DYNAMICAL SYSTEMS APPROACH TO F(R) GRAVITY

Sulona Kandhai
University of Cape Town, South Africa

Supervised by Prof. Peter Dunsby

FIELD EQUATIONS OF F(R) GRAVITY

- Field equations are derived from the generalised form of the Einstein Hilbert Lagrangian:

$$S = \int \left[f(R) + \mathcal{L}_M \right] \sqrt{-g} d^4x$$

- $f(R)$ is a generic function of the Ricci scalar taking the form:
    - $f(R) = R + g(R)$ so that we regain GR when $g(R) = 0$

COSMOLOGICAL FIELD EQUATIONS FOR F(R) GRAVITY - FLRW METRIC

Modified Friedmann Equation

$$H^2 = \frac{1}{3f'}\left(\rho + \frac{1}{2}\left(Rf' - f\right) - 3H\dot{R}f''\right)$$

Modified Raychaudhury Equation

$$2\dot{H} + 3H^2 = -\frac{1}{f'}\left(P + 2H\dot{R}f'' + \frac{1}{2}\left(f - Rf'\right) + \dot{R}^2 f''' + \ddot{R}f''\right)$$

Trace Equation of modified field equations

$$3\ddot{R}f'' = \rho(1 - 3w) + f'R - 2f - 9Hf''\dot{R} - 3f'''\dot{R}^2.$$

HU-SAWICKI MODEL

The Hu-Sawicki model is a class of broken power law functions which are constructed to evade solar system tests, provide the late time accelerated expansion observed.

$$f(R) = R - m^2 \frac{c_1 \left(\frac{R}{m^2}\right)^n}{c_2 \left(\frac{R}{m^2}\right)^n + 1}$$

$$m^2 \equiv \frac{\kappa^2 \bar{\rho}_0}{3} = (8315 Mpc)^{-2} \left(\frac{\Omega_m h^2}{0.13}\right)$$

- The parameters of the model:
    - $n>0$
    - c_1 and c_2 are dimensionless parameters

HU-SAWICKI MODEL

The Hu-Sawicki model is a class of broken power law functions which are constructed to evade solar system tests, provide the late time accelerated expansion observed.

$$f(R) = R - m^2 \frac{c_1 \left(\frac{R}{m^2}\right)^n}{c_2 \left(\frac{R}{m^2}\right)^n + 1}$$

$$m^2 \equiv \frac{\kappa^2 \bar{\rho}_0}{3} = (8315 Mpc)^{-2} \left(\frac{\Omega_m h^2}{0.13}\right)$$

- The parameters of the model:
  - n>0
  - c_1 and c_2 are dimensionless parameters

In the high curvature limit the function f(R) clearly reduces to:

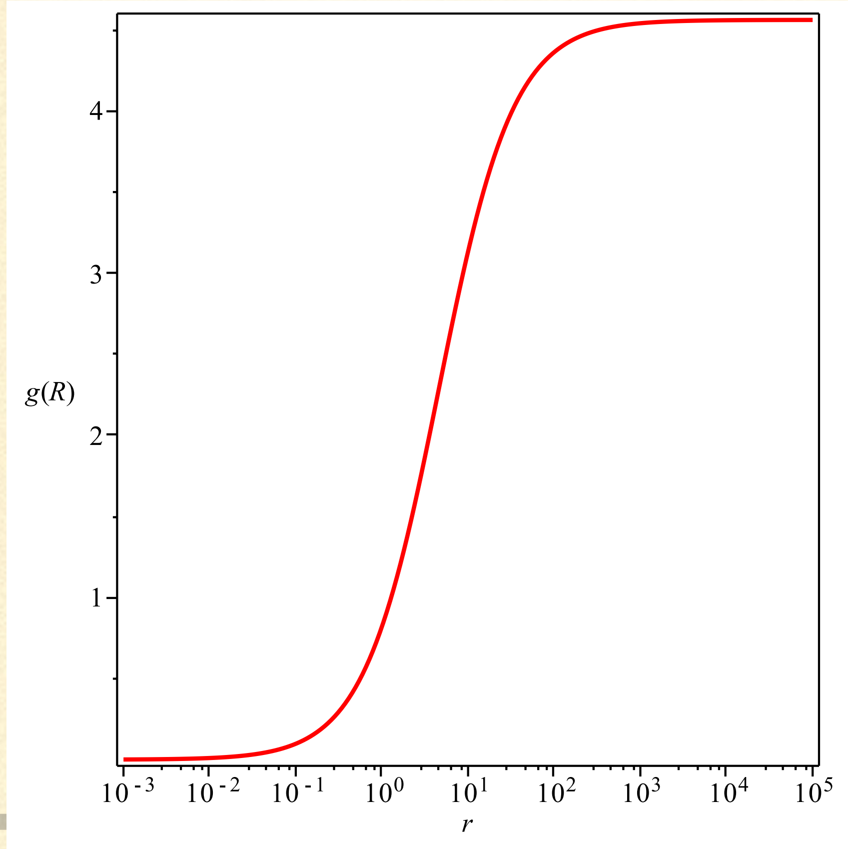
$$\lim_{\frac{m^2}{R} \to 0} f(R) \approx R - \frac{c_1}{c_2} m^2$$

HU-SAWICKI MODEL

The Hu-Sawicki model is a class of broken power law functions which are constructed to evade solar system tests, provide the late time accelerated expansion observed.

$$f(R) = R - m^2 \frac{c_1 \left(\frac{R}{m^2}\right)^n}{c_2 \left(\frac{R}{m^2}\right)^n + 1}$$

$$m^2 \equiv \frac{\kappa^2 \bar{\rho}_0}{3} = (8315 Mpc)^{-2} \left(\frac{\Omega_m h^2}{0.13}\right)$$



In the high curvature limit the function f(R) clearly reduces to:

$$\lim_{\frac{m^2}{R} \to 0} f(R) \approx R - \frac{c_1}{c_2} m^2$$

This term encapsulates the cosmological constant behaviour

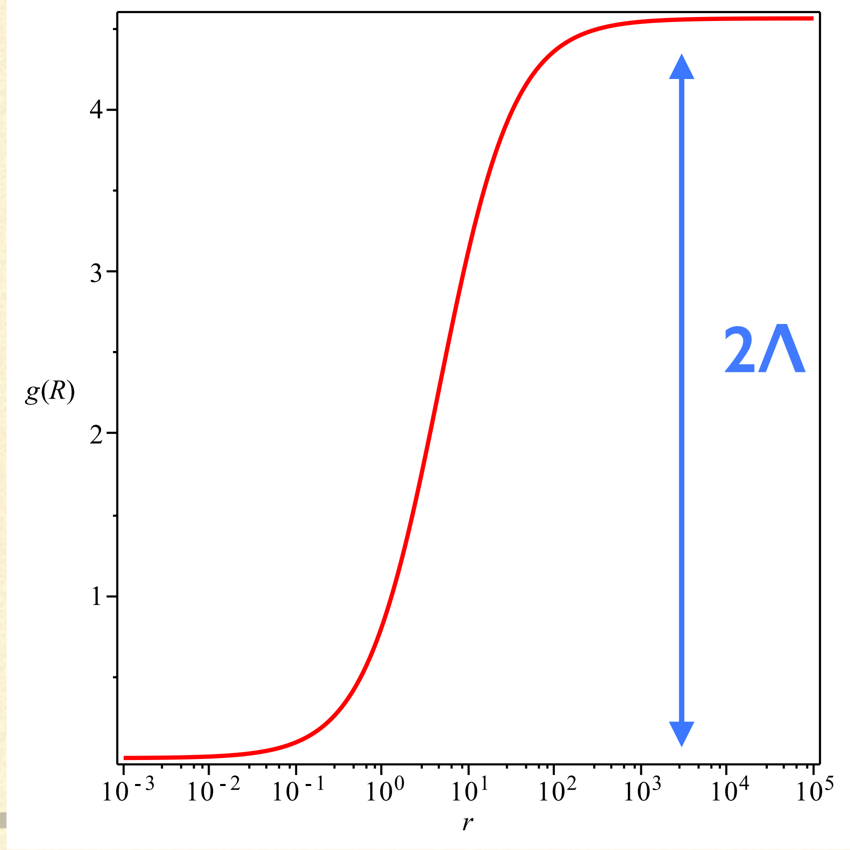
To require the simulation of late time ΛCDM behaviour we must set it equal to 2Λ

HU-SAWICKI MODEL

The Hu-Sawicki model is a class of broken power law functions which are constructed to evade solar system tests, provide the late time accelerated expansion observed.

$$f(R) = R - m^2 \frac{c_1 \left(\frac{R}{m^2}\right)^n}{c_2 \left(\frac{R}{m^2}\right)^n + 1}$$

$$m^2 \equiv \frac{\kappa^2 \bar{\rho}_0}{3} = (8315 Mpc)^{-2} \left(\frac{\Omega_m h^2}{0.13}\right)$$



