Area deficits and gravitational energy

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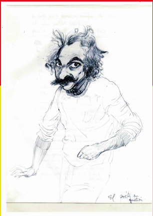
Traveling through Pedro's universes,
Madrid, 3rd December 2018

In Memoriam, Pedro Félix González Díaz (1947 – 2012)



En homenaje a Pedro

Introduction

Gravity is curvature, is geometry.

- Direct detection of gravitational waves from a black-hole binary (in 2015)
- Physics Nobel Prize 2017
- Measured with a laser interferometer able to "feel" tiny geometric disturbances
- Surely, these waves carry energy and momentum!

- Equivalence Principle: the gravitational field can be made to vanish along any causal curve
- That implies that the gravitational energy-momentum can be set to zero anywhere at will
- Gravitational energy is not localizable

No $t_{\mu\nu}$!

Equivalence principle implies that there is no energy-momentum *tensor* for the gravitational field.

There are definitions of "total energy-momentum" for isolated systems, and other global interesting energy-momentum quantities, but how to quantify the energy that affected the LIGO/VIRGO interferometer?

Introduction

Gravity is curvature.

How does curvature affect area/volume?

Pauli:Theory of Relativity

In an arbitrary [n-dimensional] Riemannian manifold, [the volume of a hyper-sphere of radius ℓ] becomes a complicated function of ℓ. We can imagine it to be expanded in a power series in ℓ and retain only the [first non-trivial] term. This gives

$$V = \Omega_n \ell^n \left(1 + \frac{\mathcal{R}}{6(n+2)}\ell^2 + \dots\right)$$

[...] Differentiating, one obtains [...] the formula for the surface of the sphere

$$A = n\Omega_n \ell^{n-1} \left(1 + \frac{\mathcal{R}}{6n}\ell^2 + \dots\right)$$

Here, V is the volume of the small ball, A is the "area" of its boundary, ℓ its radius, and $\mathcal{R}$ the scalar curvature of the space at the ball's center.

$$\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

is the volume of the unit n-sphere.

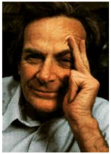
Introduction

Matter generates gravity
(ergo curvature)

What does matter do to geometry?

The Feynman lectures, vol.2

The rule that Einstein gave for the curvature is the following: If there is a region of space with matter in it and we take a sphere small enough that the density ϱ *of matter inside it is effectively constant, then the radius excess for the sphere is proportional to the mass inside the sphere. Using the definition of excess radius, we have*

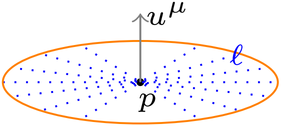


$$\delta\ell|_A = \ell - \sqrt{\frac{A}{4\pi}} = \frac{G}{3c^2} M \left(= \frac{G}{3c^2} \frac{4\pi}{3} \varrho \ell^3 \right)$$

Here M is the mass inside the sphere, and $\delta\ell|_A$ is the "excess" radius to keep the area fixed.

Spatial geodesic balls

Choose $p \in \mathcal{M}$ and then choose $u^\mu \in T_p\mathcal{M}$, $u^\mu u_\mu = -1$.



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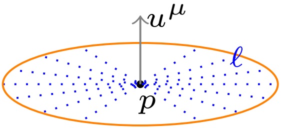

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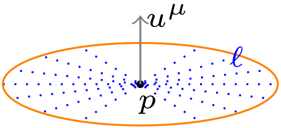
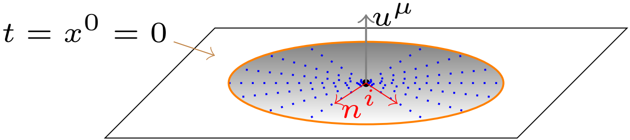


- Take RNC $\{x^\mu\}$ based at p and adapt them so that $u^\mu = \delta^\mu_0$
- The spatial geodesic ball lies on the hypersurface $t \equiv x^0 = 0$ and the spacelike geodesics generating it have

$$x^\mu = r n^\mu, \quad u_\mu n^\mu = 0, \quad \Longrightarrow n^\mu = n^i \delta^\mu_i$$

where r is the affine parameter and we set $\delta_{ij} n^i n^j = 1$



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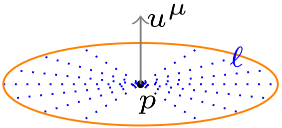

