{(\kappa^2/3)(\mu_M^2/2 + \mu_R)} \\
k = -1 \;\rightarrow \quad a = & \quad b_0 \sinh\left(\eta - \eta_0\right) - \tfrac{\kappa^2}{6}\,\mu_M \\
& \text{with } b_0 = \sqrt{(\kappa^2/3)(\mu_R - \mu_M^2/2)}
\end{aligned}
\tag{14}
$$

The $k = +1$ closed model yields a cycloid, the $k = -1$ open model a "hyper-cycloid", and the $k = 0$ flat model expands linearly in conformal time[1].

[1] Notice that these solutions are very much analogues of the solutions of the Lemaître-Tolman-Bondi dust model

The radiation content enters the previous derivations only as a constant in the Friedmann equation which in turn define the scale coefficient b_0 in the previous solutions. The Raychaudhuri equation when written in conformal time effaces the underlying role of radiation, and thus the latter effectively only enters as an integration constant.

More general models

We wish to consider both anisotropic, but spatially homogeneous metrics, as well inhomogeneous metrics.

We first consider shear free, irrotational, geodesic models which are a sort of minimum extension with regard to the FLRW universes. The metric reads

$$\mathrm{d}s^2 = -dt^2 + A^2(t, x^\mu)\, h_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta \ , \tag{15}$$

where the coordinates can be chosen such that $h_{\alpha\beta}$ is diagonal, and for which the expansion scalar, $(\nabla_a u^a)$, is

$$\theta = 3\frac{\dot{A}}{A} \ . \tag{16}$$

[G.F.R. Ellis & Van Elkst, NATO Sci.Ser.C 541 (1999) 1-116,
A. A. Coley and D. J. McManus, Class. Quantum Grav. 11 (1994) 1261]

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More general models

The field equations read

$$3\left(\frac{\dot{A}}{A}\right)^2 + \frac{^{(3)}R}{2A^2} - 2\frac{\nabla^2 A}{A^3} + \frac{\nabla A.\nabla A}{A^4} = \rho \tag{17}$$

$$2\frac{\ddot{A}}{A} = -\frac{\rho + 3p}{3} \tag{18}$$

$$2\partial_t \left[\nabla_\alpha \ln A\right] = q_\alpha \tag{19}$$

$$^{(3)}R_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}{}^{(3)}R - \nabla_\alpha\nabla_\beta \ln A + \nabla_\alpha \ln A \nabla_\beta \ln A + \frac{1}{3}\left[\nabla^2 \ln A - \nabla \ln A.\nabla \ln A\right] h_{\alpha\beta} = \pi_{\alpha\beta} \tag{20}$$

More general models

and

$$\dot{\rho} + 3\frac{\dot{A}}{A}(\rho + p) + \frac{\nabla^{\alpha} q_{\alpha}}{A} = 0 \tag{21}$$

$$\partial_t q_{\alpha} + 3\frac{\dot{A}}{A} q_{\alpha} + \nabla_{\alpha} p + \frac{\nabla^{\beta}\pi_{\alpha\beta}}{A^2} = 0 \, . \tag{22}$$

[A. A. Coley and D. J. McManus, Class. Quantum Grav. 11 (1994) 1261]

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More general models in conformal time

Using $\mathrm{d}\eta = \mathrm{d}t/A$ the field equations read

$$3\left(\frac{A'}{A}\right)^2 + {}^{(3)}R - 2\frac{\nabla^2 A}{A} + \frac{\nabla A.\nabla A}{A^2} = \rho\, A^2 \quad (23)$$

$$\frac{A''}{A} + \frac{{}^{(3)}R}{3} = -\frac{\rho - 3p}{6}\, A^2 + \frac{2}{3}\frac{\nabla^2 A}{A} + \frac{\nabla A.\nabla A}{3A^2} \quad (24)$$

$$2\partial_\eta \left[\nabla_\alpha \ln A\right] = q_\alpha\, A \quad (25)$$

$$\begin{aligned} {}^{(3)}R_{\alpha\beta} - \frac{1}{3}\, h_{\alpha\beta}{}^{(3)}R - \nabla_\alpha \nabla_\beta \ln A + \nabla_\alpha \ln A \nabla_\beta \ln A \\ + \frac{1}{3}\left[\nabla^2 \ln A - \nabla \ln A.\nabla \ln A\right] h_{\alpha\beta} = \pi_{\alpha\beta} \end{aligned} \quad (26)$$