In the high curvature limit the function f(R) clearly reduces to:

$$\lim_{\frac{m^2}{R} \to 0} f(R) \approx R - \frac{c_1}{c_2} m^2$$

This term encapsulates the cosmological constant behaviour

To require the simulation of late time ΛCDM behaviour we must set it equal to 2Λ

HU-SAWICKI MODEL

- setting the plateau of the corrective term equal to 2Λ we obtain the following relationship between the constants and the energy densities of the cosmological constant and the matter content:

$$\frac{c_1}{c_2} \approx 6\frac{\Omega_\Lambda}{\Omega_m} = 6\frac{(1-\Omega_m)}{\Omega_m}$$

DYNAMICAL SYSTEMS APPROACH TO F(R) GRAVITY

- Need to express the cosmological equations (Modified Friedmann, modified Raychaudhury)
  - in terms of a set of generalised dynamical variables
  - as a set of autonomous differential equations

- USE A COMPACT PHASE SPACE
  - Define a POSITIVE DEFINITE normalisation, D, to bring all solutions to exist within a finite compact volume in the general phase space
  - Define a normalised time variable as:

$$\frac{d}{d\tau} \equiv \frac{1}{D}\frac{d}{dt}$$

  - Define a set of normalised, **general dynamical system** variables

DYNAMICS OF THE GENERAL F(R) - COMPACT ANALYSIS

POSITIVE NORMALISATION

- Any negative contribution to the Friedmann equation must be absorbed into the normalisation
- LHS defines a positive definite quantity that we use as the normalisation

$$\left(3H + \frac{3}{2}\frac{\dot{f}'}{f'}\right)^2 + \frac{3}{2}\frac{f}{f'} = \frac{3\rho_m}{f'} \quad + \tfrac{3}{2}R + \left(\tfrac{3}{2}\tfrac{\dot{f}}{f'}\right)^2$$

$$D^2 = \left(3H + \frac{3}{2}\frac{\dot{f}'}{f'}\right)^2 + \frac{3}{2}\frac{f}{f'}$$

DYNAMICS OF THE GENERAL F(R) - COMPACT ANALYSIS

DYNAMICAL SYSTEMS VARIABLES

- From the other terms in the Friedmann equation, we can now neatly define the dynamical system variables

$$\left(3H + \frac{3}{2}\frac{\dot{f}'}{f'}\right)^2 + \frac{3}{2}\frac{f}{f'} = \frac{3\rho_m}{f'} \quad + \frac{3}{2}R + \left(\frac{3}{2}\frac{\dot{f}}{f'}\right)^2$$

$$x = \frac{3}{2}\frac{\dot{f}'}{f'}\frac{1}{D} \qquad y = \frac{3}{2}\frac{f}{f'}\frac{1}{D^2}$$

$$v = \frac{3}{2}\frac{R}{D^2} \qquad \Omega = \frac{3\rho_m}{f'}\frac{1}{D^2}$$

$$Q = \frac{3H}{D} \qquad D^2 = \left(3H + \frac{3}{2}\frac{\dot{f}'}{f'}\right)^2 + \frac{3}{2}\frac{f}{f'}$$

system constraints

Friedmann $$1 = \Omega + x^2 + v$$

Normalisation $$1 = (Q + x)^2 + y$$

$$-1 \leq x \leq 1, \quad 0 \leq \Omega \leq 1, \quad -2 \leq Q \leq 2,$$
$$0 \leq v \leq 1, \quad 0 \leq y \leq 1$$

PROPAGATION EQUATIONS

Differentiating each dynamical variable, substituting in the modified cosmological field equations and then eliminating variables y and Ω using the constraints, we obtain a 3 dimensional system of first order autonomous differential equations :

$$\frac{dv}{d\tau} = -\tfrac{1}{3}v\left[(Q+x)\left(2\,v+4\,xQ-\left(1-v-x^2\right)(1+3\,w)\right)-2\,Q-4\,x+\underline{2\,x\Gamma}\,(v-1)\right],$$

$$\frac{dx}{d\tau} = \tfrac{1}{6}\left[\underline{-2\,x^2v\Gamma}+\left(1-v-x^2\right)(1-3\,w)+2v+4\,\left(x^2-1\right)\left(1-Q^2-xQ\right)\right.$$
$$\left.+x\,(Q+x)\left(\left(1-v-x^2\right)(1+3\,w)-2\,v\right)\right],$$

$$\frac{dQ}{d\tau} = \tfrac{1}{6}\left[-4\,xQ^3+(5+3\,w)\,Qx\,(1-xQ)-Q^2\,(1-3\,w)-Qx^3\,(1+3\,w)\right.$$
$$\left.-3\,vQ\,(1+w)\,(Q+x)+2\,v\,(1-\underline{\Gamma\,Qx})\right].$$

$$\Gamma \equiv \frac{f'}{f''R}$$

this term specifies the model to be input into the above general propagation equations. Γ must be expressible in terms of the dynamical system variables to close the system

PROPAGATION EQUATIONS

Differentiating each dynamical variable, substituting in the modified cosmological field equations and then eliminating variables y and Ω using the constraints, we obtain a 3 dimensional system of first order autonomous differential equations :

$$\frac{dv}{d\tau} = -\tfrac{1}{3}v\left[(Q+x)\left(2\,v+4\,xQ-\left(1-v-x^2\right)(1+3\,w)\right)-2\,Q-4\,x+\underline{2\,x\Gamma}\,(v-1)\right],$$

$$\frac{dx}{d\tau} = \tfrac{1}{6}\left[\underline{-2\,x^2v\Gamma}+\left(1-v-x^2\right)(1-3\,w)+2v+4\,\left(x^2-1\right)\left(1-Q^2-xQ\right)\right.$$
$$\left.+x\,(Q+x)\left(\left(1-v-x^2\right)(1+3\,w)-2\,v\right)\right],$$

$$\frac{dQ}{d\tau} = \tfrac{1}{6}\left[-4\,xQ^3+(5+3\,w)\,Qx\,(1-xQ)-Q^2\,(1-3\,w)-Qx^3\,(1+3\,w)\right.$$
$$\left.-3\,vQ\,(1+w)\,(Q+x)+2\,v\,(1-\underline{\Gamma\,Qx})\right].$$

$$\Gamma \equiv \frac{f'}{f''R}$$

The above system characterises a GENERAL dynamical system for any modified gravity cosmology defined by a function f(R), which is invertible in terms of the DS variables, so we can determine Γ(x,Q,v)

DYNAMICS OF THE HU SAWICKI MODEL

Define the following relations

$$m^2 \equiv CH_0^2$$

$$r \equiv \frac{R}{H_0^2}$$

$$h \equiv \frac{H}{H_0}$$

And rewrite the HS model as:

$$f(R) = CH_0^2 \left[\left(\frac{r}{C} \right) - \frac{c_1 \left(\frac{r}{C} \right)^n}{c_2 \left(\frac{r}{C} \right)^n + 1} \right]$$

DYNAMICS OF THE HU SAWICKI MODEL

Define the following relations

$$m^2 \equiv CH_0^2$$
$$r \equiv \frac{R}{H_0^2}$$
$$h \equiv \frac{H}{H_0}$$

And rewrite the HS model as:

$$f(R) = CH_0^2 \left[\left(\frac{r}{C}\right) - \frac{\left(\frac{r}{C}\right)}{c_2 \left(\frac{r}{C}\right) + 1} \right]$$

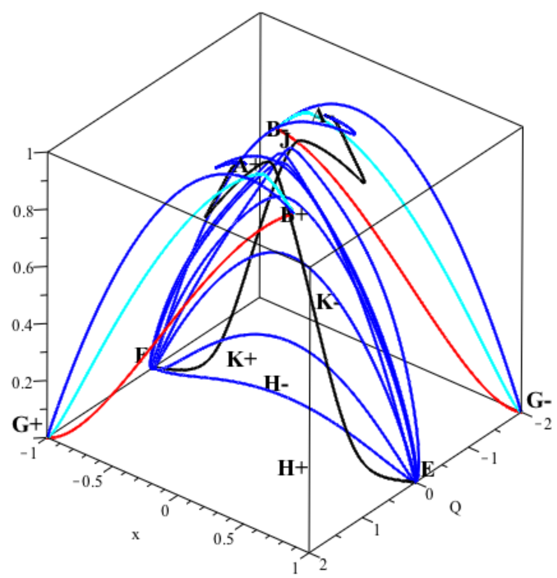
a compact dynamical systems analysis can only be done for for a very special HS model:

$$n = 1, c_1 = 1$$

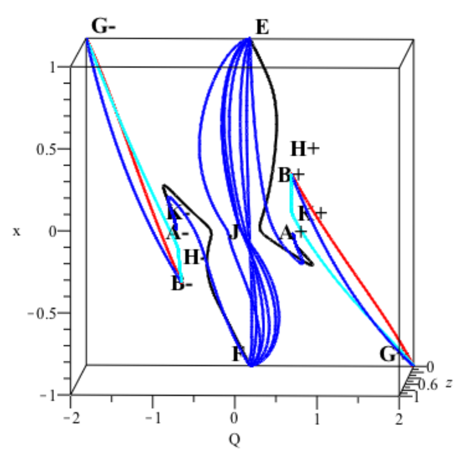
$$\Gamma \equiv \frac{f'}{f''R} = \frac{1}{2} \frac{vy}{(v-y)^2}$$