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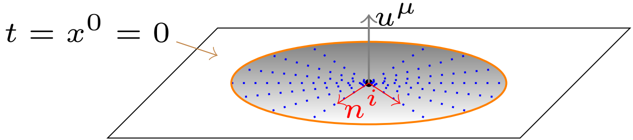

- $\{\theta^A\}$ are local coordinates on the ball's boundary, $n^i(\theta^A)$

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- A calculation at linear order in the curvature gives, for the volume of the geodesic ball (d is the spacetime dimension)

$$V - V^\flat = \Omega_{d-2}\ell^{d-2}\left(\delta\ell_1 - \frac{\mathcal{R}}{6(d^2-1)}\ell^3\right) := \delta^{(1)}V$$

where $V^\flat = \Omega_{d-1}\ell^{d-1} = \Omega_{d-2}\ell^{d-1}/(d-1)$ is the volume of a radius ℓ round ball in Euclidean space;

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- And for the area

$$A - A^\flat = \Omega_{d-2}\ell^{d-3}\left((d-2)\delta\ell_1 - \frac{\mathcal{R}}{6(d-1)}\ell^3\right) := \delta^{(1)}A$$

where $A^\flat = \Omega_{d-2}\ell^{d-2}$ has the same meaning. $\mathcal{R}$ is the intrinsic scalar curvature of the $t = 0$ hypersurface at p.

Using the Einstein field equations

- Note: at first order, the volume and area depend only on the spherically symmetric "excess" $\delta\ell_1$, and not on the direction-dependent $\tilde{\delta}\ell_1(\theta_A)$.

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- We also recover Feynman's interesting remark by keeping, instead, the area A fixed ($A = A^\flat$), noticing that ($G_{\mu\nu}$ is the Einstein tensor)

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- Then

$$\delta\ell_1|_A = \frac{\ell^3}{3(d-1)(d-2)} G_{00} = \frac{8\pi G}{c^4} \frac{\ell^3}{3(d-1)(d-2)} T_{00}$$

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- The area deficit is in both cases proportional to the energy density (at the center of the ball), but the proportionality factor is different. What is the correct factor?

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- This correctness is based on the use of a Bekenstein-Hawking entropy $Ac^3/4G\hbar$, and on an entanglement entropy which is stationary for a conformal field theory when the Einstein equations hold.
- He even argued that Einstein's field equations could be deduced from the above expressions by assuming an equilibrium condition for the vacuum entanglement entropy !

A relationship between area deficit and energy density!

- Thus, the right relation between area deficit and energy density at first order is taken to be

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- Can this relationship between area deficit and energy density be taken as a guiding principle, valid in more general situations?

Vacuum!

What does pure gravity do to geometry?

T Jacobson, JMM Senovilla, A Speranza, *Class. Quantum Grav* **35** *(2018) 085005*

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- Alternatively, area deficits could help provide a notion of quasilocal energy for the gravitational field.
- At second order, the volume of a geodesic ball and the area of its boundary receive corrections depending quadratically on the curvature.

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$$D_{ijkl} = F_{ijkl} + \frac{1}{d-3}(E_{ik}h_{jl} - E_{jk}h_{il} - E_{il}h_{jk} + E_{jl}h_{ik})$$

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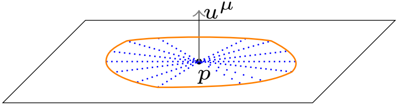
- Observe: F_{ijkl} vanishes in $d = 4$, in which case D_{ijkl} is equivalent to E_{ij}, and E_{ij} and $B_{ij} \equiv \frac{1}{2}\epsilon_{jkl}H_i{}^{kl}$ are simply referred to as the electric and magnetic parts relative to u^α.

The ball at second order



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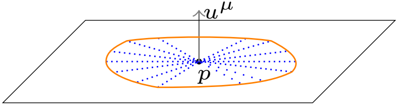

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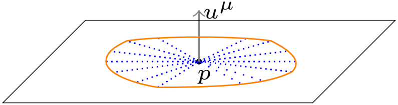

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- As a function defined on the $(d-2)$-sphere, $\tilde{\delta}\ell_1$ can be expanded in spherical harmonics. Letting s denote the "spin," we have

$$\tilde{\delta}\ell_1 = \sum_{s=1}^{\infty} Y_{i_1 \ldots i_s} n^{i_1} \ldots n^{i_s}$$

where $Y_{i_1 \ldots i_s}$ are totally symmetric and traceless for $s > 1$.

