Soluciones de problemas 1 de EDII(r) (2011)

1 $\boxed{\begin{array}{l}3x^2u_y+u_x=x^5\\ u(x,0)=x^3\end{array}}$ $\frac{dy}{dx}=3x^2$ $\begin{cases}\xi=y-x^3\\ \eta=x\end{cases}$ $\to u_\eta=\eta^5,\ u=\frac{\eta^6}{6}+p^*(\xi)=\frac{x^6}{6}+p^*(y-x^3)\to p^*(-x^3)=x^3-\frac{x^6}{6}$

O bien, $\begin{cases}\xi=y-x^3\\ \eta=y\end{cases}$ $\to u_\eta=\frac{1}{3}x^3=\frac{\eta-\xi}{3}\ ,\ u=\frac{\eta^2}{6}-\frac{\xi\eta}{3}+p(\xi)=\frac{yx^3}{3}-\frac{y^2}{6}+p(y-x^3)\to p(-x^3)=x^3$

$\to\ p^*(v)=-v-\frac{v^2}{2}\ \forall v$ ó $p^*(v)=-v\ \forall v\to$ $\boxed{u=\frac{yx^3}{3}-\frac{y^2}{6}-y+x^3}$ A pesar de la tangencia en $(0,0)$ $[\Delta=3x^2]$, hay solución única.

$\boxed{\begin{array}{l}(2y-x)u_y+xu_x=2y\\ u(1,y)=0\end{array}}$ $\frac{dy}{dx}=\frac{2y}{x}-1\to y=Cx^2+x\to\begin{cases}\xi=\frac{y}{x^2}-\frac{1}{x}\\ \eta=x\end{cases}\to xu_\eta=2y,\ u_\eta=2+2\xi\eta\to$

$u=2\eta+\xi\eta^2+p(\xi)=x+y+p(\frac{y}{x^2}-\frac{1}{x})\to\ 1+y+p(y-1)=0\to p(v)=-v-2,$ $\boxed{u=\frac{1}{x}-\frac{y}{x^2}+x+y-2}$.

Solución única (era $\Delta=-1\cdot1\neq0$).

$\boxed{\begin{array}{l}u_y+xu_x=-x^2e^{-y}\\ u(-1,y)=0\end{array}}$ $\frac{dy}{dx}=\frac{1}{x}\to\log|x|-y=C,$ $\begin{cases}\xi=x\,e^{-y}\\ \eta=y\end{cases}\to u_\eta=-\xi^2e^\eta,\ u=p(\xi)-\xi^2e^\eta=p(x\,e^{-y})-x^2e^{-y}$

[con $\eta=x\to u_\eta=-x\,e^{-y}=-\xi,\ u=p(\xi)-\xi\eta=p(x\,e^{-y})-x^2e^{-y}$ como antes].

$u(-1,y)=p(-e^{-y})-e^{-y}=0\to p(v)=-v$ (para $v<0$) $\to$ $\boxed{u=-(x+x^2)\,e^{-y}}$ (vale para todo x)

$\boxed{\begin{array}{l}xu_y-yu_x=2xyu\\ u(x,0)=x\end{array}}$ $\frac{dy}{dx}=-\frac{x}{y}\to\begin{cases}\xi=x^2+y^2\\ \eta=y\end{cases}\to u_\eta=2\eta u,\ u=p(\xi)\,e^{\eta^2}=p(x^2+y^2)\,e^{y^2}\to p(x^2)=x$

$p(v)=\pm\sqrt{v}\ \to$ $\boxed{u=\pm\sqrt{x^2+y^2}\,e^{y^2}\quad\begin{array}{l}x>0\\ x<0\end{array}}$ [problemas en el origen: $\Delta=x=0$ si $x=0$]

$\boxed{\begin{array}{l}u_x-u_y=\frac{x-y}{xy}u\\ u(x,1)=x\end{array}}$ $\begin{cases}\xi=x+y\\ \eta=x\end{cases}\to u_\eta=\frac{2\eta-\xi}{\eta(\xi-\eta)}u\ ,\ u=\frac{p(\xi)}{\eta(\xi-\eta)}=\frac{p(x+y)}{xy}\to\frac{p(x+1)}{x}=x\to$ $\boxed{u=\frac{(x+y-1)^2}{xy}}$ $[\Delta=-1]$

$\boxed{\begin{array}{l}u_y+3y^2u_x=\frac{2u}{y}+6y^4x\\ u(x,1)=x^2\end{array}}$ $\begin{cases}\xi=x-y^3\\ \eta=y\end{cases}\to u_\eta=\frac{2u}{\eta}+6\eta^4(\xi+\eta^3)\ ,\ u=p(\xi)\eta^2+\eta^2(\xi+\eta^3)^2=p(x-y^3)y^2+x^2y^2$

O bien, $\begin{cases}\xi=x-y^3\\ \eta=x\end{cases}\to u_\eta=\frac{2u}{3(\eta-\xi)}+2(\eta-\xi)^{2/3}\eta\ ,\ u=p(\xi)(\eta-\xi)^{2/3}+\eta^2(\eta-\xi)^{2/3}$ (||)

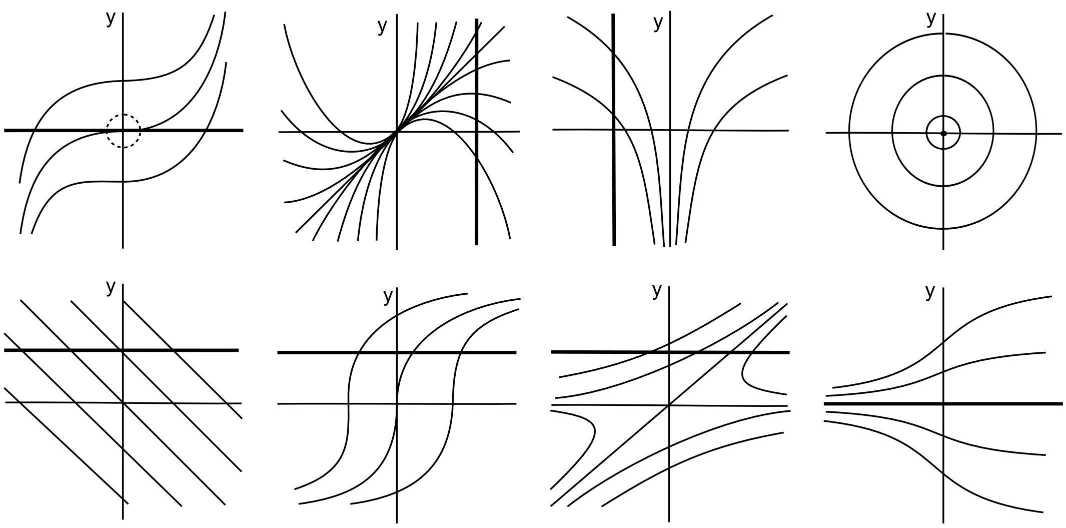
$u(x,1)=p(x-1)+x^2=x^2\to p(v)=0$ (para todo v, $\Delta=-1$) $\to$ $\boxed{u=x^2y^2}$

$\boxed{\begin{array}{l}yu_y+(2y-x)u_x=x\\ u(x,1)=0\end{array}}$ $\frac{dy}{dx}=\frac{y}{2y-x}$ (exacta u homogénea) o mejor $\frac{dx}{dy}=-\frac{x}{y}+2$ lineal $\to xy-y^2=C$, $\begin{cases}\xi=xy-y^2\\ \eta=y\end{cases}$

$\to\eta u_\eta=\frac{\xi+\eta^2}{\eta}\ ,\ u=p(\xi)-\frac{\xi}{\eta}+\eta=p(xy-y^2)+2y-x\to p(v)=v-1\ \forall v\to$ $\boxed{u=xy-y^2+2y-x-1}$ $[\Delta=1]$

$\boxed{\begin{array}{l}yu_y+e^{x^2}u_x=2x\\ u(x,0)=0\end{array}}$ $\frac{dy}{dx}=e^{-x^2}y\to y=C\,e^{\int_0^x e^{-s^2}ds},$ $\begin{cases}\xi=y\,e^{-\int_0^x e^{-s^2}ds}\\ \eta=x\end{cases}\to u_\eta=2\eta\,e^{-\eta^2}\ ,\ u=p(\xi)-e^{-\eta^2},$

$u=p(y\,e^{-\int_0^x e^{-s^2}ds})-e^{-x^2},\ u(x,0)=p(0)-e^{-x^2}=0$ no tiene solución $[\Delta=0\cdot1-e^{x^2}\cdot0\equiv0]$.



2 $\boxed{yu_y - xu_x = u+2x}$ $\frac{dy}{dx} = -\frac{y}{x} \to y = \frac{C}{x} \to \begin{cases} \xi = xy \\ \eta = x \end{cases} \to u_\eta = -\frac{u}{\eta} - 2$, $u = \frac{p(\xi)}{\eta} - \eta = \frac{p(xy)}{x} - x$.

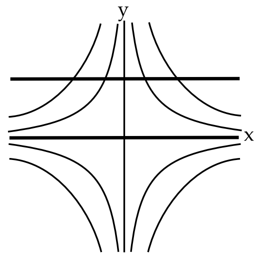
O bien, $\begin{cases} \xi = xy \\ \eta = y \end{cases} \to u_\eta = \frac{u}{\eta} + \frac{2\xi}{\eta^2}$, $u = p^*(\xi)\eta + \eta \int \frac{2\xi}{\eta^3} d\eta = p^*(\xi)\eta - \frac{\xi}{\eta} = p^*(xy)y - x$.

i) $u(x,0) = \frac{p(0)}{x} - x = -x \to$ toda $p \in C^1$ con $p(0)=0$ lo cumple, por ejemplo, $\begin{matrix} p(v)\equiv 0 \to u = x \\ p(v) = v \to u = y - x \end{matrix}$.

O bien, $u(x,0) = -x = -x$ para toda $p^* \in C^1$; eligiendo $p^*(v) \equiv 0, 1$ obtenemos las de arriba.

ii) $u(x,2) = \frac{p(2x)}{x} - x = 7x$, $p(2x) = 8x^2 \to p(v) = 2v^2 \to \boxed{u(x,y) = 2xy^2 - x}$.

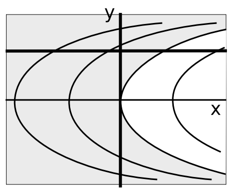
O bien, $u(x,2) = 2p^*(2x) - x = 7x$, $p^*(2x) = 4x \to p^*(v) = 2v$ ↗.



El dibujo de las características muestra que para i) había problemas de unicidad [dato sobre característica] y que había solución única para ii) [no hay tangencia]. El Δ nos lo confirma:

i) $\Delta = 1 \cdot 0 - 0 \cdot (-x) \equiv 0$, ii) $\Delta = 1 \cdot 2 - 0 \cdot (-x) = 2 \neq 0$.

3 $\boxed{u_y + 2yu_x = 3xu}$ con: i) $u(x,1) = 1$, ii) $u(0,y) = 0$.



$\frac{dy}{dx} = \frac{1}{2y} \to x - y^2 = K \to \begin{cases} \xi = x - y^2 \\ \eta = y \end{cases} \to u_\eta = 3xu = (3\xi + 3\eta^2)u$

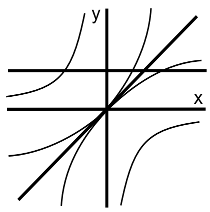
$\to u = p(\xi) e^{3\xi\eta + \eta^3} = p(x-y^2)\, e^{3xy - 2y^3}$.

[Es bastante más largo con $\begin{cases} \xi = x - y^2 \\ \eta = x \end{cases} \to 2yu_\eta = 3xu$, $u_\eta = \frac{3\eta}{[\xi - \eta]^{1/2}} u$, ...].

i) $p(x-1)\, e^{3x-2} = 1$, $p(v) = e^{-3v-1} \to \boxed{u = e^{3xy - 3x - 2y^3 + 3y^2 - 1}} = e^{(y-1)(3x - 2y^2 + y + 1)}$ (única; $\Delta \equiv 1$).

ii) $u(0,y) = p(-y^2)\, e^{-2y^3} = 0 \to p(v) \equiv 0$, si $v \leq 0$, pero indeterminada si $v > 0 \to u \equiv 0$, si $x \leq y^2$, indeterminada si $x > y^2 \to$ solución única excepto en un entorno del origen ($\Delta = -2y$).