Fixed points for $v = y = 0$ appear in DSA for any f(R)

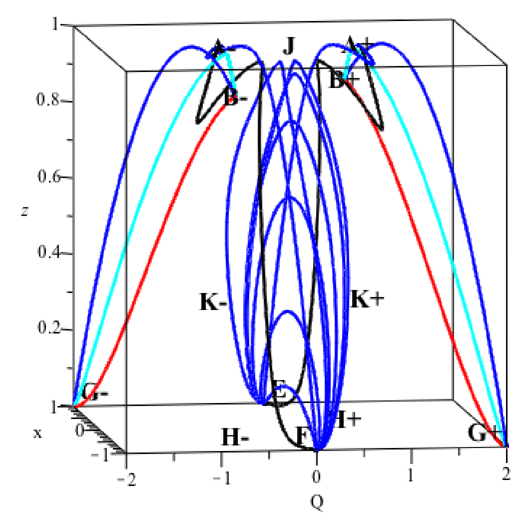
PHASE SPACE



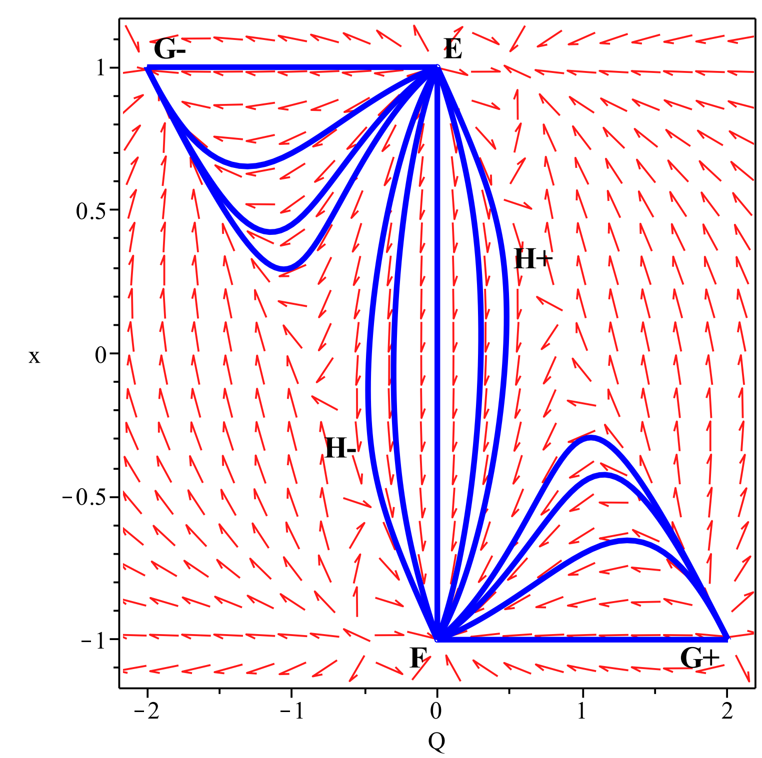
A
B
J
A+
B+
K-
F
K+
H-
G+
G-
H+
E
x
Q



G-
E
H+
B+
K+
A+
J
A-
H-
B-
F
G
x
Q
z



J
A+
B-
B+
K-
K+
E
H+
G-
H-
F
G+
z
x
Q



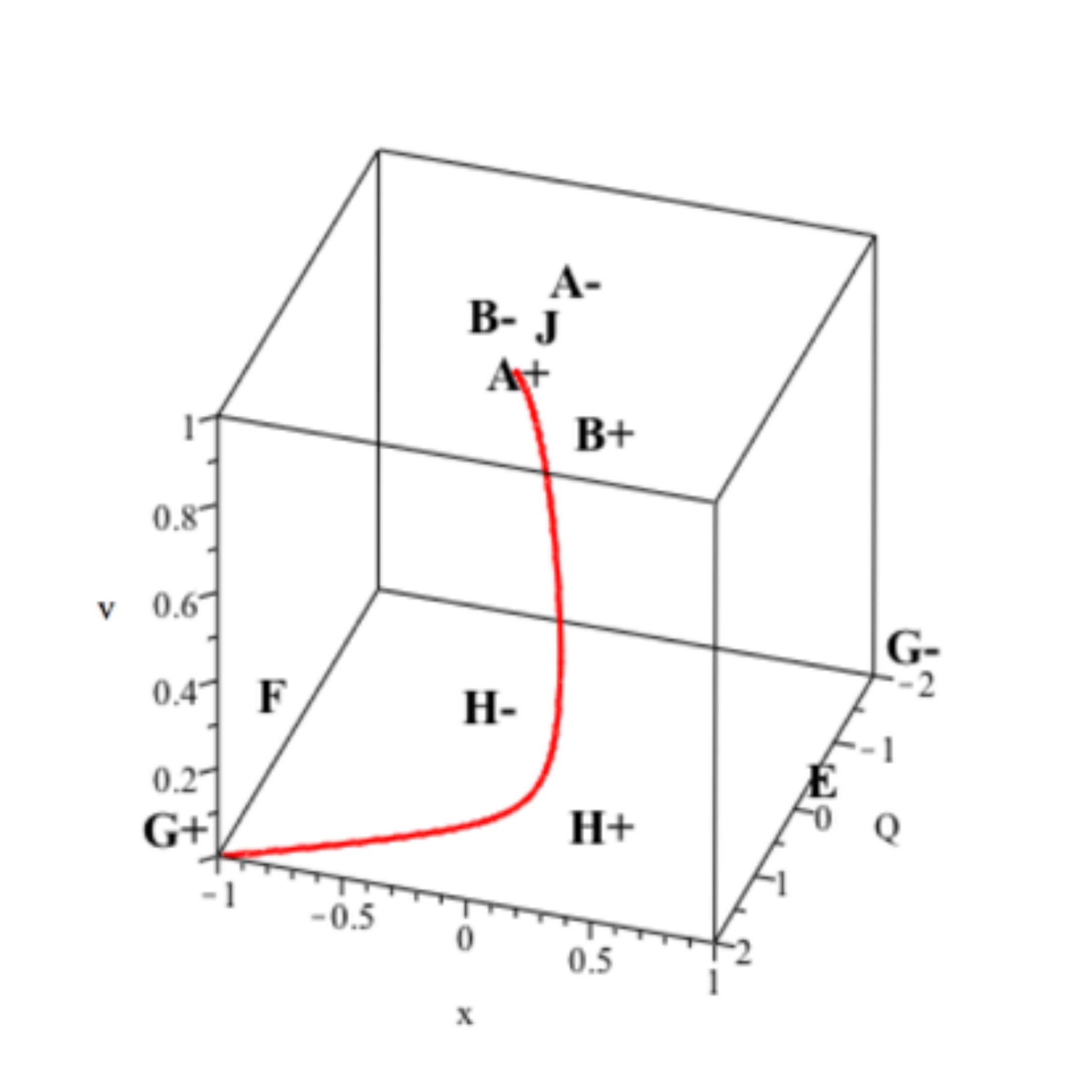
G-
E
H+
H-
F
G+
x
Q

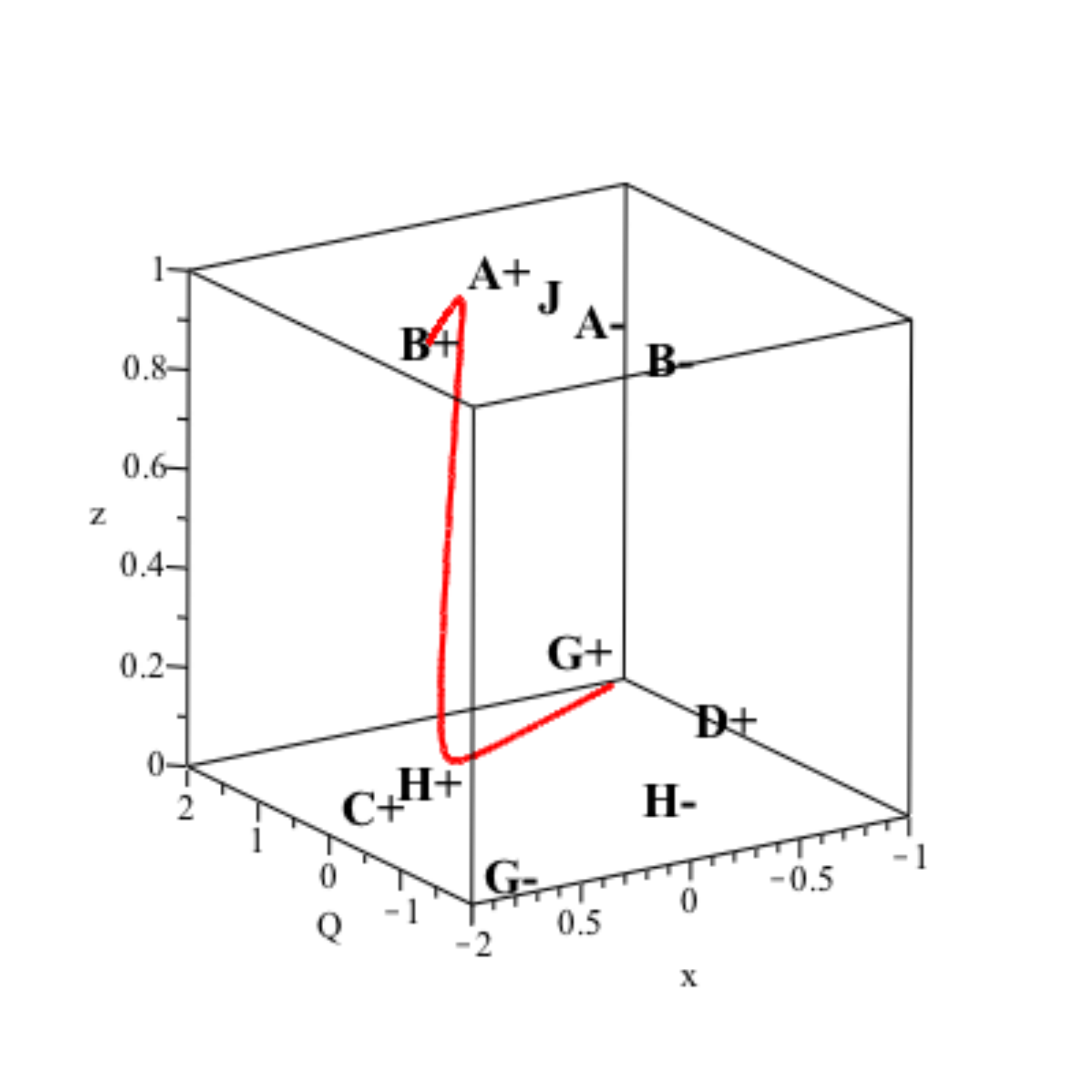
RESULTS - DSA TO HS

Table 1. Fixed points and associated exact solutions

Point	Coordinate (Q, x, y, v, Ω)	Scale factor evolution, $a(t)$
$\mathcal{A}_\pm$	$\left[\pm\frac{1}{\sqrt{2}}, 0, \frac{1}{2}, 1, 0\right]$	$a(t) = a_0 e^{H_0(t-t_0)}$
$\mathcal{B}_\pm$	$\left[\pm\frac{2}{3}, \pm\frac{1}{3}, 0, \frac{8}{9}, 0\right]$	$a(t) = a_0 e^{H_0(t-t_0)}$
$\mathcal{C}(w=-1)$	$\left[0, \sqrt{\frac{3(w+1)}{1+3w}}, -\frac{2}{1+3w}, 0, -\frac{2}{1+3w}\right]$	$a(t) = a_0$
$\mathcal{D}(w=-1)$	$\left[0, -\sqrt{\frac{3(w+1)}{1+3w}}, -\frac{2}{1+3w}, 0, -\frac{2}{1+3w}\right]$	$a(t) = a_0$
$\mathcal{E}$	$[0, 1, 0, 0, 0]$	$a(t) = a_0$
$\mathcal{F}$	$[0, -1, 0, 0, 0]$	$a(t) = a_0$
$\mathcal{G}_\pm$	$[\pm 2, \mp 1, 0, 0, 0]$	$a(t) = a_0\left(2H_0(t-t_0)+1\right)^{\frac{1}{2}}$
$\mathcal{H}_\pm(w \leq \frac{2}{3})$	$\left[\mp\frac{2}{3(w-1)}, \pm\frac{3w-1}{3(w-1)}, 0, 0, -\frac{4}{9}\frac{3w-2}{(w-1)^2}\right]$	$a(t) = a_0\left(2H_0(t-t_0)+1\right)^{\frac{1}{2}}$
$\mathcal{J}$	$[0, 0, 1, 1, 0]$	$a(t) = a_0$
$\mathcal{K}_\pm$	$\left[\pm\frac{\sqrt{6}}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right]$	$a(t) = a_0\left(\frac{3}{2}H_0(t-t_0)+1\right)^{\frac{2}{3}}$

RESULTS - DSA TO HS





EXPANSION HISTORY