Shear-free models without heat flux

Restrict to the family of locally rotationally symmetric (LRS) Bianchi I,Bianchi III and Kantowski-Sachs models

$$ds^2 = -dt^2 + a^2(t)\, dr^2 + b^2(t)\, \left[d\theta^2 + f^2(\theta)\, d\phi^2\right] \qquad , \tag{27}$$

where

$$f(\theta) = \begin{cases} \sin\theta & \text{if } {}^3R \text{ is positive -- KS} \\ \theta & \text{if } {}^3R \text{ is null -- BI} \\ \sinh\theta & \text{if } {}^3R \text{ is negative -- BIII} \end{cases} \qquad . \tag{28}$$

${}^3R$ is the Ricci scalar of spatial hypersurfaces and is ${}^3R = \frac{2k}{b^2}$, where $k = 0, \pm 1$. The homogeneous models associated with the values $k = 0, -1$ belong to Bianchi types I and III, respectively, and to the value $k = +1$ corresponds the Kantowski-Sachs model.

*[J. P. Mimoso and P. Crawford, Class. Quantum Grav. 10 (1993) 315;
A. A. Coley and D. J. McManus, Class. Quantum Grav. 11 (1994) 1261]*

Shear free LRS

In conformal time, the field equations become

$$3(A')^2 + \frac{k}{b^2}A^4 = \kappa^2\left(\mu_M A + \mu_R\right) + 3\sigma^2 A^4 \ . \tag{29}$$

and

$$3A'' + \frac{k}{b^2}A^3 = -2\sigma^2 A^3 - \frac{1}{2}\mu_M \tag{30}$$

where

$$\theta = \frac{\dot{a}}{a} + 2\frac{\dot{b}}{b} = 3\frac{\dot{A}}{A} \tag{31}$$

$$\sqrt{2}\sigma = (\sigma_{ab}\sigma^{ab})^{1/2} = \frac{1}{\sqrt{3}}(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}) = \frac{1}{\sqrt{3}}(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}) \quad . \tag{32}$$

Notice that the existence of shear makes the integration procedure a lot more complicated than for the FLRW models.

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Shear free LRS

The necessary and sufficient condition for the invariance of shear-free condition is

$$\Pi_{ab} = \frac{k}{A^2}\mathrm{diag}\,(0,0,1,1) = 2E_{ab}\ . \qquad (33)$$

and we have only to deal with the equations

$$3(A')^2 + kA^2 = \kappa^2\left(\mu_M A + \mu_R\right)\ . \qquad (34)$$

and

$$3A'' + kA = -\frac{1}{2}\,\mu_M \qquad (35)$$

where we have used the fact that $a = b$ due to the vanishing of the shear. We see that the latter equations reproduce those of FLRW

Shear free LRS

The solutions are once again

$$
\begin{aligned}
k = +1\ ; \rightarrow \quad a = & \quad b_0 \sin\left(\eta - \eta_0\right) + \tfrac{\kappa^2}{6}\, \mu_M \\
& \text{with } b_0 = \sqrt{(\kappa^2/3)(\mu_M^2/2 + \mu_R)} \\
k = 0 \ \rightarrow \quad a = & \quad b_0 \left(\eta - \eta_0\right)^2 + \frac{\kappa^2 \mu_M}{6} \\
& \text{with } b_0 = \sqrt{(\kappa^2/3)(\mu_M^2/2 + \mu_R)} \\
k = -1 \ \rightarrow \quad a = & \quad b_0 \sinh\left(\eta - \eta_0\right) - \tfrac{\kappa^2}{6}\, \mu_M \\
& \text{with } b_0 = \sqrt{(\kappa^2/3)(\mu_R - \mu_M^2/2)}
\end{aligned}
\tag{36}
$$