4 $\boxed{y^2 u_y + x^2 u_x = x^2 + y^2}$ $\frac{dy}{dx} = \frac{y^2}{x^2} \to \frac{1}{y} - \frac{1}{x} = C \to \begin{cases} \xi = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy} \\ \eta = y \text{ (por ejemplo)} \end{cases} \to y^2 u_\eta = x^2 + y^2$



$u_\eta = \frac{1}{(1-\xi\eta)^2} + 1 \left[x = \frac{\eta}{1 - \xi\eta} \right] \to u = \frac{1}{\xi(1-\xi\eta)} + \eta + p(\xi) = \frac{x^2}{x-y} + y + p\left(\frac{x-y}{xy}\right)$

$u(x,1) = \frac{x^2+x-1}{x-1} + p\left(\frac{x-1}{x}\right) = x + 1$, $p\left(\frac{x-1}{x}\right) = -\frac{x}{x-1} \to p(v) = -\frac{1}{v}$, $u = \frac{x^2 + xy - y^2 - xy}{x-y} = \boxed{x+y}$.

Solución única porque no es tangente $y=1$ en ningún punto a las características:

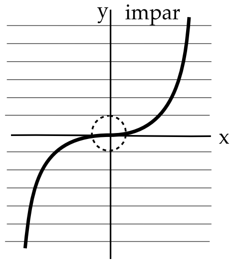
$\Delta = 1 \cdot 1 - 0 \cdot x^2 = 1 \neq 0\ \forall x$ [o porque las características crecen estrictamente en $y > 0$].

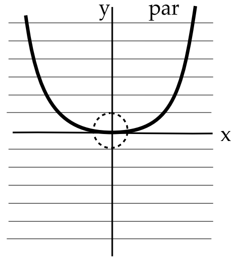
Hay tres características sencillas: $y=0$, $x=0$ e $y=x$ (para $C=0$), pero las tres anulan denominadores. Para dar datos en $y=0$ ponemos: $u = \frac{x^2}{x-y} + y + p^*\left(\frac{xy}{x-y}\right)$. $u(x,0) = x + p^*(0) = x$ lo cumple toda $p^* \in C^1$ con $p^*(0) = 0$. Eligiendo $p^*(v) = -v$ obtenemos la de antes $u = x+y$, con $p^*(v) \equiv 0$ se tiene $u = \frac{x^2 + xy - y^2}{x-y}$, . . .

[Si hallamos $\Delta = 0 \cdot 1 - 0 \cdot x^2 \equiv 0 \to$ es característica]. [Un dato $u\left(x, \frac{x}{1+x}\right) = \cdots$ evita ceros de denominadores].

5 $\boxed{\begin{matrix} u_x + yu = y^2 \\ u(x,x^n) = x^n \end{matrix}}$ $u = p(y)\, e^{-xy} + y$ [$y=C$ características]

$u(x,x^n) = p(x^n) e^{-x^{n+1}} + x^n = x^n \to p(x^n) = 0$



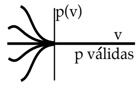


[Hay tangencia en el origen $\forall n \geq 2$, $\Delta = -nx^{n-1}$].

Si n impar: $p(v) \equiv 0\ \forall v \to u = y$, solución única.

Si n par: $p(v) \equiv 0\ \forall v \geq 0$, indeterminada si $v < 0$

$\to u = p(y)\, e^{-xy} + y$, con $p \in C^1$ de la forma:



6 $\boxed{A(x,y,u)u_y+B(x,y,u)u_x=C(x,y,u)}$ $\quad \frac{du}{dx}=\frac{C}{B}\,,\ \frac{dy}{dx}=\frac{A}{B}\to\ \begin{matrix}\eta(x,y,u)=c_1\\ \xi(x,y,u)=c_2\end{matrix}$ [para cada c_1,c_2 hay una curva característica en el espacio].

Si $\eta(x,y,u)=p[\xi(x,y,u)]$ define implícitamente $u=u(x,y)$ [si va bien el teorema de la función implícita]

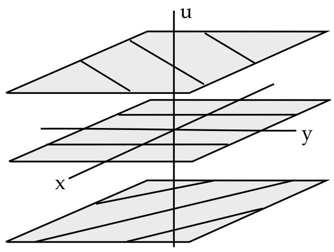
$$\partial_x\to\eta_x+\eta_u u_x=p'(\xi)[\xi_x+\xi_u u_x]\to\ u_x=\frac{p'(\xi)\xi_x-\eta_x}{\eta_u-p'(\xi)\xi_u}\ ;\quad \partial_y\to\ u_y=\frac{p'(\xi)\xi_y-\eta_y}{\eta_u-p'(\xi)\xi_u}\ .\ \text{ Por tanto:}$$

$$Au_y+Bu_x=\frac{1}{\eta_u-p'(\xi)\xi_u}\left[p'(\xi)(A\xi_y+B\xi_x)-(A\eta_y+B\eta_x)\right]=\frac{1}{\eta_u-p'(\xi)\xi_u}\left[p'(\xi)(-C\xi_u)-(-C\eta_u)\right]=C\,,$$

pues: $\xi_x+\xi_y\frac{dy}{dx}+\xi_u\frac{du}{dx}=\frac{1}{B}[B\xi_x+A\xi_y+C\xi_u]=0$ (igual para η, y análogo para $\xi(x,y,u)=q[\eta(x,y,u)]$).

$\boxed{u_y+uu_x=0}$ $\quad \frac{du}{dx}=0\ \to\ u=c_1$

$\qquad\qquad \frac{dy}{dx}=\frac{1}{u}=\frac{1}{c_1}\to y=\frac{x+c_2}{c_1}=\frac{x+c_2}{u}\ \to\ yu-x=c_2$



[las características están contenidas en planos $u=cte$ y cada uno son rectas]

Solución general: (1) $\boxed{yu-x=p(u)}$ o (2) $\boxed{u=q(yu-x)}$.

i) $u(x,0)=x\ \to\ -x=p(x)\ \to\ yu-x=-u\ \to\ \boxed{u=\frac{x}{y+1}}$ [de (2) igual].

ii) $u(0,y)=0\ \to\ 0=p(0)$; para cada $p\in C^1$ con $p(0)=0$ una solución distinta, por ejemplo: $p\equiv 0\to u=\frac{x}{y}$, $p(v)=v\to u=\frac{x}{y-1}$, $p(v)=-v\to u=\frac{x}{y+1}$, ...

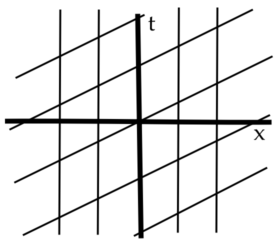
(2) da más: $q(0)=0$; $q\equiv 0\to u\equiv 0$ [no recogida en (1)], ... Problemas por ser $(0,y,0)$ característica.

7 $\boxed{e^x u_{xx}+e^y u_{yy}=u}$ Elíptica en $\mathbf{R}^2$ $\quad \frac{dx}{dt}=\pm i\,e^{(x-y)/2},\ e^{-x/2}\pm i\,e^{-y/2}=C\ \begin{cases}\alpha=e^{-x/2}\\ \beta=e^{-y/2}\end{cases}\to\ u_{\alpha\alpha}+u_{\beta\beta}+\frac{u_\alpha}{\alpha}+\frac{u_\beta}{\beta}=4u$

$\boxed{u_{xx}-3yu_x+2y^2u=y}$ Parabólica en forma normal. $\lambda^2-3y\lambda+2y^2=0\to$ $\boxed{u=p(y)e^{xy}+q(y)e^{2xy}+\frac{1}{2y}}$

$\boxed{u_{xx}+4u_{xy}-5u_{yy}+6u_x+3u_y=9u}$ Hiperbólica $\begin{cases}\xi=x-\frac{y}{5}\\ \eta=x+y\end{cases}$ ó $\begin{cases}\xi=5x-y\\ \eta=x+y\end{cases}\ \to\ 4u_{\xi\eta}+3u_\xi+u_\eta=u$ no resoluble

8 (E) $\boxed{u_{tt}+2u_{xt}=2}$ Hiperbólica $\begin{cases}\xi=x-2t\\ \eta=x\end{cases}\to\begin{cases}u_{tt}=4u_{\xi\xi}\\ u_{tx}=-2u_{\xi\xi}-2u_{\xi\eta}\end{cases}\to$



$u_{\xi\eta}=-\frac{1}{2}\ \to\ u=p(\xi)+q(\eta)-\frac{\xi\eta}{2}=\boxed{u=p(x-2t)+q(x)-\frac{x^2}{2}+xt}$

i) $y=0$ no tangente a las características $\to$ solución única con:

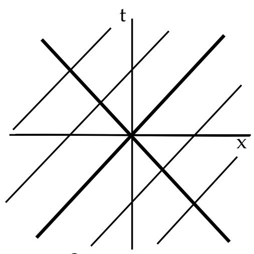
$\begin{cases}0=u(x,0)=p(x)+q(x)-\frac{1}{2}x^2 & q(x)=\frac{1}{4}x^2-C\\ 0=u_t(x,0)=-2p'(x)+x\to p(x)=\frac{1}{4}x^2+C\nearrow & \end{cases}\ \to\ \boxed{u=t^2}$

ii) $u(0,t)=0$, $u_x(0,t)=t$ son datos sobre característica: $\left.\begin{matrix}p(-2t)+q(0)=0\\ p'(-2t)+q'(0)+t=t\end{matrix}\right\}$ Cada q con $q'(0)=0$, y $p(t)\equiv -q(0)$ da una solución distinta (infinitas).

[(E) se puede resolver también: $u_t=v$, $v_t+2v_x=2$, ... O bien: $[u_t+2u_x]_t=2$, $u_t+2u_x=2t+q(x)$, ...]

9 $\boxed{u_{tt}+2u_{xt}+u_{xx}+u=0}$ De coeficientes constantes y parabólica. $x-\frac{B}{2A}t=x-t=K$ características.

$\begin{cases}\xi=x-t\\ \eta=t\end{cases}\to\begin{cases}u_t=-u_\xi+u_\eta\\ u_x=u_\xi\end{cases},\ \begin{cases}u_{tt}=u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}\\ u_{xt}=-u_{\xi\xi}+u_{\xi\eta}\\ u_{xx}=u_{\xi\xi}\end{cases}\ \to\ u_{\eta\eta}+u=0$



$\to\ u=p(\xi)\cos\eta+q(\xi)\,\text{sen}\,\eta$, $\boxed{u=p(x-t)\cos t+q(x-t)\,\text{sen}\,t}$

[Eligiendo $\eta=x$ se llega a $u_{\eta\eta}+u=0\ \to\ u=p^*(x-t)\cos x+q^*(x-t)\,\text{sen}\,x$].

$u_t(x,t)=\left[q(x-t)-p'(x-t)\right]\cos t-\left[p(x-t)+q'(x-t)\right]\text{sen}\,t$

$\begin{cases}p(2x)\cos x-q(2x)\,\text{sen}\,x=0\ _{[\bullet]}\ \to[2p'(2x)-q(2x)]\cos x-[p(2x)+2q'(2x)]\,\text{sen}\,x=0\\ [q(2x)-p'(2x)]\cos x+[p(2x)+q'(2x)]\,\text{sen}\,x=1\end{cases}$

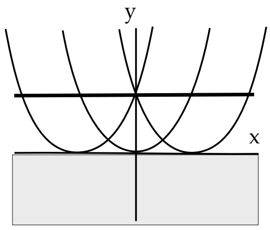
$\overset{1^a+2\times 2^a}{\longrightarrow}\ q(2x)\cos x+p(2x)\,\text{sen}\,x=2\ _{[\circ]}$; $\quad\begin{matrix}[\bullet]\times\cos+[\circ]\times\text{sen}\to p(2x)=2\,\text{sen}\,x\to p(v)=2\,\text{sen}\,\frac{v}{2}\\ [\circ]\times\cos-[\bullet]\times\text{sen}\to q(2x)=2\cos x\to p(v)=2\cos\frac{v}{2}\end{matrix}$

$\to\ u=2\,\text{sen}\,\frac{x-t}{2}\cos t+2\cos\frac{x-t}{2}\,\text{sen}\,t=\boxed{2\,\text{sen}\,\frac{x+t}{2}}$ (solución única).