We use a finite dynamical system to study the expansion behaviour of the HS model we analysed in the compact phase space. We can do this because the interesting de Sitter and radiation like fixed points are all finite points. To do finite, NON COMPACT analysis we use the following non compact DS variables:

$$x = \frac{\dot{f}'}{f'}\frac{1}{hH_0}, \qquad v = \frac{1}{6}\frac{R}{h^2H_0^2},$$

$$y = \frac{1}{6}\frac{f}{f'}\frac{1}{h^2H_0^2}, \qquad \Omega = \frac{1}{3}\frac{\rho_m}{f'}\frac{1}{h^2H_0^2}$$

$$1 = \Omega + v - x - y$$

$$\Gamma \equiv \frac{f'}{f''R} = \frac{1}{2}\frac{vy}{(v-y)^2}$$

Propagation equations

$$\frac{dx}{dz} = \frac{1}{(z+1)}\left[(-1+3\,w)\,\Omega + x^2 + (1+v)\,x - 2v + 4y\right],$$

$$\frac{dy}{dz} = -\frac{1}{(z+1)}\left[vx\Gamma - xy + 4\,y - 2\,yv\right],$$

$$\frac{dv}{dz} = -\frac{v}{(z+1)}\left[(x\Gamma + 4 - 2\,v)\right],$$

$$\frac{d\Omega}{dz} = \frac{1}{(z+1)}\left[\Omega\,(-1+3\,w+x+2\,v)\right],$$

$$\frac{dh}{dz} = \frac{h}{z+1}\left[2-v\right]$$

EXPANSION HISTORY

Setting the initial conditions of the DS variables below to their value at z=0, we integrate the system

$$x = \frac{\dot{f}'}{f'}\frac{1}{hH_0}, \qquad v = \frac{1}{6}\frac{R}{h^2H_0^2},$$

$$y = \frac{1}{6}\frac{f}{f'}\frac{1}{h^2H_0^2}, \qquad \Omega = \frac{1}{3}\frac{\rho_m}{f'}\frac{1}{h^2H_0^2}$$

$$1 = \Omega + v - x - y$$

Propagation equations

$$\frac{dx}{dz} = \frac{1}{(z+1)}\left[(-1+3\,w)\,\Omega + x^2 + (1+v)\,x - 2v + 4y\right],$$

$$\frac{dy}{dz} = -\frac{1}{(z+1)}\left[vx\Gamma - xy + 4\,y - 2\,yv\right],$$

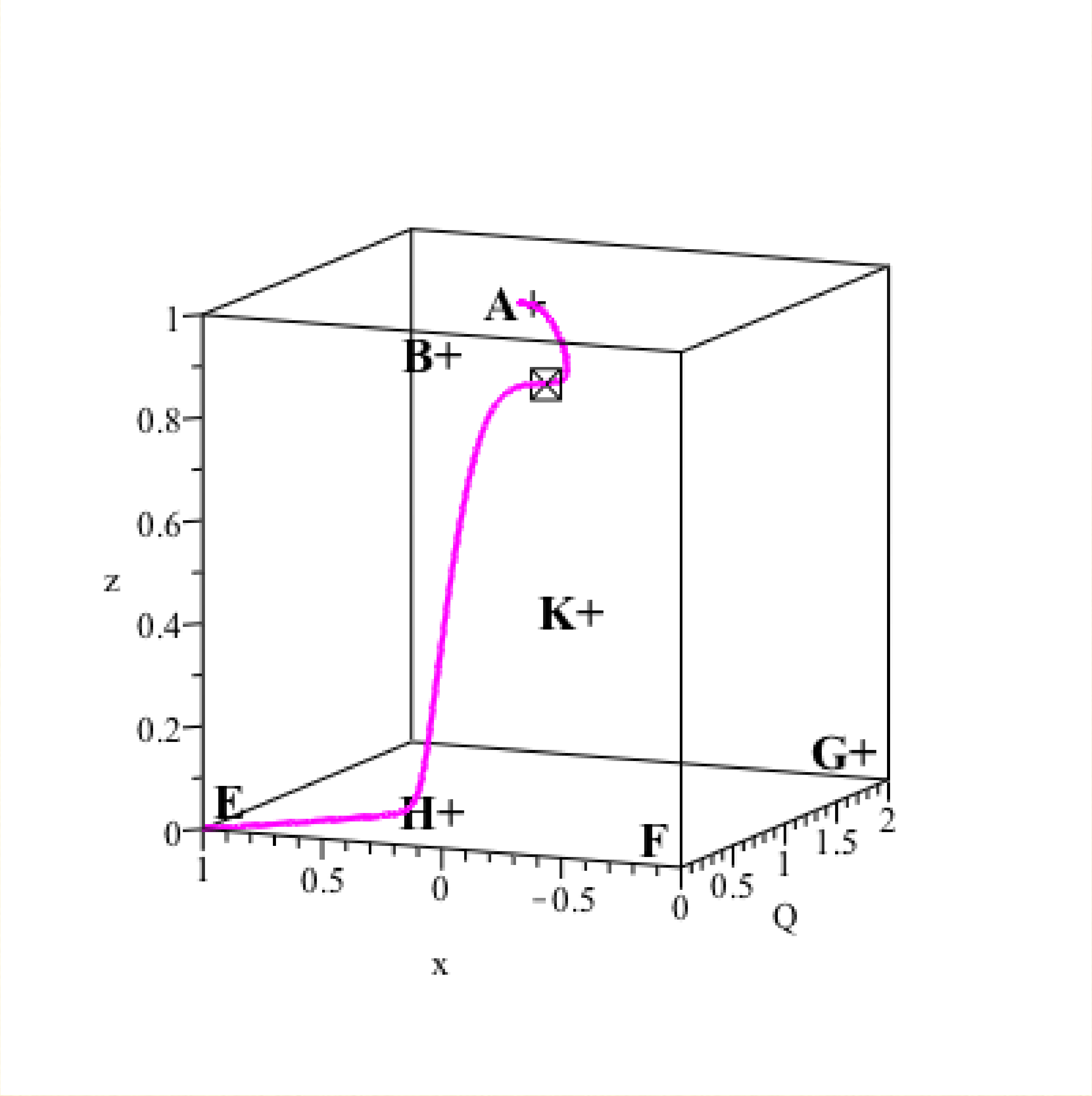
$$\frac{dv}{dz} = -\frac{v}{(z+1)}\left[(x\Gamma + 4 - 2\,v)\right],$$