Volume of geodesic balls at quadratic order

The volume of the ball at this order (with $R_{\mu\nu} = 0$ and $\delta\ell_1 = 0$) is

$$V = V^\flat + \underbrace{\frac{\Omega_{d-2}\ell^{d+3}}{15(d^2-1)(d+3)}\left[-\frac{D^2}{8} - \frac{H^2}{2} + \frac{E^2}{3}\right]}_{O(2)}$$

$$+ \underbrace{\Omega_{d-2}\ell^{d-3}\left[\ell\delta\ell_2 + (d-2)\sum_{s=1}^{\infty} c_s Y^2_{[s]} - \frac{\ell^3}{3(d^2-1)} Y^{ij} E_{ij}\right]}_{O(2)}$$

where c_s are known constant factors depending on d and s.

$$(Y^2_{[s]} \equiv Y_{i_1\ldots i_s} Y^{i_1\ldots i_s},\ E^2 \equiv E_{ij}E^{ij},\ \text{and so on})$$

Area of geodesic balls at quadratic order

Similarly, the area of the ball's boundary at this order is

$$A = A^\flat + \underbrace{\frac{\Omega_{d-2}\ell^{d+2}}{15(d^2-1)}\left[-\frac{D^2}{8} - \frac{H^2}{2} + \frac{E^2}{3}\right]}_{O(2)}$$

$$\underbrace{+\,\Omega_{d-2}\ell^{d-4}\left[(d-2)\ell\delta\ell_2 + \sum_{s=1}^{\infty} b_s Y_{[s]}^2 - \frac{\ell^3 d}{3(d^2-1)} Y^{ij} E_{ij}\right]}_{O(2)}$$

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- Only the spin-2 deformation gives a different contribution to the area in curved space than in flat space: the term $Y^{ij} E_{ij}$
- Thus, $Y_{[s]}$ for all $s \neq 2$ cannot be fixed in terms of the local gravitational field at this order in perturbations, and only the component of Y_{ij} aligned with E_{ij} contributes differently than in flat space, hence

$$Y_{ij} = \gamma E_{ij}$$

With this in mind, setting $Y_{i_1 \ldots i_s} = 0$ for all $s \neq 2$ and $Y_{ij} = \gamma E_{ij}$, and using the explicit value of b_2, we can rewrite

$$
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Observe that the magenta terms alone give an expression which is not negative definite.

(Unless $d = 4$, where they reduce to $-B^2 - E^2/6$).

Does $\delta A^{(2)}$ provide gravitational energy formula?

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What should we expect as the correct answer at this quadratic order, and in vacuum?

Required properties

There are several desirable and expected properties for the proper deficit $-\delta^{(2)} A$ if it is to describe gravitational strength:

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3. It should be the timelike component (with respect to u^{μ}) of a tensor field

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6. The dominant property also guarantees that the 'momentum density' vector (the tensor contracted on u^μ on all indices but one) is future-pointing timelike or null. This momentum density points in the direction of propagation of the putative energy

Interlude: Bel-Robinson super-energy tensor

There is a unique (symmetric) tensor with the above properties (JMMS , Class. Quantum Grav. **17** (2000) 2799):

the **generalized Bel-Robinson tensor** $T_{\alpha\beta\mu\nu}$.

Recall: the electromagnetic field ($d = 4$)

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho} \star F_{\nu}{}^{\rho}\right)$

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Local tensor describing gravitational strength

- the paradigmatic such tensor is the **Bel-Robinson tensor** given in 4 dimensions by

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- The Bel-Robinson tensor is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!
- It looks related somehow to the energy-momentum properties of the the gravitational field, but its physical dimensions (L^{-4}) are wrong
- is there any relation with gravitational energy?

Quasilocal energy in the small sphere limit ($d = 4$)

- Take any of the (many) definitions of quasilocal energy E for closed surfaces and apply it to a very small sphere of radius r. Then one can prove that at first non-trivial order in r one gets

$$E = \frac{4\pi}{3} r^3 T_{00} + O(r^4)$$

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  where $\mathcal{T}_{0000}$ is the timelike component of the Bel-Robinson tensor (the "super-energy density").
- Analogously, the gravitational momentum vector of a small sphere leads to T_{0i} and, in vacuum, to $\mathcal{T}_{000i}$. The energy flux of a gravitational plane wave, for instance, travels in the direction of $\mathcal{T}_{000i}$.