$\begin{cases}p(0)\cos x+q(0)\,\text{sen}\,x=0\to p(0)=q(0)=0\\ \left[q(0)-p'(0)\right]\cos x-\left[p(0)+q'(0)\right]\text{sen}\,x=0\end{cases}\ \to p(0)=p'(0)=q(0)=q'(0)=0\ \to$ Por ejemplo,

$p(v)=v^2$, $q(v)\equiv 0\ \to\ \boxed{u=(x-t)^2\cos t}$; ó $p(v)=1-\cos v$, $q(v)\equiv 0\ \to\ \boxed{u=[1-\cos(x-t)]\cos t}$; ...

10 a) $\boxed{4yu_{yy}-u_{xx}+2u_y=0}$ $B^2-4AC=16y \to$ $y>0$ hiperbólica, $y<0$ elíptica



$\boxed{y>0}$: $\frac{dx}{dy}=\pm\frac{\sqrt{16y}}{8y}=\pm\frac{1}{2\sqrt{y}} \to x\pm\sqrt{y}=C$ características $[\ y=(x-C)^2\ ]$.

$\begin{cases}\xi=x+\sqrt{y}\\ \eta=x-\sqrt{y}\end{cases} \to \boxed{u_{\xi\eta}=0} \to u=p(\xi)+q(\eta)=\boxed{p(x+\sqrt{y})+q(x-\sqrt{y})}$

Los datos iniciales de b) son para esta región:

$\begin{cases}u(x,1)=p(x+1)+q(x-1)=2x & \to p'(x+1)+q'(x-1)=2\\ u_y(x,1)=\frac{1}{2}p'(x+1)-\frac{1}{2}q'(x-1)=x & \to p'(x+1)-q'(x-1)=2x\end{cases} \to p'(v)=v\,,\ p(v)=\frac{v^2}{2}+k\,,\ q(v)=-\frac{v^2}{2}-k$

$u=\frac{1}{2}\big[(x+\sqrt{y})^2-(x-\sqrt{y})^2\big]=\boxed{2x\sqrt{y}}$.

$\boxed{y<0}$: $\frac{dx}{dy}=\pm\frac{i}{2\sqrt{-y}}$, $x\pm i\sqrt{-y}=C$ $\begin{cases}\xi=x\\ \eta=\sqrt{-y}\end{cases}$ $\begin{cases}u_x=u_\xi\\ u_y=-\frac{(-y)^{-1/2}}{2}u_\eta\end{cases}$ $\begin{cases}u_{xx}=u_{\xi\xi}\\ u_{yy}=-\frac{1}{4y}u_{\eta\eta}+\frac{(-y)^{-1/2}}{4y}u_\eta\end{cases}$, $\boxed{u_{\xi\xi}+u_{\eta\eta}=0}$.

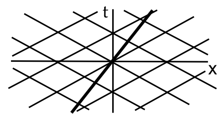
11 $\boxed{\begin{array}{l}u_{tt}-4u_{xx}=2\\ u(x,x)=x^2\,,\ u_t(x,x)=x\end{array}}$. i) Copiando de los apuntes características y cambio:

$\begin{cases}\xi=x+2t\\ \eta=x-2t\end{cases} \to -16u_{\xi\eta}=2 \to u_\xi=p^*(\xi)-\frac{\eta}{8} \to u=p(\xi)+q(\eta)-\frac{\xi\eta}{8}=p(x+2t)+q(x-2t)+\frac{4t^2-x^2}{8}$

$\begin{cases}u(x,x)=p(3x)+q(-x)+\frac{3x^2}{8}=x^2 \to 3p'(3x)-q'(-x)=\frac{5x}{4}\\ u_y(x,x)=2p'(3x)-2q'(-x)+x=x \to q'(-x)=p'(3x)\end{cases} \to p'(3x)=\frac{5x}{8}\,,\ p'(v)=\frac{5v}{24}\,,\ p(v)=\frac{5v^2}{48} \to$

$q(-x)=x^2-\frac{3x^2}{8}-\frac{15x^2}{16}=\frac{25x^2}{16}$, $q(v)=-\frac{5v^2}{16}$. $u=\frac{5(x+2t)^2-15(x-2t)^2+24t^2-6x^2}{48}=\boxed{\frac{1}{3}\big[5xt-t^2-x^2\big]}$.

ii) $w=u-t^2 \to \begin{cases}w_{tt}-4w_{xx}=0\\ w(x,x)=0\,,\ w_t(x,x)=-x\end{cases}$, $w=p(x+2t)+q(x-2t)$ [apuntes].



$\begin{cases}w(x,x)=p(3x)+q(-x)=0\\ u_y(x,x)=2p'(3x)-2q'(-x)=-x\end{cases} \to p'(3x)=\frac{x}{4}\,,\ p(v)=\frac{v^2}{24}\,,\ q(v)=-\frac{3v^2}{8}\,,\ \dots$

12 (E) $\boxed{Au_{yy}+Bu_{xy}+Cu_{xx}+Du_y+Eu_x+Fu=G(x,y)}$ si no es parabólica, $B^2-4AC\neq 0$.

$u=\mathrm{e}^{py}\mathrm{e}^{qx}w \to \begin{cases}u_y=[pw+w_y]\,\mathrm{e}^{py+qx}\\ u_x=[qw+w_x]\,\mathrm{e}^{py+qx}\end{cases}$ $\begin{cases}u_{yy}=[p^2w+2pw_y+w_{yy}]\,\mathrm{e}^{py+qx}\\ u_{xy}=[pqw+pw_x+qw_y+w_{xy}]\,\mathrm{e}^{py+qx}\\ u_{xx}=[q^2w+2qw_x+w_{xx}]\,\mathrm{e}^{py+qx}\end{cases} \to$

$$Aw_{yy}+Bw_{xy}+Cw_{xx}+(2pA+qB+D)w_y+(2qC+pB+E)w_x$$
$$+(p^2A+pqB+q^2C+pD+qE+F)w=\mathrm{e}^{-py-qx}G(x,y)$$

Si $\left.\begin{array}{l}2pA+qB+D=0\\ 2qC+pB+E=0\end{array}\right\} \to \boxed{p=\frac{2CD-BE}{B^2-4AC}}$, $\boxed{q=\frac{2AE-BD}{B^2-4AC}}$, desaparecen los términos en w_y y w_x.

Y la ecuación se convierte en: (E*) $\boxed{Aw_{yy}+Bw_{xy}+Cw_{xx}+\big[\frac{AE^2+CD^2-BDE}{B^2-4AC}+F\big]w=\mathrm{e}^{-py-qx}G(x,y)}$

Por tanto, si las constantes son tales que el corchete se anula, la ecuación tampoco tiene término en w. [Y si es hiperbólica se puede reducir a $u_{\xi\eta}=G^*$, y se puede resolver].

$\boxed{u_{xy}+2u_y+3u_x+6u=1} \to B^2-4AC=1$, $p=-3$, $q=-2$, $[\ \ ]=[\frac{-6}{1}+6]=0$;

$u=\mathrm{e}^{-3y-2x}w \to w_{xy}=\mathrm{e}^{3y+2x} \to w=\frac{1}{6}\mathrm{e}^{3y+2x}+p(x)+q(y) \to \boxed{u=\frac{1}{6}+\mathrm{e}^{-3y}\mathrm{e}^{-2x}[p(x)+q(y)]}$.

Si (E) es parabólica se puede poner en la forma canónica: $u_{\eta\eta}+D^*u_\eta+E^*u_\xi+F^*u=G^*(\xi,\eta)$.

Si $E^*=0$, la ecuación (lineal de segundo orden con coeficientes constantes en η) es resoluble.

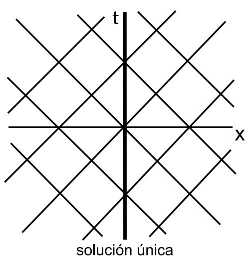
$u=\mathrm{e}^{py}\mathrm{e}^{qx}w \to w_{\eta\eta}+(2p+D^*)w_\eta+E^*w_\xi+(p^2+pD^*+qE^*+F^*)w=\mathrm{e}^{-py-qx}G^*(\xi,\eta)\equiv G^{**}(\xi,\eta)$

Si $E^*\neq 0$, con $\boxed{p=-\frac{D^*}{2}}$, $\boxed{q=\frac{1}{E^*}\big[\frac{D^{*2}}{4}-F^*\big]}$ se convierte en la del calor: $\boxed{w_\eta+E^*w_{\xi\xi}=G^{**}(\xi,\eta)}$.

13 $\boxed{u_{tt}-u_{xx}+Du_t+Eu_x=4}$ a] Las características de esta ecuación son las de la ecuación de ondas (sólo dependen de las derivadas de segundo orden):

$\begin{cases}\xi=x+t\\ \eta=x-t\end{cases} \to -4u_{\xi\eta}+(D+E)u_\xi+(E-D)u_\eta=4 \to$ **si** $\boxed{E=\pm D}$ **es resoluble.**

(Con cambios $u=\mathrm{e}^{pt}\mathrm{e}^{qx}w$ no se consigue nada más).



solución única

b] $u_{\xi\eta}-u_\eta=-1 \xrightarrow{u_\eta=v} v_\xi=v-1 \xrightarrow{v_p\text{ a ojo}} v=p^*(\eta)\mathrm{e}^\xi+1 \to u=q(\xi)+p(\eta)\mathrm{e}^\xi+\eta$

$\to \boxed{u=q(x+t)+p(x-t)\,\mathrm{e}^{x+t}+x-t}$ $(\,u_x=q'+p'\,\mathrm{e}^{x+t}+p\,\mathrm{e}^{x+t}+1\,)$:

$\begin{cases}u(0,t)=q(t)+p(-t)\,\mathrm{e}^t-t=\mathrm{e}^{2t} \to q'(t)-p'(-t)\mathrm{e}^t+p(-t)\mathrm{e}^t=1+2\mathrm{e}^{2t}\\ u_x(0,t)=q'(t)+p'(-t)\,\mathrm{e}^t+p(-t)\,\mathrm{e}^t+1=2 \qquad p'(-t)=-\mathrm{e}^t,\ p'(v)=-\mathrm{e}^{-v}\end{cases}$

$\to p(v)=\mathrm{e}^{-v}+K \to q(t)=t-K\mathrm{e}^t \to u=x+t+\mathrm{e}^{-x+t}\mathrm{e}^{x+t}+x-t$, $\boxed{u=2x+\mathrm{e}^{2t}}$.

Soluciones de problemas 2 de EDII(r) (2011)

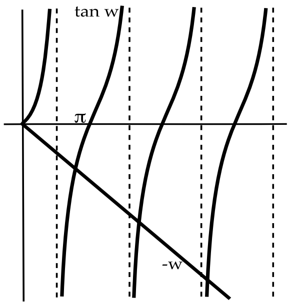
1 $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y(0)=y'(1)=0 \end{array}}$ $\lambda\geq 0$ (teor 1). $\lambda=0:\ y=c_1+c_2x\ ,\ \left.\begin{array}{l} y(0)=c_1=0 \\ y'(1)=c_2=0 \end{array}\right\}\to y\equiv 0$. $\lambda=0$ no autovalor.

$\lambda>0:\ y=c_1\cos wx+c_2\operatorname{sen} wx\ ,\ \left.\begin{array}{l} y(0)=c_1=0 \\ y'(1)=wc_2\cos w=0 \end{array}\right\}\to \lambda_n=\frac{(2n-1)^2\pi^2}{2^2},\ y_n\equiv\{\operatorname{sen}\frac{(2n-1)\pi x}{2}\}\ ,\ n=1,2,\dots$.

$\boxed{\begin{array}{l} y''+\lambda y=0 \\ y(-1)=y(1)=0 \end{array}}\ \overset{s=x+1}{\longrightarrow}\ \left.\begin{array}{l} y''+\lambda y=0 \\ y(0)=y(2)=0 \end{array}\right\}\to\lambda_n=\frac{n^2\pi^2}{2^2},\ y_n\equiv\{\operatorname{sen}\frac{n\pi s}{2}\}=\{\operatorname{sen}(\frac{n\pi x}{2}+\frac{n\pi}{2})\}\ ,\ n=1,2,\dots$.

Directamente ($\lambda>0$): $\left.\begin{array}{l} c_1\cos w-c_2\operatorname{sen} w=0 \\ c_1\cos w+c_2\operatorname{sen} w=0 \end{array}\right\}\to \left|\quad\right|=\operatorname{sen}2w=0\ ,\ \lambda_n=\frac{n^2\pi^2}{2^2}\ \to\ \begin{array}{l} n \text{ par, } c_1=0\to\operatorname{sen} \\ n \text{ impar, } c_2=0\to\cos \end{array}$.