$$\frac{d\Omega}{dz} = \frac{1}{(z+1)}\left[\Omega\,(-1+3\,w+x+2\,v)\right],$$

$$\frac{dh}{dz} = \frac{h}{z+1}\left[2-v\right]$$

EXPANSION HISTORY

$z = 0$



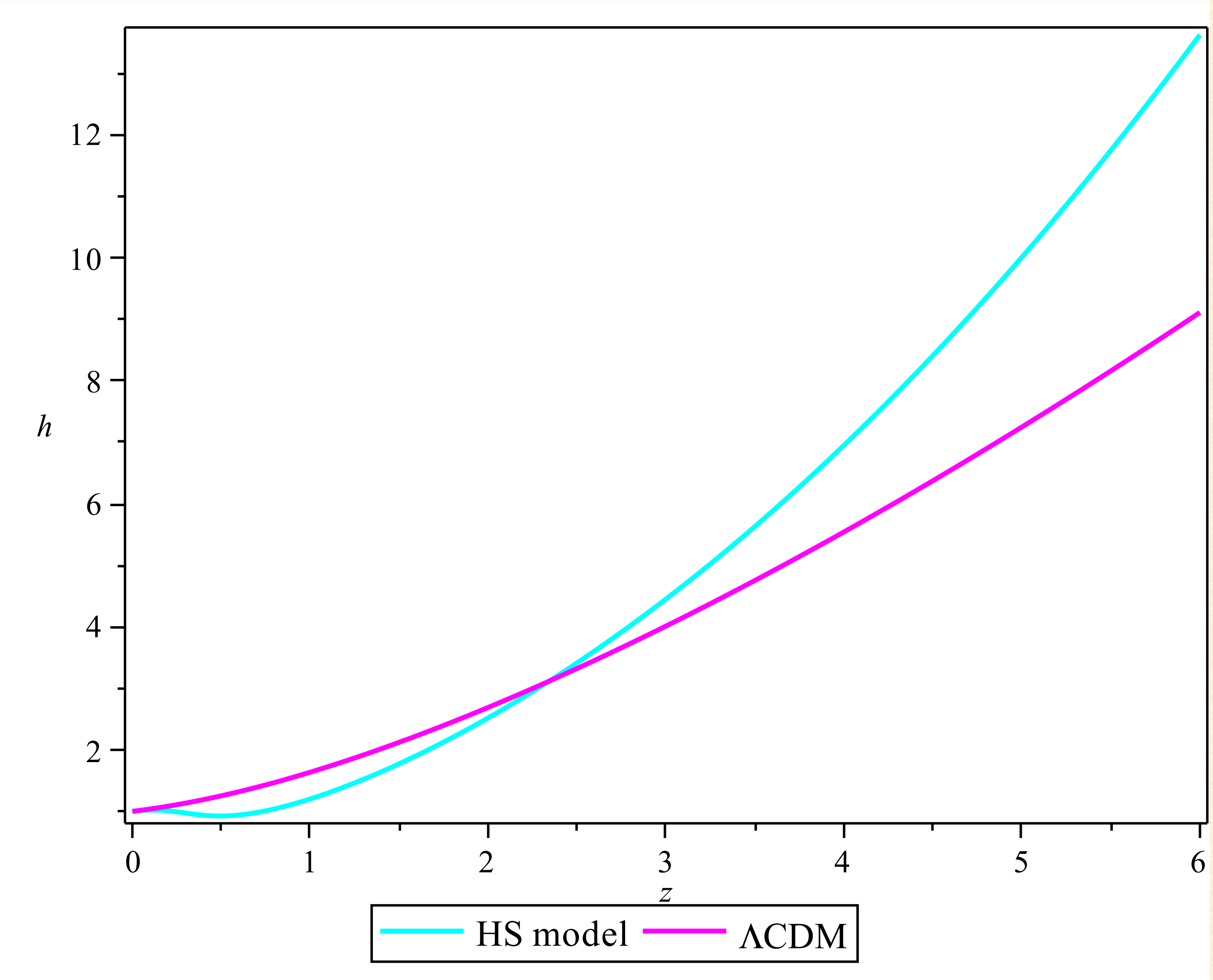
SUMMARY OF EXPANSION HISTORY FOR $z = 0$

We see that this case (n=1, c1=1) of the HS model does not do well to mimic the ΛCDM model :

SUMMARY OF EXPANSION HISTORY FOR $z = 0$

We see that this case (n=1, c1=1) of the HS model does not do well to mimic the ΛCDM model :

1. There are large deviations in the Hubble parameter



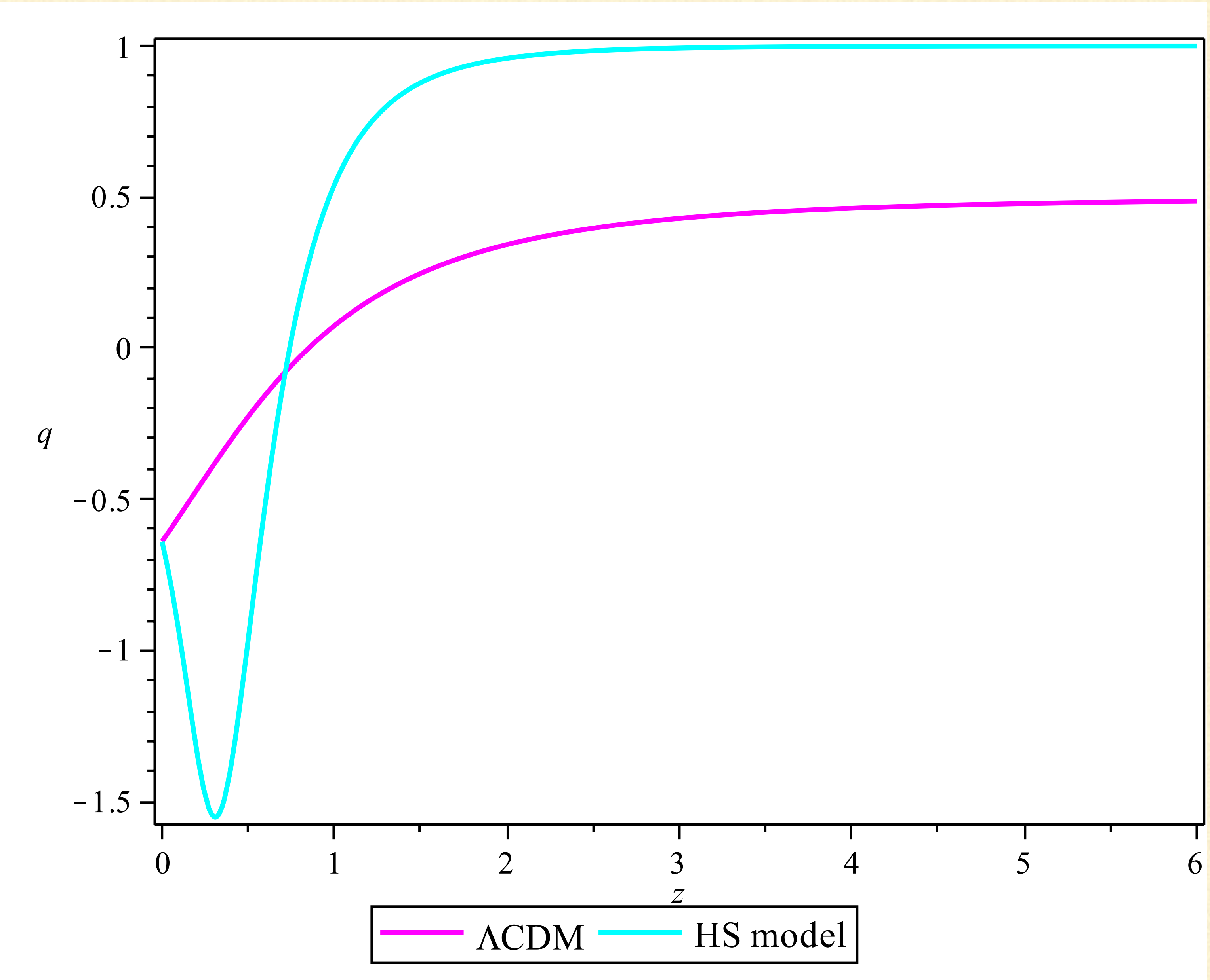
SUMMARY OF EXPANSION HISTORY FOR $z = 0$