Spherically symmetric and Inhomogeneous

Consider now the spherically symmetric inhomogeneous metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{[(ar)']^2}{1 - k(r)\,r^2}\,\mathrm{d}r^2 - (ar)^2\,\mathrm{d}\Omega^2 \; , \tag{37}$$

where $k(r)$ can take positive, negative or vanishing values.
The field equations can be cast as

$$\frac{\dot{a}^2}{a^2} + \frac{k(r)}{a^2} + (ar)^3\,\frac{\partial_r\left(\frac{\dot{a}^2}{a^2} + \frac{k(r)}{a^2}\right)}{\partial_r(ar)^3} = \frac{8}{3}\pi G\rho(t,r) + \frac{\Lambda}{3} \tag{38}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k(r)}{a^2} = -8\pi Gp + \Lambda \; . \tag{39}$$

More general models

Defining the conformal time η analogously to the FLRW case $\mathrm{d}\eta = \mathrm{d}t/a(t,r)$ so that

$$\mathrm{d}s^2 = a^2(r,t)\left[\mathrm{d}\eta^2 - \frac{[(ar)']^2}{a^2(1-k(r)\,r^2)}\,\mathrm{d}r^2 - (ar)^2\,\mathrm{d}\Omega^2\right] , \tag{40}$$

we have

$$\frac{a''}{a} + k(r) = \frac{\Lambda}{3}\,a^2 + \frac{8\pi G}{6}\,(\rho - 3p)\,a^2 + \frac{ra}{6}\,\frac{\partial_r\left(\frac{\dot{a}^2}{a^2} + \frac{k(r)}{a^2}\right)}{\partial_r(ar)}\,a^2 . \tag{41}$$

and for dust

$$a'^2 + k(r)a^2 = \frac{M(r)}{a} + \frac{\Lambda}{3}\,a^2 . \tag{42}$$

Dynamical system of the spherically symmetric model with an imperfect fluid

$$\left(\frac{\Theta}{3}+\sigma\right)^{\cdot} = -\left(\frac{\Theta}{3}+\sigma\right)^{2} - \frac{\kappa^2}{6}\left(\rho + 3\left(P - 2\Pi\right)\right) + \frac{\Lambda}{3} - \left(\Xi + \frac{\kappa^2}{2}\Pi\right), \tag{43}$$

$$\dot{\sigma} = -\,\sigma\,\Theta + \sigma\left(\frac{\Theta}{3}+\sigma\right)\left(\Xi - \frac{\kappa^2}{2}\Pi\right), \tag{44}$$

$$\left(\Xi + \frac{\kappa^2}{2}\Pi\right)^{\cdot} = -\frac{\kappa^2}{2}\sigma\left(\rho + P - 2\Pi\right) - \left[2\left(\Xi + \frac{\kappa^2}{2}\Pi\right) + \left(\Xi - \frac{\kappa^2}{2}\Pi\right)\right]\times \\ \times\left(\frac{\Theta}{3}+\sigma\right), \tag{45}$$

$$\Xi + \frac{\kappa^2}{2}\Pi = \eta + \sigma\left(\frac{\Theta}{3}+\sigma\right), \tag{46}$$

$$\left(\frac{\Theta}{3}+\sigma\right)^{2} = \frac{\kappa^2}{3}\rho - \frac{{}^{(3)}R}{6} + \frac{\Lambda}{3} + 2\sigma\left(\frac{\Theta}{3}+\sigma\right), \tag{47}$$

(48)

$$(P-2\Pi)' = 6\Pi\frac{r'}{r} - (\rho + P - 2\Pi)\,\frac{\alpha'}{\alpha}, \tag{49}$$

$$\left(\frac{\Theta}{3}+\sigma\right)' = -\,3\sigma\frac{r'}{r}, \tag{50}$$

$$\frac{\kappa^2}{6}\rho' = -\,\frac{\left(\left(\Xi + \frac{\kappa^2}{2}\Pi\right) r^3\right)'}{r^3}\,. \tag{51}$$

So, defining $\eta = \mathrm{d}t \times \exp\left(-\int \theta\, \mathrm{d}s\right)$ so that $\dot{A}/A = \theta/3$, and restricting to shear-free spherical flow we have the constraint equation

$$\Xi = \frac{\kappa^2}{2}\Pi \tag{52}$$

[J.P. Mimoso, M. Le Delliou and F. Mena, Phys.Rev. D88 (2013) 043501]

and we derive once again

$$\frac{A''}{A} + \frac{{}^{(3)}R}{2} A^2 = -\frac{\kappa^2}{6} (\rho - 3P) A^2 + \frac{\Lambda}{3}, \tag{53}$$

$$\frac{A'^2}{A^2} + \frac{{}^{(3)}R}{2} A^2 = \frac{\kappa^2}{3} \rho A^2 + \frac{\Lambda}{3} A^2 . \tag{54}$$