$\boxed{\begin{array}{l} y''+\lambda y=0 \\ y(0)=y(1)+y'(1)=0 \end{array}}$ $\lambda\geq 0$ (t1). $\lambda=0:\ \left.\begin{array}{l} c_1=0 \\ 2c_1+c_2=0 \end{array}\right\}$ no autovalor.



$\lambda>0:\ y=c_1\cos wx+c_2\operatorname{sen} wx\ ,\ \left.\begin{array}{l} c_1=0 \\ c_2(\operatorname{sen} w+w\cos w)=0 \end{array}\right\}\to\tan w_n=-w_n$

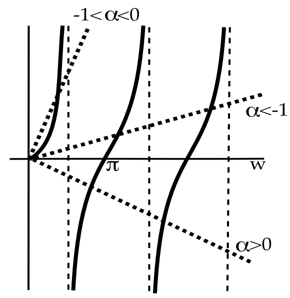
$\lambda_n=w_n^2\ (\lambda_1\approx 4.116,\dots),\ \ y_n\equiv\{\operatorname{sen} w_nx\}\ ,\ n=1,2,\dots$.

$\boxed{\begin{array}{l} x^2y''+xy'+[\lambda x^2-\frac{1}{4}]y=0 \\ y(1)=y(4)=0 \end{array}}$ $[xy']'-\frac{y}{4x}+\lambda xy=0\ ,\ \lambda>0$. Casi Bessel: $s=\sqrt{\lambda}\,x\to s^2y''+sy'+[s^2-\frac{1}{4}]y=0$

$y=c_1\frac{\cos\sqrt{\lambda}x}{\sqrt{x}}+c_2\frac{\operatorname{sen}\sqrt{\lambda}x}{\sqrt{x}}\ \overset{\text{c.c.}}{\to}\ \lambda_n=\frac{n^2\pi^2}{9}\ ,\ y_n=\{\frac{1}{\sqrt{x}}\operatorname{sen}\frac{n\pi(x-1)}{3}\}\ ,\ n=1,2,\dots$

[O bien: $u=\sqrt{x}\,y\to\ \left.\begin{array}{l} u''+\lambda u=0 \\ u(1)=u(4)=0 \end{array}\right\}\overset{x=s+1}{\to}\ \left.\begin{array}{l} u''+\lambda u=0 \\ u(0)=u(3)=0 \end{array}\right\}\to u=\operatorname{sen}\frac{n\pi s}{3}=\operatorname{sen}\frac{n\pi(x-1)}{3}$].

2 $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y'(0)-\alpha y(0)=y(1)=0 \end{array}}$ $\lambda>0$: Si $\alpha=0$, $\lambda_n=\frac{(2n-1)^2\pi^2}{2^2},\ y_n\equiv\{\cos\frac{(2n-1)\pi x}{2}\}$.



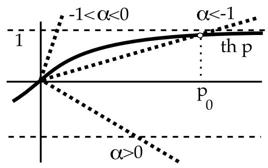
Si $\alpha\neq 0$: $\left.\begin{array}{l} wc_2-\alpha c_1=0 \\ c_1\cos w+c_2\operatorname{sen} w=0 \end{array}\right\}\ \tan w_n=-\frac{w_n}{\alpha}\ ,\ y_n\equiv\{\alpha\operatorname{sen} w_nx+w_n\cos w_nx\}$.

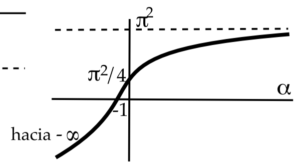
$\lambda=0:\ \left.\begin{array}{l} c_2-\alpha c_1=0 \\ c_1+c_2=0 \end{array}\right\}\to$ Autovalor si $\alpha=-1$, con autofunción $y_0\equiv\{1-x\}$.

$\lambda<0:\ y=c_1\,\mathrm{e}^{px}+c_2\,\mathrm{e}^{-px}\ ,\ \left.\begin{array}{l} (p-\alpha)c_1-(p+\alpha)c_2=0 \\ c_1\,\mathrm{e}^{p}+c_2\,\mathrm{e}^{-p}=0 \end{array}\right\}\to$

$p[\mathrm{e}^p+\mathrm{e}^{-p}]+\alpha[\mathrm{e}^p-\mathrm{e}^{-p}]=0\to\operatorname{th}p=-\frac{p}{\alpha}$

Si $\alpha<-1$ hay un $\lambda=-p_0^2$ $[\,y_0\equiv\{\alpha\operatorname{sh}p_0x+p_0\operatorname{ch}p_0x\}]$.





El menor autovalor es negativo si $\alpha<-1$, 0 si $\alpha=-1$ y positivo si $\alpha>-1$ (para $\alpha=0$ es $\lambda=\frac{\pi^2}{4}$).

3 a) $\boxed{f(x)=1}$ Su serie en senos es: $\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\operatorname{sen}(2m-1)\pi x}{2m-1}$.

Tiende hacia la extensión 2-periódica de $f(x)=\begin{cases}-1, & -1<x<0 \\ 1, & 0<x<1\end{cases}$,

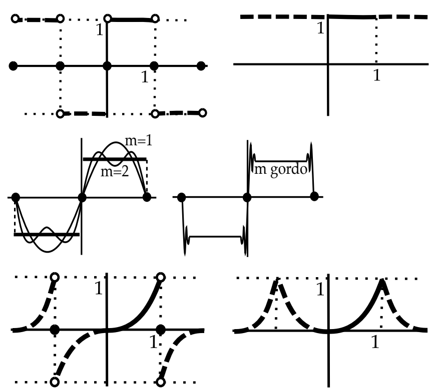
y la suma es 0 si $x\in\mathbf{Z}$. Cerca de ellos convergerá mal.

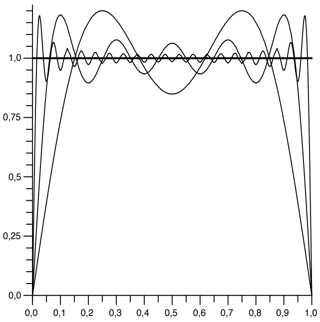
La serie en cosenos es la propia constante $1=1+0+0+\cdots$ (es uno de los elementos de la base de Fourier).

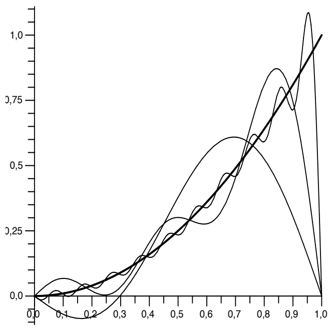
b) $\boxed{f(x)=x^2}$ $=\sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{\pi n}+\frac{4[(-1)^n-1]}{\pi^3n^3}\right]\operatorname{sen}n\pi x$.

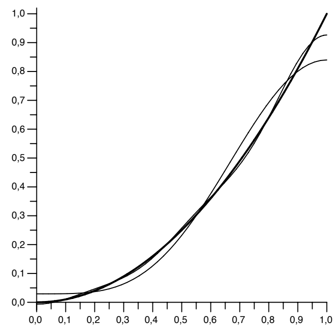
[En la serie en senos aparecerán picos cerca de 1].

$x^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$ converge uniformemente en $[0,1]$.




a) sen, $m=2,5,20$


b) sen, $n=2,5,20$


b) cos, $n=2,5$

4 i) Para desarrollar en cosenos basta escribir $\boxed{\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x}$, $x\in[0,\pi]$.

(La 'serie' claramente 'converge uniformemente' en todo $[0,\pi]$ hacia la f dada).

ii) Los coeficientes de la serie en senos vienen dados por las fórmulas de los apuntes:

$$b_n = \frac{2}{\pi}\int_0^\pi \cos^3 x \,\mathrm{sen}\, nx\, dx = \frac{3}{2\pi}\int_0^\pi \cos x \,\mathrm{sen}\, nx\, dx + \frac{1}{2\pi}\int_0^\pi \cos 3x \,\mathrm{sen}\, nx\, dx$$

$$= \frac{3}{4\pi}\int_0^\pi [\mathrm{sen}(n+1)x + \mathrm{sen}(n-1)x]\, dx + \frac{1}{4\pi}\int_0^\pi [\mathrm{sen}(n+3)x + \mathrm{sen}(n-3)x]\, dx$$

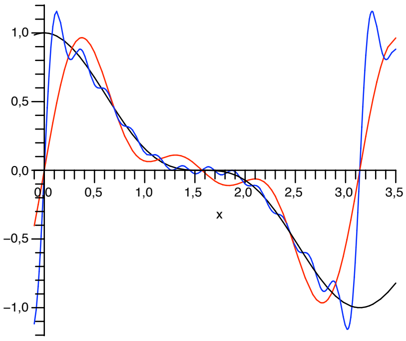
$$= -\frac{3}{4\pi}\Big[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\Big]_0^\pi - \frac{1}{4\pi}\Big[\frac{\cos(n+3)x}{n+3} + \frac{\cos(n-3)x}{n-3}\Big]_0^\pi$$

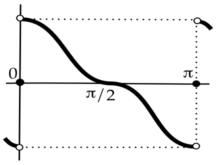
$$= \frac{3}{4\pi}\Big[\frac{1+(-1)^n}{n+1} + \frac{1+(-1)^n}{n-1}\Big] + \frac{1}{4\pi}\Big[\frac{1+(-1)^n}{n+3} + \frac{1+(-1)^n}{n-3}\Big] = \frac{3n}{2\pi}\,\frac{1+(-1)^n}{n^2-1} + \frac{n}{2\pi}\,\frac{1+(-1)^n}{n^2-9} = \frac{2n(n^2-7)[1+(-1)^n]}{\pi(n^2-1)(n^2-9)}$$

Esta serie converge hacia los puntos de continuidad de la extensión impar y 2π-periódica de f, es decir, converge hacia $\cos^3 x$ en $(0,\pi)$ y converge a 0 (evidentemente) cuando $x=0,\pi$:

$$\boxed{\cos^3 x = \sum_{m=1}^{\infty} \frac{8m(4m^2-7)}{\pi(4m^2-1)(4m^2-9)}\,\mathrm{sen}\, 2mx} \ , \ x\in(0,\pi) .$$

La función límite es la de abajo y a la derecha están los dibujos (Maple) de las sumas parciales con $m=3$ y 10. Hay convergencia uniforme en cualquier $[a,b]\subset(0,\pi)$.





5 a) $\boxed{f(x)=\mathrm{sen}^2 x} = \frac{1}{2} - \frac{1}{2}\cos 2x$, ya desarrollada $[\,a_0=\frac{1}{2}\,,\ a_2=-\frac{1}{2}$ y los demás a_n y los b_n son $0\,]$.

b) $\boxed{f(x)=|\mathrm{sen}\, x|}$ par $\to$ $b_n=0$. $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|\mathrm{sen}\, x|\,dx=\frac{2}{\pi}\int_0^\pi \mathrm{sen}\, x\, dx=\frac{4}{\pi}$. $a_1=\frac{2}{\pi}\int_0^\pi \mathrm{sen}\, x\cos x\, dx=0$.

$$a_n=\frac{2}{\pi}\int_0^\pi \mathrm{sen}\, x\cos nx\, dx=\frac{1}{\pi}\int_0^\pi[\mathrm{sen}(1+n)x+\mathrm{sen}(1-n)x]dx=-\frac{1}{\pi}\Big[\frac{\cos(1+n)x}{1+n}+\frac{\cos(1-n)x}{1-n}\Big]_0^\pi$$

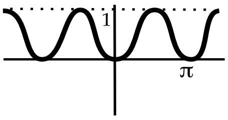
$$=\frac{1}{\pi}\Big[\frac{1+\cos n\pi}{1+n}+\frac{1+\cos n\pi}{1-n}\Big]=\frac{2}{\pi}\,\frac{1+(-1)^n}{1-n^2}\to |\mathrm{sen}\, x| = \frac{2}{\pi}+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{1-4m^2}$$

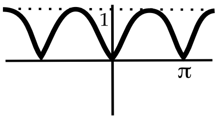
c) $\boxed{f(x)=\mathrm{sen}\,\frac{x}{2}}$ impar. $a_n=0$. $b_n=\frac{2}{\pi}\int_0^\pi \mathrm{sen}\,\frac{x}{2}\,\mathrm{sen}\, nx\, dx=\frac{1}{\pi}\int_0^\pi\Big[\cos\frac{(1-2n)x}{2}-\cos\frac{(1+2n)x}{2}\Big]dx=\frac{8}{\pi}\,\frac{(-1)^n n}{1-4n^2}$.