We see that this case (n=1, c1=1) of the HS model does not do well to mimic the ΛCDM model :

2. For f(R) gravity, in these dynamical variables, the deceleration parameter looks like:

$$q = 1 - v$$

while the deceleration parameter shows general behaviour that favours a late time accelerated epoch of expansion, it has the wrong asymptotic behaviour



EXPANSION HISTORY
$z = 0$

Total Effective equation of state

Friedmann

$$H^2 = \frac{\rho_m}{3f'} + \frac{R}{6} - \frac{1}{6}\frac{f}{f'} - H\frac{\dot{f}'}{f'}$$

$$3H^2 \quad = \frac{\rho_m}{f'} + 3H^2\left[v - y - x\right]$$

$$\Longrightarrow \rho_{DE,eff} = 3H^2\left[v - y - x\right]$$

From Raychaudhury

$$P_{tot} = H^2(2q - 1)$$

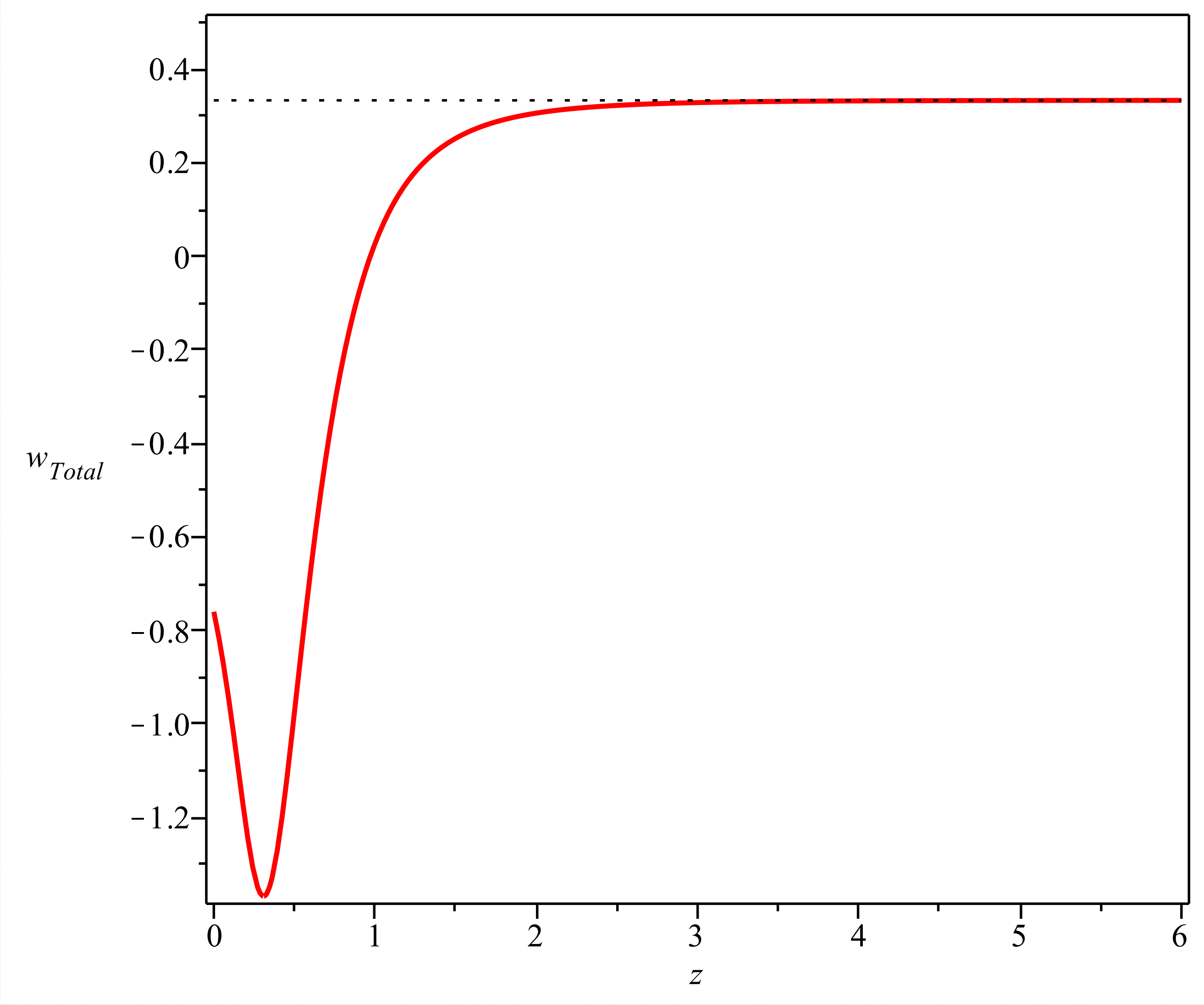
$$P_{tot} = P_{DE} = H^2(1 - 2v)$$

$$w_{total} = \frac{P_{total}}{\rho_{total}} = \frac{P_{tot}}{\rho_m + \rho_{DE,eff}}.$$

SUMMARY OF EXPANSION HISTORY FOR $z = 0$

We see that this case (n=1, c1=1) of the HS model does not do well to mimic the ΛCDM model :

3. The total effective equation of state has the wrong asymptotic behaviour. It represents a radiation like fluid being present until very late in the universe, in an analysis which was done entirely for dust.

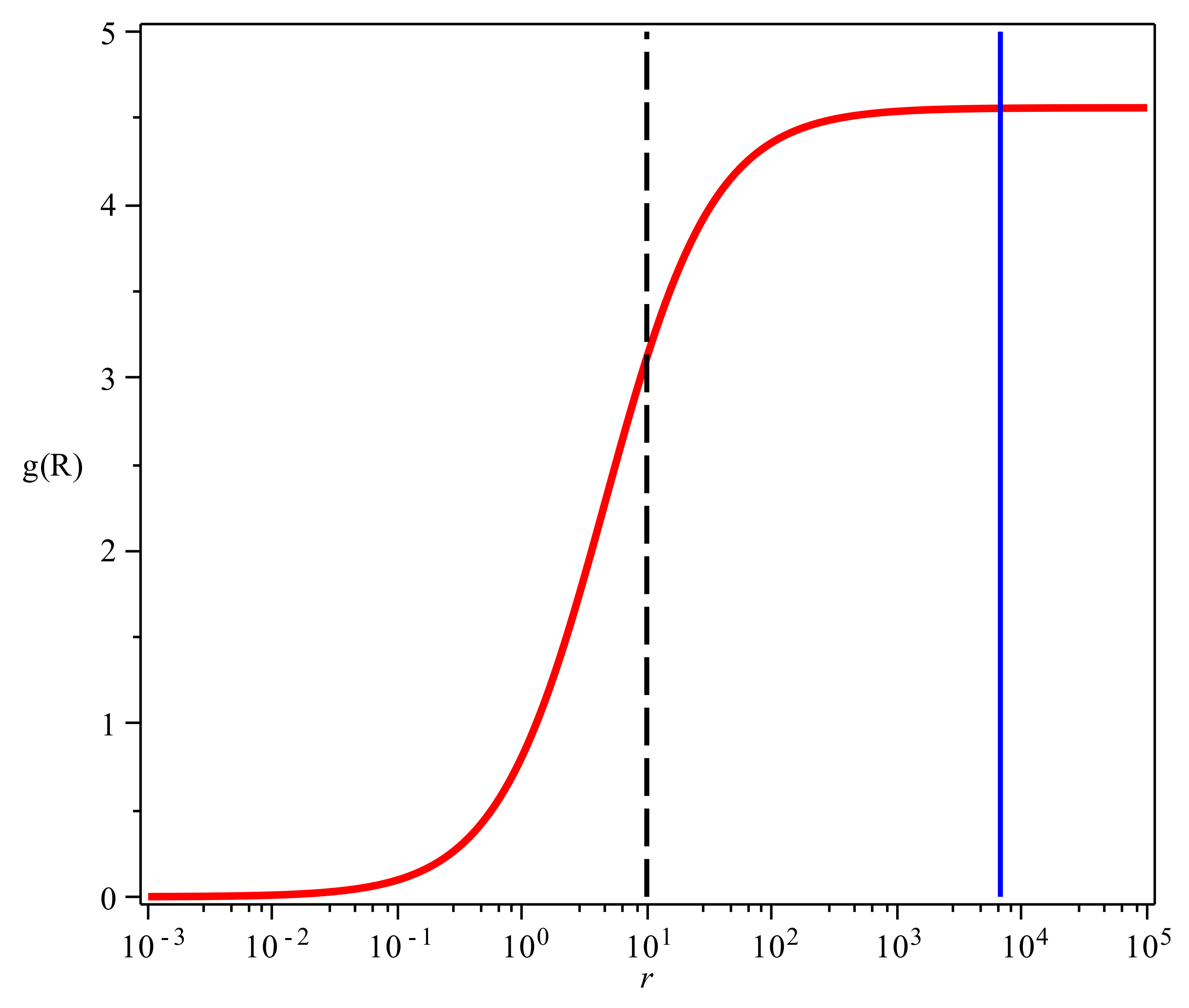


This is consistent with the findingsof the dynamical systems analysis, as we found, that orbits most likely originated from radiation like fixed points and evolved toward de Sitter like points.