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- The corresponding totally timelike component (Bel-Robinson energy density) is

$$W := T_{0000} = \frac{1}{2}\left(E^2 + H^2 + \frac{D^2}{4}\right)$$

$$(W = E^2 + B^2 \text{ if } d = 4.)$$

The area deficit in terms of W

$$\delta^{(2)}A = \frac{\Omega_{d-2}\ell^{d+2}}{(d^2-1)}\left[-\frac{W}{15} + E^2\left(\gamma^2(d^2-3d+4) - \frac{\gamma d}{3} + \frac{1}{18}\right) + \frac{(d-2)(d^2-1)}{\ell^5}\delta\ell_2\right]$$

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- The freedom encoded in γ and $\delta\ell_2$ is obviously enough to get something proportional to W,
- Generically the 2nd-order radius variation $\delta\ell_2$ has to be nonzero for this to occur.
- Oddly enough, precisely when $d = 4$ and $\gamma = \gamma_0 = 1/12$, the E^2 coefficient vanishes, leaving

$$\delta^{(2)} A|_\ell = -\frac{\Omega_2 \ell^6}{225} W$$

if the radius is held constant.

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- In summary, how to be sure that a given deformed ball (a volume limited by an area) is the "same" as a corresponding ball in flat spacetime?

Fixing the ball deformation

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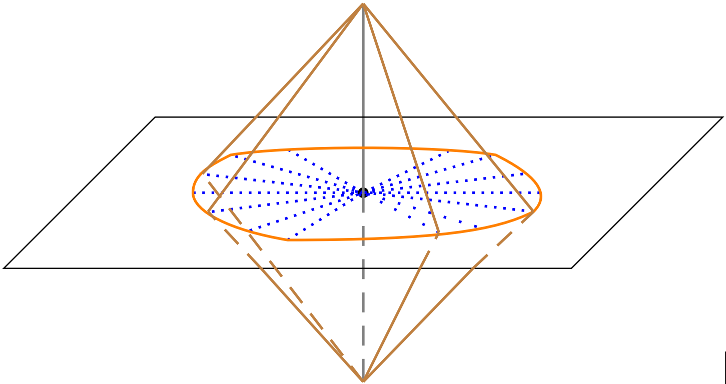
A particularly natural way to define the ball deformation is to choose its shape to ensure that the ball is the base of a small causal diamond.



The ball as the base of a causal diamond

- The calculation, assuming $R_{\mu\nu} = 0$ and at linear order in the curvature gives

$$t = 0, \qquad r = \ell \left(1 + \frac{1}{6} \ell^2 n^i n^j E_{ij} \right),$$

where ℓ / c is the (future and past) proper times of the central geodesics.

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- It leaves $\delta \ell_2$ free, as this is just an ambiguity in the value ℓ / c of the proper time corresponding to the apexes of the cones.

Two further independent arguments

- The trace of the 2nd fundamental form of the ball's boundary is (at this order)

$$K = \frac{d-2}{\ell} + \frac{1}{\ell^2} \sum_{s \neq 2} \left[(d-2)(s-2) + s(s-1)\right] Y_{i_1 \ldots i_s} n^{i_1} \ldots n^{i_s}$$

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- That is, $\gamma = 1/6$ as before!

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It seems that we must compare geodesic balls that are the base of a small causal diamond in their respective spacetimes.

This provides an intrinsic definition, independent of the spacetime, of the boundary of the ball.

The area deficit with $\gamma = 1/6$

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- Again: What is to be compared?

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- Altogether, this is a little puzzling!

Other possibilities

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- My favorite bet at present: keep the causal diamond construction, but forget about geodesic balls: define the co-dimension 2 "surface" as the diamond spacelike boundary and then try to control the volume by considering all possible hypersurfaces with such boundary
- An interesting idea is to <u>maximize</u> the volume enclosed by such a boundary (in flat space one knows that this is a round ball).

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Is the latter described by the Bel-Robinson W?

Gracias Pedro.