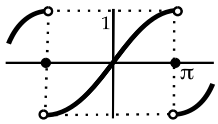
d) $\boxed{f(x)=\begin{cases}-\pi, \text{ si } -\pi\le x<0\\ \mathrm{sen}\, x, \text{ si } 0\le x<\pi\end{cases}}$ $\quad a_n=\frac{1}{\pi}\int_0^\pi \mathrm{sen}\, x\cos nx\, dx-\int_{-\pi}^0\cos nx\, dx$, $n=0,1,\ldots$

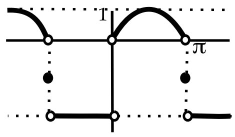
$b_n=\frac{1}{\pi}\int_0^\pi \mathrm{sen}\, x\,\mathrm{sen}\, nx\, dx-\int_{-\pi}^0 \mathrm{sen}\, nx\, dx$, $n=1,2,\ldots$

$\frac{1}{2}a_0=\frac{1}{\pi}-\frac{\pi}{2}$; $a_1=0$; $a_n=\frac{1}{\pi}\,\frac{1+(-1)^n}{1-n^2}$, $n=2,3,\ldots$; $b_1=\frac{5}{2}$; $b_n=\frac{1-(-1)^n}{n}$, $n=2,3,\ldots$.







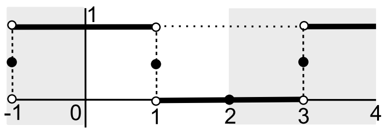


6 $\boxed{f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & 1<x\le 2\end{cases}}$ Si $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}$, es $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$. En este caso:

$$a_0=\frac{2}{2}\int_0^1 dx=1\,,\ a_n=\int_0^1\cos\frac{n\pi x}{2}\,dx=\frac{2}{n\pi}\,\mathrm{sen}\,\frac{n\pi}{2}=\begin{cases}0\,,\ n \text{ par}\\ \frac{2(-1)^m}{(2m+1)\pi}\,,\ n=2m+1\end{cases} \to \boxed{\frac{1}{2}+\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos\frac{(2m+1)\pi x}{2}} .$$

i) En $x=1$ es f discontinua y la serie tenderá hacia $\frac{1}{2}[f(1^-)+f(1^+)]=\frac{1}{2}$, como se comprueba fácil:

$$\frac{1}{2}+\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos\frac{(2m+1)\pi}{2}=\frac{1}{2}\ \text{ [los cosenos se anulan].}$$



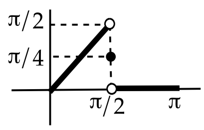
ii) Como tiende en todo **R** hacia la extensión par y 4-periódica de f, en $x=2$ ha de tender hacia $f(2)=0$. Sustituyendo:

$$\frac{1}{2}+\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos(2m+1)\pi=\frac{1}{2}-\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}=0\,,$$ ya que la última serie $1-\frac{1}{3}+\frac{1}{5}+\cdots=\arctan 1=\frac{\pi}{4}$.

7 Autovalores y autofunciones conocidos: $\lambda_n=\frac{(2n-1)^2}{2^2}$, $y_n=\Big\{\mathrm{sen}\,\frac{(2n-1)x}{2}\Big\}$, $n=1,2,\ldots$. $\langle y_n,y_n\rangle=\frac{\pi}{2}$.

$$\to c_n=\frac{2}{\pi}\int_0^\pi f(x)\,\mathrm{sen}\,\frac{(2n-1)x}{2}\,dx=\frac{2}{\pi}\int_0^{\pi/2}x\,\mathrm{sen}\,\frac{(2n-1)x}{2}\,dx$$

$$=-\Big[\frac{4x\cos\frac{(2n-1)x}{2}}{\pi(2n-1)}\Big]_0^{\pi/2}+\frac{4}{\pi(2n-1)}\int_0^{\pi/2}\cos\frac{(2n-1)x}{2}\,dx=\frac{8\,\mathrm{sen}\,\frac{(2n-1)\pi}{4}}{\pi(2n-1)^2}-\frac{2\cos\frac{(2n-1)\pi}{4}}{2n-1} .$$



La serie converge hacia $f(x)$ en los $x\in(0,\pi)$ en que f es continua (en los extremos no lo sabemos), y hacia $\frac{1}{2}[f(x^+)+f(x^-)]$ en los que f es discontinua. Por tanto:

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}c_n\,\mathrm{sen}\,\frac{(2n-1)\pi}{4}=\sum_{n=1}^{\infty}\Big[\frac{8\,\mathrm{sen}^2(\frac{n\pi}{2}-\frac{\pi}{4})}{\pi(2n-1)^2}-\frac{\mathrm{sen}\,\frac{(2n-1)\pi}{2}}{2n-1}\Big]=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}-\frac{\pi}{4}\ \Rightarrow\ \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8} .$$

8 $x=\sum_{m=1}^{\infty} c_n \operatorname{sen}\frac{(2n-1)\pi x}{2}$ $\begin{matrix}[0,1]\\ r=1\end{matrix}$ $c_n=2\int_0^1 x\operatorname{sen}\frac{(2n-1)\pi x}{2}dx=\frac{8(-1)^{n+1}}{\pi^2(2n-1)^2}$

$x=\sum_{m=1}^{\infty} c_n \operatorname{sen}\frac{n\pi(x+1)}{2}$ $\begin{matrix}[-1,1]\\ r=1\end{matrix}$ $\int_{-1}^1 \operatorname{sen}^2\frac{n\pi(x+1)}{2}=1$, $c_n=\int_{-1}^1 x\operatorname{sen}\frac{n\pi(x+1)}{2}\,dx=-\frac{2[1+(-1)^n]}{n\pi}$

$x=\sum_{m=1}^{\infty} c_n \operatorname{sen} w_n x$, $\tan w_n=-w_n$ $\begin{matrix}[0,1]\\ r=1\end{matrix}$ $\int_0^1 \operatorname{sen}^2 w_n x\,dx=\frac{1}{2}-\frac{\operatorname{sen}2w_n}{4w_n}=\frac{1+\cos^2 w_n}{2}=\frac{2+w_n^2}{2(1+w_n^2)}$,

$\int_0^1 x\operatorname{sen} w_n x\,dx=\frac{\operatorname{sen} w_n}{w_n^2}-\frac{\cos w_n}{w_n}=\frac{2\operatorname{sen} w_n}{w_n^2}=\frac{-2\cos w_n}{w_n}\to c_n=\frac{4(1+w_n^2)\operatorname{sen} w_n}{(2+w_n^2)w_n^2}$

$x=\sum_{m=1}^{\infty}\frac{c_n}{\sqrt{x}}\operatorname{sen}\frac{n\pi(x-1)}{2}$ $\begin{matrix}[1,4]\\ r=x\end{matrix}$ $\int_1^4 \operatorname{sen}^2\frac{n\pi(x-1)}{2}\,dx=\frac{3}{2}$, $c_n=\frac{2}{3}\int_1^4 x^{3/2}\operatorname{sen}\frac{n\pi(x-1)}{2}\,dx$ no elemental.

9 $\boxed{\begin{matrix}y''+2y'+\lambda y=0\\ y(0)+y'(0)=y(1/2)=0\end{matrix}}$ En forma autoadjunta: $(y'e^{2x})'+\lambda e^{2x}y=0$ [problema de S-L regular].
$\mu^2+2\mu+\lambda=0\to\mu=-1\pm\sqrt{1-\lambda}$. En principio, puede haber λ negativos.

$\lambda<1$, $\sqrt{1-\lambda}=p\to y=c_1e^{(p-1)x}+c_2e^{-(p+1)x}\to$ $\begin{matrix}y(0)+y'(0)=p(c_1-c_2)=0\\ y(\frac{1}{2})=(c_1e^{p/2}+c_2e^{-p/2})e^{-1/2}=0\end{matrix}\to c_1=c_2=0$.

$\lambda=1\to y=(c_1+c_2x)e^{-x}\to$ $\begin{matrix}y(0)+y'(0)=c_2=0\\ y(\frac{1}{2})=(c_1+\frac{c_2}{2})e^{-1/2}=0\end{matrix}\to c_1=c_2=0$.

$\lambda>1$, $\sqrt{\lambda-1}=w\to y=(c_1\cos wx+c_2\operatorname{sen} wx)\,e^{-x}\to y(0)+y'(0)=c_2w=0\to c_2=0\to$
$y(\frac{1}{2})=c_1\cos\frac{w}{2}e^{-1/2}=0\to w_n=(2n-1)\pi$, $\lambda_n=1+(2n-1)^2\pi^2$, $y_n=\{e^{-x}\cos(2n-1)\pi x\}$, $n=1,2,\dots$

Por tanto: $1=\sum_{n=1}^{\infty}\frac{\langle 1,y_n\rangle}{\langle y_n,y_n\rangle}y_n$, con $\langle 1,y_n\rangle=\int_0^{1/2}e^x\cos(2n-1)\pi x\,dx$, $\langle y_n,y_n\rangle=\int_0^{1/2}\cos^2(2n-1)\pi x\,dx$

Como $\int_0^{1/2}\cos^2 bx=\frac{1}{2}\int_0^{1/2}(1+\cos 2bx)=\frac{1}{4}+\frac{\operatorname{sen} b}{4b}\to\langle 1,y_n\rangle=\frac{1}{4}$ e $\int e^x\cos bx\,dx=\frac{(\cos bx+b\operatorname{sen} bx)e^x}{1+b^2}$,

concuimos que: $1=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)\pi e^{1/2}-1}{1+(2n-1)^2\pi^2}\,e^{-x}\cos(2n-1)\pi x$.

10 $\boxed{\begin{matrix}\left([1-x^2]y'\right)'+\lambda y=0\\ y(0)=0,\ y\text{ acotada en }1\end{matrix}}$ Los P_{2n-1} son las únicas soluciones de la ecuación de Legendre que pasan por el origen y están acotados en $x=1$ →

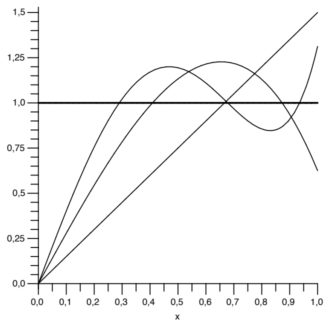
$\lambda_n=2n(2n-1)$, $y_n=\{P_{2n-1}\}$, $n\in\mathbf{N}$. $\int_{-1}^1 P_n^2=\frac{2}{2n+1}\to\int_0^1 P_{2n-1}^2=\frac{1}{4n-1}$.

$1=\sum_{n=1}^{\infty}c_nP_{2n-1}(x)$
$r(x)=1$

$c_1=3\int_0^1 x\,dx=\frac{3}{2}$

$c_2=7\int_0^1\left[\frac{5}{2}x^3-\frac{3}{2}x\right]dx=-\frac{7}{8}$

$c_3=11\int_0^1\left[\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\right]dx=\frac{11}{16}$



11 $\boxed{\begin{matrix}xy''-y'=x^2-a\\ y'(2)=y'(4)=0\end{matrix}}$ La homogénea se puede resolver como Euler: $\lambda(\lambda-1)-\lambda=0\to y=c_1+c_2x^2$,
o haciendo $y'=v\to v'=\frac{v}{x}\to v=Ce^{\ln x}=Cx\to y=c_1+c_2x^2$ $[\,y'=2c_2x\,]$.

Imponiendo los datos a esta solución: $\begin{matrix}y'(2)=4c_2=0\\ y'(4)=8c_2=0\end{matrix}\to c_2=0$ y c_1 indeterminado.

El homogéneo tiene, pues, infinitas soluciones $y_h=\{1\}$ y el no homogéneo tendrá infinitas o ninguna.

En forma autoadjunta: $y''-\frac{1}{x}y'=x-\frac{a}{x}\xrightarrow{\times e^{-\log x}}\left(\frac{1}{x}y'\right)'=1-\frac{a}{x^2}$. Hallemos la integral:

$\int_2^4 1\cdot(1-\frac{a}{x^2})\,dx=2+\left[\frac{a}{x}\right]_2^4=2-\frac{a}{4}\to$ Si $\boxed{a=8}$ tiene infinitas soluciones. [Si $a\neq 8$, ninguna].