INITIAL VALUE OF CORRECTION FUNCTION: g(R)

Choosing the initial parameters to be exactly equal to their ΛCDM values TODAY is what compromised the agreement between the Hu-Sawicki model and the ΛCDM model.

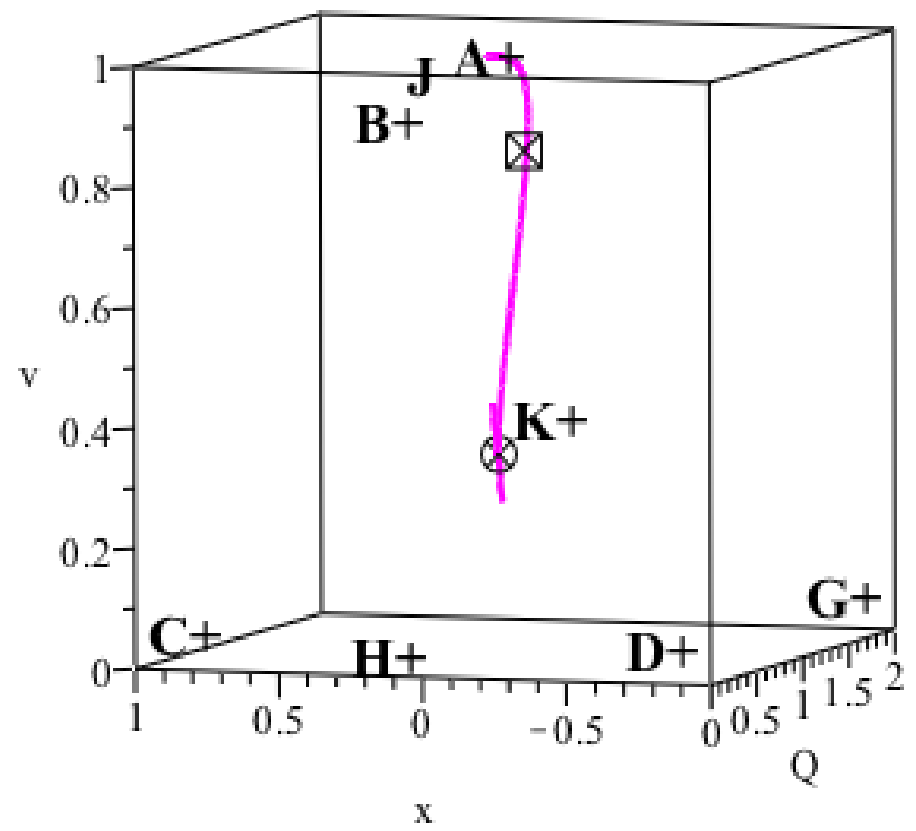
In order for the model to mimic ΛCDM, the initial value of the Ricci scalar should lie on the plateau corresponding to the constant value of g(R)



EXPANSION HISTORY $z = 20$

To demonstrate the effects of placing the initial value of the correction on its function's plateau:

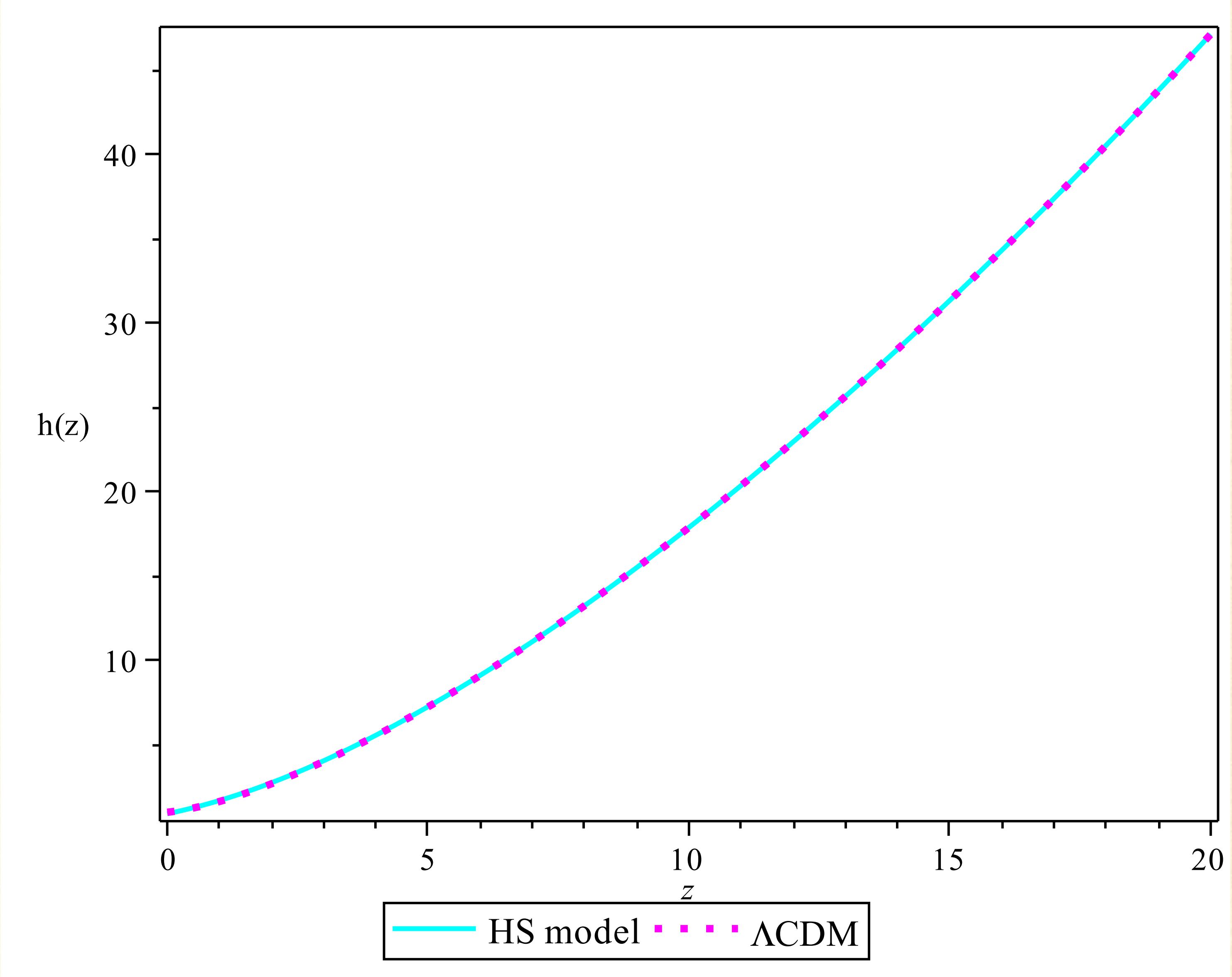
This trajectory in the phase space immediately shows an improvement, it evolves from an initially matter dominated era toward a de Sitter like expansion era.