Conformal Transformation

Consider the rescaling tranformation

$$g_{ab} \longrightarrow \bar{g}_{ab} = \Omega^2 \, g_{ab} \tag{55}$$

It follows that

$$\bar{R}_{ab} = R_{ab} + P_{ab} \tag{56}$$

where the P_{ab} tensor is the "Ricci" tensor constructed from the connection $\gamma^a{}_{bc}$. The form of this extra Ricci is (Wald, 1983)

$$\begin{aligned} P_{ab} &= -(D-2)\nabla_a \nabla_b \log\Omega - g_{ab}\nabla^c\nabla_c \log\Omega + \\ &+ (D-2)\nabla_a \log\Omega \nabla_b \log\Omega - (D-2)\nabla^c \log\Omega \nabla_c \log\Omega \end{aligned} \tag{57}$$

and, thus, the trace of equation (56) yields

$$\begin{aligned} \bar{R} &\equiv \bar{R}^a_a = \Omega^{-2}\, R - \Omega^{-2}\Big[2(D-2)\nabla^a\nabla_a \log\Omega - \\ &- (D-2)(D-1)\nabla^a \log\Omega \nabla_a \log\Omega\Big] \,. \end{aligned} \tag{58}$$

1+3 threading

Using a timelike vector field u^a (such that $u^a\,u_a = -1$), we split the metric into a part parallel to that vector and an orthogonal part h_{ab}

$$g_{ab} = -u_a\,u_b + h_{ab} \tag{59}$$

According to (55), we then have

$$\bar{u}^a = \Omega^{-1}\ u^a \quad . \tag{60}$$

We can relate the covariant derivatives of u^a and $\bar{u}^a$. We obtain

$$\bar{\nabla}_b\,\bar{u}_a = \Omega\ [\nabla_b\,u_a - \gamma^c{}_{ba}\,u_c] + u_a\,\nabla_b\,\Omega \tag{61}$$

1+3 threading

So we obtain

- for the expansion tensor

$$\bar{\theta}_{ab} = \Omega \left[\theta_{ab} + \frac{\dot{\Omega}}{\Omega} h_{ab} \right] \tag{62}$$

- for the shear

$$\bar{\sigma}_{ab} = \Omega \, \sigma_{ab} \tag{63}$$

- for the expansion scalar

$$\bar{\theta} = \frac{1}{\Omega} \left(\theta + 3 \frac{\dot{\Omega}}{\Omega} \right) \tag{64}$$

The relation between the conformally transformed times is

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}() = \tilde{u}^a \tilde{\nabla}_a() = \frac{u^a}{\Omega}\left(\nabla_a() + \gamma^c_{ad}(\Omega)()\right) = \frac{\mathrm{d}}{\mathrm{d}t}() + \frac{u^a}{\Omega}\gamma^c_{ad}(\Omega)() \qquad (65)$$

Using this into Raychaudhuri eqn. it is possible to show that, in order for the equation to reproduce with a conformally transformed time the remarkable role of the radiation component, we require

$$\frac{\dot{\Omega}}{\Omega} = \frac{\theta}{3} \qquad (66)$$

which defines the general criterion to prescribe a *generalized conformal time* in models that extend the FLRW models:

$$\mathrm{d}\eta = \mathrm{d}t/\Omega = \exp - \left(\int \frac{\theta}{3}\,\mathrm{d}s\right)\mathrm{d}t \qquad (67)$$

Conclusions

- Conformal time plays central role in cosmology

Conclusions

- It is related to the underlying invariance of the speed of light for geodesic models

Conclusions

- It provides a way of deriving exact solutions combining radiation with other non-interacting fluids, namely with the matter components devoid of pressure.

Conclusions

- Work in progress, or "under construction" in www-ish...

Conclusions

- Thank you for listening!

Boostrap

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