[Se llega a lo mismo imponiendo los datos en la solución genera de la no homogénea $\cdots y=c_1+c_2x^2+\frac{1}{3}x^3+ax\cdots$].

12 $\boxed{\begin{matrix}x^2y''-ay=3x-4\\ y(1)+y'(1)=y(2)=0\end{matrix}}$ i) Para $a=2$ es Euler con $\mu(\mu-1)-2=0\to y=c_1x^2+c_2x^{-1}$, $y'=2c_1x-c_2x^{-2}$

$\to\begin{cases}y(1)+y'(1)=3c_1=0\\ y(2)=4c_1+\frac{c_2}{2}=0\end{cases}\to c_1=c_2=0\to$ [P] tiene **solución única**.

[Se podría calcular esta solución. Con la fvc o tanteando $\cdots$ $y=c_1x^2+c_2x^{-1}-\frac{3x}{2}+2\xrightarrow{\text{datos}}y=\frac{1}{3}x^2-\frac{3}{2x}-\frac{3x}{2}+2$].

ii) Para $a=0$ también es de Euler, o podemos ‘resolverla’ así: $y''=0\to y'=c_1$, $y=c_1x+c_2$

$\to\begin{cases}y(1)+y'(1)=2c_1+c_2=0\\ y(2)=2c_1+c_2=0\end{cases}\to$ **el homogéneo tiene infinitas soluciones** $y_h=\{x-2\}$.

La ecuación en forma S-L es $(y')'=\frac{3}{x}-\frac{4}{x^2}$. Que [P] tenga infinitas o ninguna depende de:

$\int_1^2(x-2)(\frac{3}{x}-\frac{4}{x^2})dx=\int_1^2(3-\frac{10}{x}+\frac{8}{x^2})dx=3-10\left[\ln x\right]_1^2-\left[\frac{8}{x}\right]_1^2=7-10\ln 2\neq 0$. **No tiene solución**.

[Verlo directamente lleva más tiempo: $y''=\frac{3}{x}-\frac{4}{x^2}\to y'=3\ln x+\frac{4}{x}+c_1\xrightarrow{\text{partes}}y=3x\ln x-3x+4\ln x+c_1x+c_2\to$
$\begin{cases}y(1)+y'(1)=2c_1+c_2+1=0\\ y(2)=2c_1+c_2+10\ln 2-6=0\end{cases}$, sistema que no tiene solución porque $1\neq 10\ln 2-6$].

13 $\boxed{\begin{array}{l} y''+y'+\lambda y=1-x \\ y'(0)=y'(2)=0 \end{array}}$ En forma Sturm-Liouville: $(e^x y')'+\lambda e^x y=(1-x)\,e^x \quad (p,r>0)$.

i] $q\equiv 0$, $\alpha\alpha'=\beta\beta'=0 \Rightarrow$ los autovalores del problema homogéneo son $\geq 0 \Rightarrow$ si $\lambda=-2$ el problema homogéneo no tiene más solución que la trivial y el no homogéneo solución única.

[Es fácil ver directamente que $\lambda=-2$ no es autovalor: $\mu^2+\mu-2=(\mu-1)(\mu+2) \to$ $y=c_1e^x+c_2e^{-2x}$, $y'=c_1e^x-2c_2e^{-2x} \to \begin{array}{l} c_1-2c_2=0 \\ c_1e^2-2c_2e^{-4}=0 \end{array} \to c_1=c_2=0$].

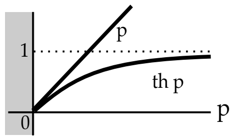
ii] Para el homogéneo:

$\mu(\mu-1) \to y=c_1+c_2e^{-x}$, $y'=-c_2e^{-x} \to \begin{array}{l} -c_2=0 \\ -c_2e^{-2}=0 \end{array} \to c_2=0$, c_1 indederminado $\to y_h=\{1\}$.

Como $\int_0^2 1(1-x)e^x dx=(1-x)e^x\big]_0^2+\int_0^2 e^x dx=-e^2-1+e^2-1=-2\neq 0$, no tiene solución.

[Se podría comprobar a partir de la solución general de la no homogénea: $y=c_1+c_2e^{-x}+2x-\frac{1}{2}x^2$].

14 [P] $\boxed{\begin{array}{l} y''+\lambda y=x \\ y(0)=y'(1)-y(1)=0 \end{array}}$ Ecuación en forma autoadjunta. Como $\beta\beta'<0$ puede haber $\lambda\leq 0$.



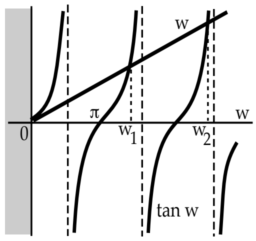
$\lambda<0:\ y=c_1e^{px}+c_2e^{-px} \to \left.\begin{array}{l} c_1=-c_2 \\ c_2(p[e^p+e^{-p}]-[e^p-e^{-p}])=0 \end{array}\right\} \to y\equiv 0$

[no existe $p>0$ con $p=\text{th}\,p$, pues $(\text{th}\,p)'(0)=1$].

$\lambda=0:\ y=c_1+c_2x \to \left.\begin{array}{l} c_1=0 \\ c_1+c_2=c_2 \end{array}\right\} \to \lambda_0=0$ autovalor con $y_0=\{x\}$.

$\lambda>0:\ y=c_1\cos wx+c_2\,\text{sen}\,wx \to \left.\begin{array}{l} c_1=0 \\ c_2(\text{sen}\,w-w\cos w)=0 \end{array}\right\}$

Hay infintos w_n con $w_n=\tan w_n \to \lambda_n=w_n^2$, $y_n=\{\text{sen}\,w_nx\}$.



Por tanto: Si $\lambda\neq\lambda_n$ hay solución única de [P].

Si $\lambda=0$, como $\int_0^1 x\,x\,dx\neq 0$, [P] no tiene solución.

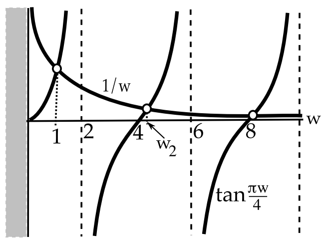
Si $\lambda=\lambda_n$, $n=1,2,\dots$, $\int_0^1 x\,\text{sen}\,w_nx\,dx=\frac{1}{w_n^2}[\text{sen}\,w_n-w_n\cos w_n]=0 \to$ infinitas soluciones.

[Podíamos ahorrarnos el cálculo de esta integral pues y_0 e y_n son ortogonales].

15 $\boxed{\begin{array}{l} y''+\lambda y=\cos 3x \\ y'(0)=y'(\frac{\pi}{4})+y(\frac{\pi}{4})=0 \end{array}}$ $\begin{array}{l} \alpha\cdot\alpha'=0 \\ \beta\cdot\beta'>0 \end{array} \Rightarrow \lambda\geq 0$. $\lambda=0:\ y=c_1+c_2x \to \left.\begin{array}{l} y'(0)=c_2=0\downarrow \\ y'(\frac{\pi}{4})+y(\frac{\pi}{4})=c_1=0 \end{array}\right\}$ $\lambda=0$ no autovalor.

$\lambda>0:\ y=c_1\cos wx+c_2\,\text{sen}\,wx$. $y'(0)=0 \to c_2=0 \to y'(\frac{\pi}{4})+y(\frac{\pi}{4})=c_1\big[\cos\frac{w\pi}{4}-w\,\text{sen}\,\frac{w\pi}{4}\big]=0$.

Si el corchete es cero, c_1 queda indeterminado. Infinitos w_n cumplen $\tan\frac{\pi w_n}{4}=\frac{1}{w_n}$. A cada $\lambda_n=w_n^2$, está asociada la $y_n=\{\cos w_nx\}$.



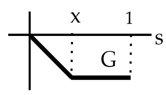
λ_1 se pueda hallar exactamente: $\boxed{\lambda_1=1 \to y_1=\{\cos x\}}$ ($\tan\frac{\pi}{4}=1$).

Si $\cos 3x=c_1\cos x+\sum_{n=2}^{\infty}c_n\cos w_nx$, el primer coeficiente c_1 es:

$$c_1=\frac{\langle\cos 3x,\cos x\rangle}{\langle\cos x,\cos x\rangle}=\frac{\int_0^{\pi/4}\cos 3x\cos x\,dx}{\int_0^{\pi/4}\cos^2 x\,dx}=\frac{\int_0^{\pi/4}(\cos 4x+\cos 2x)\,dx}{\int_0^{\pi/4}(1+\cos 2x)\,dx}=\boxed{\frac{2}{\pi+2}}$$

Para i), por no ser $\lambda=0$ autovalor hay solución única del no homogéneo. Para ii), hay infinitas del homogéneo $y_h=\{\cos x\}$ y el no homogéneo no tiene solución pues: $\int_0^{\pi/4}\cos 3x\cos x\,dx=\frac{1}{4}\neq 0$.

16 $\boxed{\begin{array}{l} y''=f(x) \\ y(0)=y'(1)=0 \end{array}}$ $y=c_1+c_2x \begin{array}{l} \nearrow y_1=x \\ \searrow y_2=1 \end{array}$ $\begin{array}{l} |W|=-1 \\ p(x)=1 \end{array} \to \boxed{G(x,s)=\begin{cases} -s,\ 0\leq s\leq x \\ -x,\ x\leq s\leq 1 \end{cases}}$



La solución para $f(x)=x$: $y=-\int_0^x s^2ds-\int_x^1 sx\,ds=\boxed{\frac{x^3}{6}-\frac{x}{2}}$.

$\boxed{\begin{array}{l} x^2y''+xy'-y=f(x) \\ y(1)+y'(1)=y(2)=0 \end{array}}$ $y=c_1x+\frac{c_2}{x} \begin{array}{l} \nearrow y_1=\frac{1}{x} \\ \searrow y_2=x-\frac{4}{x} \end{array}$ $|W|=\frac{2}{x}$, $(xy')'-\frac{y}{x}=\frac{f(x)}{x} \to p(x)=x$

$\to \boxed{G(x,s)=\begin{cases} \frac{1}{2s}(x-\frac{4}{x}),\ 1\leq s\leq x \\ \frac{1}{2x}(s-\frac{4}{s}),\ x\leq s\leq 2 \end{cases}}$ $f(x)=x \to y=(\frac{x}{2}-\frac{2}{x})\int_1^x\frac{ds}{s}+\frac{1}{x}\int_x^2(\frac{s}{2}-\frac{2}{s})ds=\boxed{\frac{x\ln x}{2}-\frac{x}{4}+\frac{1-2\ln 2}{x}}$.

$\boxed{\begin{array}{l} y''+y'-2y=f(x) \\ y(0)-y'(0)=y(1)=0 \end{array}}$ $y=c_1e^x+c_2e^{-2x} \begin{array}{l} \nearrow y_1=e^x \\ \searrow y_2=e^x-e^{3-2x} \end{array}$ $|W|=3e^{3-x}$, $(e^xy')'-2e^xy=e^xf(x)$

$\to \boxed{G(x,s)=\begin{cases} \frac{1}{3}e^s(e^{x-3}-e^{-2x}),\ 0\leq s\leq x \\ \frac{1}{3}e^x(e^{s-3}-e^{-2s}),\ x\leq s\leq 1 \end{cases}}$ $\to y=\frac{e^{x-3}-e^{-2x}}{3}\int_0^x se^{2s}ds+\frac{e^x}{3}\int_x^1 s(e^{2s-3}-e^{-s})ds=$ $\boxed{\frac{(9e^2+1)e^x}{12e^3}-\frac{e^{-2x}}{12}-\frac{x}{4}-\frac{1}{4}}$.