EXPANSION HISTORY $z = 20$

To demonstrate the effects of placing the initial value of the correction on its function's plateau:

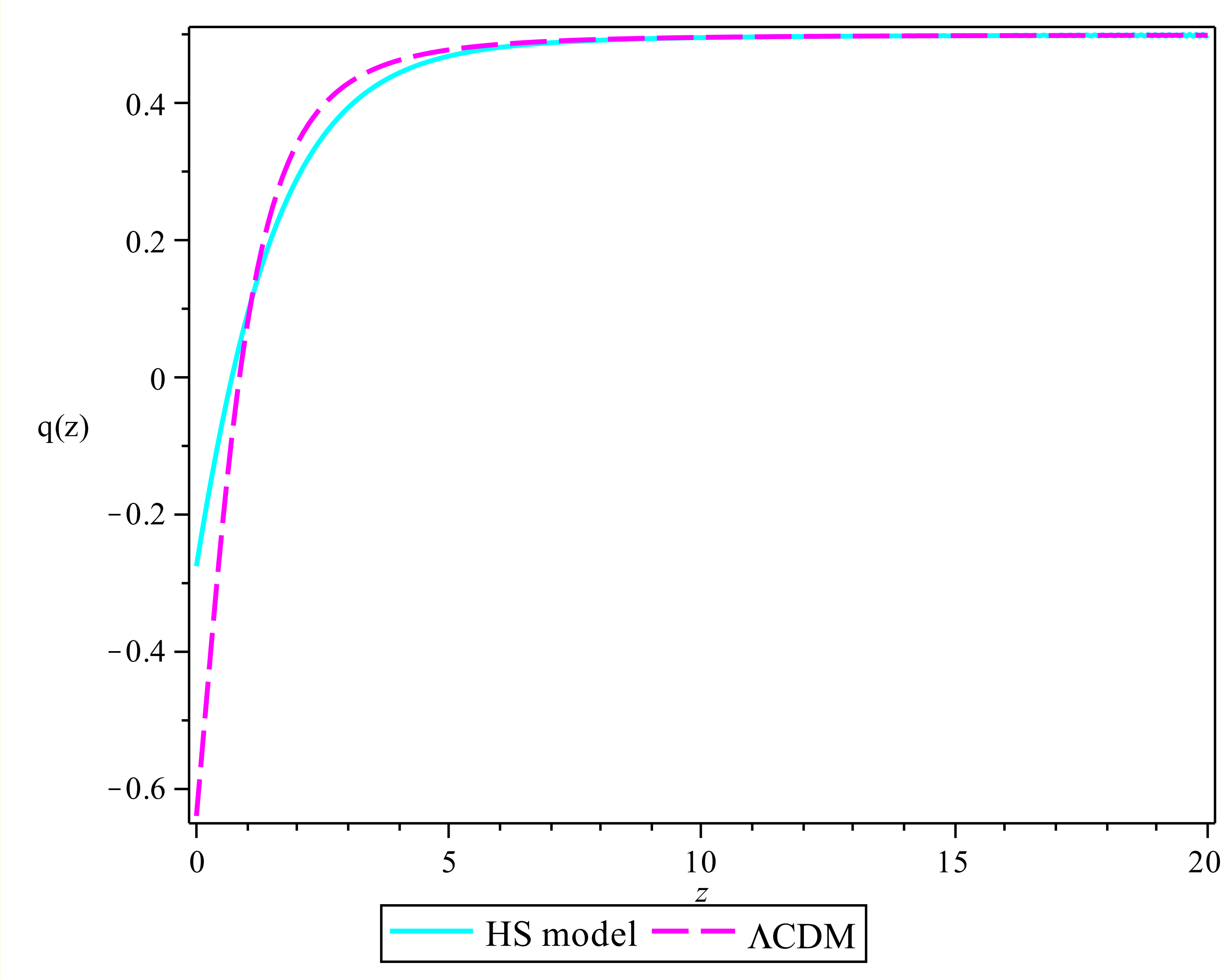
The improvement in agreement in the calculated Hubble parameter and the ΛCDM model is obvious



EXPANSION HISTORY $z = 20$

To demonstrate the effects of placing the initial value of the correction on its function's plateau:

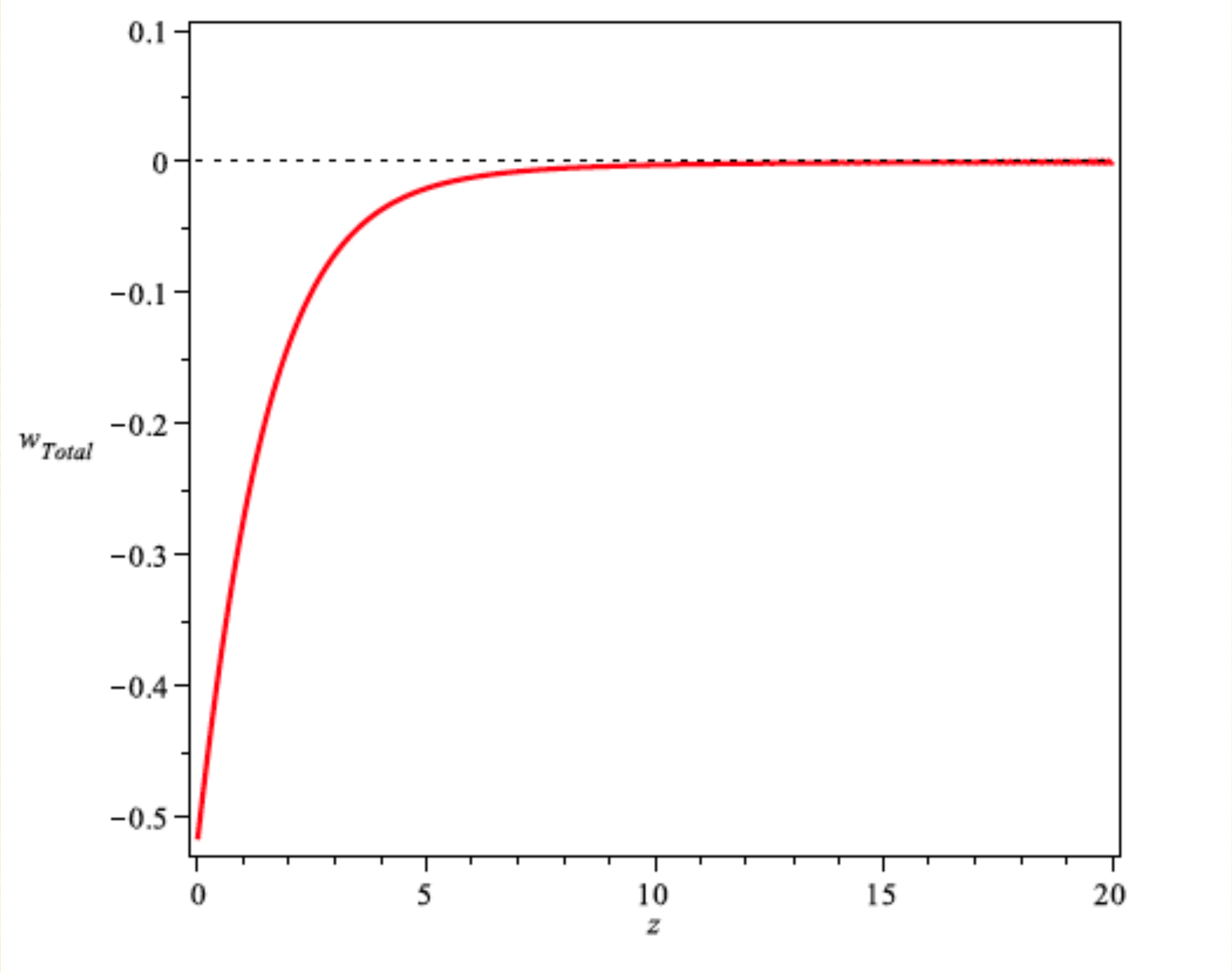
Setting the initial conditions
further back in time has also
visibly improved the asymptotic behaviour
of the deceleration parameter



EXPANSION HISTORY $z = 20$

And finally, the correct asymptotic behaviour of the total effective equation of state has been achieved.

It appears to be dominated by dust until the quite recent past, thereafter a "fluid" with negative pressure begins to dominate the energy content.



CONCLUSIONS

- A dynamical systems analysis of a specific model in the Hu-Sawicki class of f(R) models for gravity, showed that that de Sitter attractors do exist in the phase space, as well as unstable radiation like points, and a non analytic matter like point.
- This choice was limited by the chosen DS framework which only permitted an analysis of the (n=1,c1=1) case
- In order for the Hu-Sawicki model to mimic the ΛCDM model, we need to ensure that the initial value of the correction is placed comfortably on the function's plateau. Setting the initial conditions to the ΛCDM values at z=20 set g(R) equal to its plateau value, thereby almost correcting the bad behaviour.

CONCLUSIONS

- The HS model can provide the desired properties, an expansion which accelerates at low red shifts without an explicit cosmological constant term representing an energy density, but only at the expense of the present values of q_0 and H_0.
- Initial conditions must be set at high redshifts, where the behaviour is understood, and integrated out to future, where we wish to investigate viability.

DYNAMICAL SYSTEMS APPROACH TO F(R) GRAVITY

- Need to express the cosmological equations (Modified Friedmann, modified Raychaudhury)
    - in terms of a set of generalised dynamical variables
    - as a set of autonomous differential equations

- A popular method is to define a set of expansion normalised, **Hubble normalised**, variables

$$H^2 = \frac{1}{3f'}\left(\rho + \frac{1}{2}\left(Rf' - f\right) - 3H\dot{R}f''\right)$$

- But such a state space is only compact for simple, expanding models, where the theory does not allow for recollapsing, bouncing or static universes
- This translates to: there is no contribution to the respective Friedmann equation allowing for zero valued or negative H(z)