Soluciones de problemas 3 (c y o) de EDII(r) (2011)

1 a) $\left\{\begin{array}{l} u_t-u_{xx}+2tu=0\,,\ x\in(0,\frac{1}{2}),\ t>0 \\ u(x,0)=1-2x \\ u_x(0,t)=u(1/2,t)=0 \end{array}\right.$ $u(x,t)=X(x)T(t)\ \to\ \frac{X''}{X}=\frac{T'}{T}+2t=-\lambda \to \left\{\begin{array}{l} X''+\lambda X=0 \\ T'+(2t+\lambda)T=0 \end{array}\right.$

De las condiciones de contorno: $u_x(0,t)=X'(0)T(t)=0 \to X'(0)=0$
$u(\frac{1}{2},t)=X(\frac{1}{2})T(t)=0 \to X(\frac{1}{2})=0$

$\left\{\begin{array}{l} X''+\lambda X=0 \\ X'(0)=X(\frac{1}{2})=0 \end{array}\right.$ conocido: $\begin{array}{l}\lambda_n=(2n-1)^2\pi^2,\ n=1,2,\dots \\ X_n=\{\cos(2n-1)\pi x\}\end{array}$ $\to T'=-(2t+\lambda_n)T\,,\ T_n=\{e^{-t^2-(2n-1)^2\pi^2 t}\}$

Probamos: $u=\sum\limits_{n=1}^{\infty} c_n\, e^{-t^2-(2n-1)^2\pi^2 t}\cos(2n-1)\pi x$. Debe ser: $u(x,0)=\sum\limits_{n=1}^{\infty} c_n \cos(2n-1)\pi x = 1-2x\ \to$

$$c_n=\frac{2}{1/2}\int_0^{1/2}(1-2x)\cos(2n-1)\pi x\,dx=\frac{4(1-2x)}{\pi(2n-1)}\operatorname{sen}(2n-1)\pi x\Big|_0^{1/2}+\frac{8}{\pi(2n-1)}\int_0^{1/2}\operatorname{sen}(2n-1)\pi x\,dx$$
$$=\frac{8}{\pi^2(2n-1)^2}\big[-\cos(2n-1)\pi x\big]_0^{1/2}=\frac{8}{\pi^2(2n-1)^2}\ \to u(x,t)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,e^{-t^2-(2n-1)^2\pi^2 t}\cos(2n-1)\pi x$$

b) $\left\{\begin{array}{l} u_t-u_{xx}-4u_x-4u=0,\ x\in(0,\pi),\ t>0 \\ u(x,0)=e^{-2x} \\ u(0,t)=u(\pi,t)=0 \end{array}\right.$ $u=XT \to \begin{array}{l} T'+\lambda T=0 \\ \left\{\begin{array}{l} x''+4X'+(4+\lambda)X=0 \\ X(0)=X(\pi)=0\end{array}\right.\end{array}$ $\begin{array}{l}\lambda_n=n^2,\ n=1,2,\dots \\ X_n=\{e^{-2x}\operatorname{sen} nx\}\end{array}$

$u(x,t)=\sum\limits_{n=1}^{\infty} c_n\,e^{-n^2t}e^{-2x}\operatorname{sen} nx$; $u(x,0)=e^{-2x}\sum\limits_{n=1}^{\infty} c_n\operatorname{sen} nx=e^{-2x}$; $u(x,t)=\frac{4}{\pi}\sum\limits_{m=1}^{\infty}\frac{e^{-(2m-1)^2t}}{2m-1}e^{-2x}\operatorname{sen}(2m-1)x$

c) $\left\{\begin{array}{l} u_t-u_{xx}+\frac{u}{t+1}=0\,,\ x\in(0,2),\ t>0 \\ u(x,0)=\left\{\begin{array}{l}1,\ 0\le x\le 1\\ 0,\ 1<x\le 2\end{array}\right.,\ u_x(0,t)=u_x(2,t)=0 \end{array}\right.$ $u=XT\ \to \frac{X''}{X}=\frac{T'}{T}+\frac{1}{t+1}=-\lambda\ \to$ $\left\{\begin{array}{l}X''+\lambda X=0\\ X'(0)=X'(2)=0\end{array}\right.$ y $T'+\big[\lambda+\frac{1}{t+1}\big]T=0$

$\to\ \lambda_n=\frac{n^2\pi^2}{4}$, $X_n=\{\cos\frac{n\pi x}{2}\}$, $n=0,1,\dots\ \to\ T'+\big[\frac{n^2\pi^2}{4}+\frac{1}{t+1}\big]T=0\ \to\ T_n=\Big\{\frac{e^{-n^2\pi^2t/4}}{t+1}\Big\}\ \to$

$$u(x,t)=\frac{a_0}{2(t+1)}+\sum_{n=1}^{\infty}a_n\frac{e^{-n^2\pi^2t/4}}{t+1}\cos\frac{n\pi x}{2}\ \to\ u(x,0)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}=f(x),\ a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$
$$\to\ u(x,t)=\frac{1}{2(t+1)}+\frac{2}{\pi(t+1)}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\,e^{-(2m+1)^2\pi^2t/4}\cos\frac{(2m+1)\pi x}{2}\ .$$

d) $\left\{\begin{array}{l} u_t-u_{xx}=e^{-2t}\cos x\,,\ x\in(0,\frac{\pi}{2}),\ t>0 \\ u(x,0)=1\,,\ u_x(0,t)=0,\ u(\frac{\pi}{2},t)=1 \end{array}\right.$ [A una varilla inicialmente a 1^o con el extremo izquierdo aislado le metemos y sacamos calor (cada vez menos) en su interior, mientras mantenemos el extremo derecho a 1^o].

Una v salta a la vista: $v=1 \overset{w=u-1}{\longrightarrow} \left\{\begin{array}{l} w_t-w_{xx}=e^{-2t}\cos x \\ w(x,0)=0\,,\ w_x(0,t)=w(\frac{\pi}{2},t)=0\end{array}\right.$, problema no homogéneo.

Sabemos que al hacer $u=XT$ sale $X''+\lambda X=0$, que con $X'(0)=X(\frac{\pi}{2})=0$ nos da $X_n=\{\cos(2n-1)x\}$, $n=1,\dots$.

Probamos: $w(x,t)=\sum\limits_{n=1}^{\infty}T_n(t)\cos(2n-1)x\ \to\ \sum\limits_{n=1}^{\infty}\big[T_n'+(2n-1)^2T_n\big]\cos(2n-1)x=e^{-2t}\cos x$ ya desarrollada.

Hay que resolver, pues: $\left\{\begin{array}{l}T_1'+T_1=e^{-2t}\\ T_1(0)=0\end{array}\right.$ y $\left\{\begin{array}{l}T_n'+(2n-1)^2T_n=0\\ T_n(0)=0\end{array}\right.$, $n\ge 2$ $\Big[\sum\limits_{n=1}^{\infty}T_n(0)\cos(2n-1)x=0\Big]$.

La solución para $n\ge 2$ es obviamente $T_n\equiv 0$. Para la primera:

$T_{1p}=Ae^{-2t}\to\ -2A+A=1,\ T_1=Ce^{-t}-e^{-2t}$ $\big[$ó $T_1=Ce^{-t}+e^{-t}\int e^{-t}\big]$ $\overset{T_1(0)=0}{\longrightarrow} T_1(t)=e^{-t}-e^{-2t}$.

La solución del problema en u es: $u(x,t)=1+(e^{-t}-e^{-2t})\cos x$ $\big[u \underset{t\to\infty}{\longrightarrow} 1$: toda la varilla tiende a ponerse a 1^o$\big]$.

e) $\left\{\begin{array}{l} u_{tt}+4u_t-u_{xx}=0,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u(x,0)=\operatorname{sen}2x \\ u_t(x,0)=u(0,t)=u(1,t)=0 \end{array}\right.$ $u=XT\to$ $\left\{\begin{array}{l}x''+\lambda X=0\\ X(0)=X(\pi)=0\end{array}\right.$ $\begin{array}{l}\lambda_n=n^2,\ n=1,2,\dots\\ X_n=\{\operatorname{sen}nx\}\end{array}$

$\to T'+4T'+n^2T=0,\ r=-2\pm\sqrt{4-n^2}\to$

$T_1=c_1\,e^{(-2+\sqrt{3})t}+c_2\,e^{(-2-\sqrt{3})t}$, $T_2=(c_1+c_2t)\,e^{-2t}$, $T_{n\ge 3}=e^{-2t}(c_1\cos\sqrt{n^2-4}\,t+c_2\operatorname{sen}\sqrt{n^2-4}\,t)$.

$u=\sum\limits_{n=1}^{\infty}T_n(t)\operatorname{sen}nx$; $\begin{array}{l}u(x,0)=\sum T_n(0)\operatorname{sen}nx=\operatorname{sen}2x\\ u_t(x,0)=\sum T_n'(0)\operatorname{sen}nx=0\end{array}$ $\begin{array}{l}T_2(0)=c_1=1\\ T_2'(0)=c_2-2c_1=0\end{array}$ $\to u=(1+2t)\,e^{-2t}\operatorname{sen}2x$

[La cuerda con rozamiento tiende a pararse].

f) $\left\{\begin{array}{l} u_{tt}-u_{xx}=4\operatorname{sen}6x\cos 3x,\ x\in[0,\frac{\pi}{2}] \\ u(x,0)=u_t(x,0)=0,\ t\in\mathbf{R} \\ u(0,t)=u_x(\frac{\pi}{2},t)=0 \end{array}\right.$ $\left\{\begin{array}{l}x''+\lambda X=0\\ X(0)=X'(\frac{\pi}{2})=0\end{array}\right.$ $\to\ \sum\limits_{n=1}^{\infty}T_n(t)\operatorname{sen}(2n-1)x\to$

$\sum\limits_{n=1}^{\infty}\big[T_n+(2n-1)^2T_n\big]\operatorname{sen}(2n-1)x=2\operatorname{sen}3x+2\operatorname{sen}9x$; $\begin{array}{l}T_n+(2n-1)^2T_n=2\\ T_n(0)=T_n'(0)=0\end{array}$, $n=2,5$, $\begin{array}{l}u=\frac{2}{9}(1-\cos 3t)\operatorname{sen}3x\\ +\frac{2}{81}(1-\cos 9t)\operatorname{sen}9x\end{array}$

2 $\begin{cases} u_t-u_{xx}=F(t),\ x\in(0,\pi),\ t>0 \\ u(x,0)=f(x) \\ u_x(0,t)=u_x(\pi,t)=0 \end{cases}$

Homogéneo en los apuntes: $u_1=\frac{c_o}{2}+\sum_{n=1}^{\infty}c_n\,e^{-n^2t}\cos nx$, con $c_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$.

Para el otro: $u_2=T_0(t)+\sum_{n=1}^{\infty}T_n(t)\cos nx\to\begin{cases}T_0'=F(t)\\T_0(0)=0\end{cases}\to T_0(t)=\int_0^t F(s)\,ds$ (y los demás $T_n\equiv 0$).

$u=u_1+u_2\underset{t\to\infty}{\to}\frac{1}{\pi}\int_0^\pi f(x)\,dx+\int_0^\infty F(t)\,dt$. En particular, si $f(x)=\frac{1-\cos x}{2}$ y $F(t)=e^{-t}$, $u\underset{t\to\infty}{\to}1+\frac{1}{2}=\frac{3}{2}$.

3 $\begin{cases} u_t-u_{xx}=0,\ x\in(0,1),\ t>0 \\ u(x,0)=0,\ u_x(0,t)=0,\ u_x(1,t)=2e^{-t} \end{cases}$ Sabemos que al separar variables: $\begin{cases}X''+\lambda X=0\\T'+\lambda T=0\end{cases}$ [•]

Necesitamos una v. Probando parábolas (en x): $v=A(t)x+B(t)x^2\underset{c.c.}{\to}v=x^2e^{-t}$.

$w=u-v\to$ [P] $\begin{cases}w_t-w_{xx}=(x^2+2)e^{-t}\\w(x,0)=-x^2,\ w_x(0,t)=w_x(1,t)=0\end{cases}$ (ecuación no homogénea)

Hallemos v que cumpla la ecuación. Como e^{-t} está asociada a $\lambda=1$ en [•], $X''+X=0$, $v=XT\to$

$v=(c_1\cos x+c_2\operatorname{sen}x)e^{-t}\overset{c.c.}{\to}v^*=-\frac{2\cos x}{\operatorname{sen}1}e^{-t}$. $w=u-v^*\to$ [P*] $\begin{cases}w_t-w_{xx}=0\\w(x,0)=\frac{2\cos x}{\operatorname{sen}1},\ w_x(0,t)=w_x(1,t)=0\end{cases}$

$$X''+\lambda X,\ X'(0)=X'(1)=1\to\lambda_n=n^2\pi^2,\ X_n=\{\cos n\pi x\},\ n=0,1,\dots\to T_n=\{e^{-n^2\pi^2t}\}\to$$

$$w=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nT_nX_n\overset{d.i.}{\to}\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x=\frac{2\cos x}{\operatorname{sen}1}\to\frac{a_0}{2}=\int_0^1\frac{2\cos x}{\operatorname{sen}1}dx=2,$$

$$a_n=\frac{4}{\operatorname{sen}1}\int_0^1\cos x\cos n\pi x\,dx=\frac{2}{\operatorname{sen}1}\int_0^1[\cos(n\pi+1)x+\cos(n\pi-1)]\,dx=\frac{2(-1)^n}{n\pi+1}-\frac{2(-1)^n}{n\pi-1}.$$

$$u=2-\frac{2\cos x}{\operatorname{sen}1}e^{-t}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2-1}e^{-n^2\pi^2t}\cos n\pi x\underset{t\to\infty}{\to}2$$ [aislado a la izquierda y metemos cada vez menos calor por la derecha]

Más largo es hallar la solución de [P]: $w=T_0(t)+\sum_{n=1}^{\infty}T_n(t)\cos n\pi x\to$

$$T_0'+\sum_{n=1}^{\infty}[T_n'+n^2\pi^2T_n]\cos n\pi x=e^{-t}(x^2+2)=\frac{7}{3}e^{-t}+e^{-t}\sum_{n=1}^{\infty}B_n\cos n\pi x,$$

pues $\int_0^1(x^2+2)\,dx=\frac{7}{3}$, y siendo $B_n=2\int_0^1(x^2+2)\cos n\pi x\,dx=-\frac{4}{n\pi}\int_0^1 x\operatorname{sen}n\pi x\,dx=\frac{4(-1)^n}{n^2\pi^2}$.

Como $w(x,0)=T_0(0)+\sum_{n=1}^{\infty}T_n(0)\cos n\pi x=-x^2=-\frac{1}{3}-\sum_{n=1}^{\infty}B_n\cos n\pi x$, hay que resolver:

$$\begin{cases}T_0'=\frac{7}{3}e^{-t}\\T_0(0)=-\frac{1}{3}\end{cases}\to T_0=2-\frac{7}{3}e^{-t},\ \begin{cases}T_n+n^2\pi^2T_n=B_ne^{-t}\to T_n=Ce^{-n^2\pi^2t}+T_{np}\\T_n(0)=-B_n\end{cases},\ T_{np}=Ae^{-t}\to$$

$$T_{np}=\frac{B_n}{n^2\pi^2-1}e^{-t}\overset{d.i.}{\to}T_n=\frac{B_n[e^{-t}-n^2\pi^2e^{-n^2\pi^2t}]}{n^2\pi^2-1}\to u=2+(x^2-\tfrac{7}{3})e^{-t}+4\sum_{n=1}^{\infty}\frac{(-1)^n\left[\frac{e^{-t}}{n^2\pi^2}-e^{-n^2\pi^2t}\right]}{n^2\pi^2-1}\cos n\pi x\underset{t\to\infty}{\to}2$$

4 $\begin{cases} u_t-u_{xx}=0,\ x\in(0,\pi),\ t>0 \\ u(x,0)=0,\ u_x(0,t)=u_x(\pi,t)=t \end{cases}$ Casi a ojo se ve que $v=xt$ cumple las condiciones de contorno.

$$w=u-xt\to\begin{cases}w_t-w_{xx}=-x\\w(x,0)=0\\w_x(0,t)=w_x(\pi,t)=0\end{cases}\to\begin{cases}X''+\lambda X=0\\X'(0)=X'(\pi)=0\end{cases}\to X_n=\{\cos nx\},\ n=0,1,\dots\to$$

$$w=T_0(t)+\sum_{n=1}^{\infty}T_n(t)\cos nx\to T_0'+\sum_{n=1}^{\infty}[T_n'+n^2T_n]\cos nx=-x=\frac{b_0}{2}+\sum_{n=1}^{\infty}b_n\cos nx,\text{ con }b_n=-\frac{2}{\pi}\int_0^\pi x\cos nx\,dx:$$

$$b_0=-\frac{2}{\pi}\frac{\pi^2}{2}=-\pi,\quad b_n=-\frac{2}{n\pi}x\operatorname{sen}nx\Big]_0^\pi+\frac{2}{n\pi}\int_0^\pi\operatorname{sen}nx\,dx=\frac{2}{n^2\pi}[\cos n\pi-1]=\begin{cases}-4/(n^2\pi),\ n\text{ impar}\\0,\ n\text{ par}\end{cases}$$

$$\begin{cases}T_0'=-\frac{\pi}{2}\\T_0(0)=0\end{cases}\to T_0(t)=-\frac{\pi}{2}t,\quad\begin{cases}T_n'+n^2T_n=b_n\to Ce^{-n^2t}+\frac{b_n}{n^2}\\T_n(0)=0\end{cases}\to T_n(t)=\frac{b_n}{n^2}[1-e^{-n^2t}].$$

$$\boxed{u(x,t)=t(x-\tfrac{\pi}{2})-\sum_{m=1}^{\infty}\frac{4}{\pi(2m-1)^4}[1-e^{-(2m-1)^2t}]\cos(2m-1)x}$$
$\to\infty$, si $x\in(\pi/2,\pi)$
$\to 0$, si $x=\pi/2$
$\to-\infty$, si $x\in(0,\pi/2)$

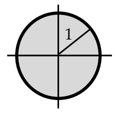
5 $\begin{cases} u_t-4u_{xx}=\cos\frac{\pi x}{2},\ x\in(0,1),\ t>0 \\ u(x,0)=T,\ u_x(0,t)=F,\ u(1,t)=T \end{cases}$ $v=Fx+T-F\to\begin{cases}w_t-4w_{xx}=\cos\frac{\pi x}{2}\\w(x,0)=F-Fx,\ u_x(0,t)=u(1,t)=0\end{cases}$

$$w_1=\sum_{n=1}^{\infty}c_ne^{-(2n-1)^2\pi^2t}\cos\frac{(2n-1)\pi x}{2},\ c_n=2F\int_0^1(1-x)\cos\frac{(2n-1)\pi x}{2}dx=\frac{8F}{\pi^2(2n-1)^2}$$

$$w_2=\sum_{n=1}^{\infty}T_n(t)\cos\frac{(2n-1)\pi x}{2}\to\begin{cases}T_1'+\pi^2T_1=1\\T_1(0)=0\end{cases}\to w_2=\frac{1}{\pi^2}(1-e^{-\pi^2t})\cos\frac{\pi x}{2}$$

$$u\underset{t\to\infty}{\to}Fx+T-F+\frac{1}{\pi^2}\cos\frac{\pi x}{2}$$

6 $\begin{cases} u_t-[u_{rr}+\frac{u_r}{r}]=0\,,\ r<1, t>0 \\ u(r,0)=0\,,\ u(1,t)=1 \end{cases}$ $\quad v=1 \underset{u=v+w}{\longrightarrow} \begin{cases} w_t-[w_{rr}+\frac{1}{r}w_r]=0 \\ w(r,0)=-1,\ w(1,t)=0 \end{cases} \to \begin{cases} T'+\lambda T=0 \\ rR''+R'+\lambda rR=0 \\ R \text{ acotada}, R(1)=0 \end{cases}$



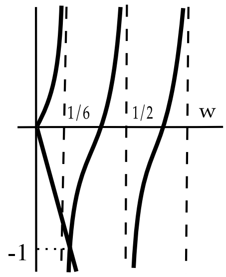
Problema singular visto en 2.2 (y 2.4): λ_n con $J_0(\sqrt{\lambda_n})=0$, y $R_n=\{J_0(\sqrt{\lambda_n}\,r)\}$; $\quad w=\sum_{n=1}^{\infty}c_n e^{-\lambda_n t}J_0(\sqrt{\lambda_n}\,r)$

$$\to \sum_{n=1}^{\infty}c_nJ_0(\sqrt{\lambda_n}\,r)=-1\ ,\ c_n=-\frac{2}{J_1^2(\sqrt{\lambda_n})}\int_0^1 rJ_0(\sqrt{\lambda_n}\,r)\,dr \to\ u=1-2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n t}}{\sqrt{\lambda_n}\,J_1(\sqrt{\lambda_n})}\,J_0(\sqrt{\lambda_n}\,r)\ ,$$

pues $\int_0^1 rJ_0(\sqrt{\lambda_n}\,r)\,dr=\frac{1}{\lambda_n}\int_0^{\sqrt{\lambda_n}}sJ_0(s)\,ds=\frac{1}{\sqrt{\lambda_n}}J_1(\sqrt{\lambda_n})$, ya que $[sJ_1]'=sJ_0$.

$u \underset{t\to\infty}{\longrightarrow} 1$ en todo el círculo, en particular, para un punto situado a 0.5 cm del centro.

7 $\begin{cases} u_t-u_{xx}-au=0,\ x\in(0,3\pi), t>0 \\ u(x,0)=1 \\ u(0,t)-4u_x(0,t)=u(3\pi,t)=0 \end{cases}$ $\quad \frac{T'-aT}{T}=\frac{X''}{X}=-\lambda \to \begin{matrix} T'=(a-\lambda)T \\ \begin{cases} X''+\lambda X=0 \\ X(0)-4X'(0)=X(3\pi)=0 \end{cases}\end{matrix}$



$\lambda=0$ no autovalor. $\lambda>0$: $\begin{cases} c_1=4c_2w \\ c_2[4w\cos 3\pi w+\operatorname{sen} 3\pi w]=0 \end{cases}$, $\tan 3\pi w_n=-4w_n\ \left[w_1=\frac{1}{4}\right]$

$$\to \lambda_n=w_n^2\ \left[\lambda_1=\tfrac{1}{16}\right],\ \ X_n=\{\operatorname{sen} w_nx+4w_n\cos w_nx\}\ \left[X_1=\{\operatorname{sen}\tfrac{x}{4}+\cos\tfrac{x}{4}\}\right].$$

$$u=c_1\,e^{(a-\frac{1}{16})t}(\operatorname{sen}\tfrac{x}{4}+\cos\tfrac{x}{4})+\sum_{n=2}^{\infty}c_ne^{(a-\lambda_n)t}X_n(x)\ ,\ \text{con}\ \ c_1=\frac{\int_0^{3\pi}X_1dx}{\int_0^{3\pi}X_1^2dx}=\frac{4[\sqrt{2}+1]}{3\pi+2}\ .$$

Si $a<\frac{1}{16}$, $u\underset{t\to\infty}{\longrightarrow}0$. Si $a=\frac{1}{16}$, $u\underset{t\to\infty}{\longrightarrow}\frac{4[\sqrt{2}+1]}{3\pi+2}(\operatorname{sen}\frac{x}{4}+\cos\frac{x}{4})$. Si $a>\frac{1}{16}$, $u\underset{t\to\infty}{\longrightarrow}\infty$ $\left[e^{(a-\frac{1}{16})t}\text{ manda y } X_1>0\right]$.

8 i] $\begin{matrix} X''+2X'+\lambda X=0 \\ X(0)=X(1)+X'(1)=0 \end{matrix}$ $\quad \mu^2+2\mu+\lambda=0,\ \mu=-1\pm\sqrt{1-\lambda}$. En forma S-L queda $\left[e^{2x}X'\right]'+\lambda e^{2x}X=0$.

Aunque $\lambda\geq 0$, hay que discutir $\lambda <,=,>1$. Llamamos $p=\sqrt{1-\lambda}$ y $w=\sqrt{\lambda-1}$.

$\lambda<1$: $X=c_1\,e^{(-1+p)x}+c_2\,e^{(-1-p)x}$, $X'=c_1(p-1)\,e^{(-1+p)x}-c_2(1+p)\,e^{(-1-p)x}$ $\to$

$\left.\begin{matrix} c_1+c_2=0 \\ c_1pe^{-1+p}-c_2pe^{-1-p}=0 \end{matrix}\right\} \to c_1pe^{-1}[e^p+e^{-p}]=0 \to c_1=c_2=0$ no autovalor.

$\lambda=1$: $X=[c_1+c_2x]\,e^{-x}$, $X'=[c_2-c_1-c_2x]\,e^{-x}$ $\to \left.\begin{matrix} c_1=0 \\ c_2=0 \end{matrix}\right\} \to X\equiv 0$. $\lambda=1$ no autovalor.

$\lambda>1$: $X=[c_1\cos wx+c_2\operatorname{sen} wx]\,e^{-x} \overset{X(0)=0}{\longrightarrow} c_1=0 \to X(1)+X'(1)=c_2\left[w\cos wx\right]e^{-x}\to$

$$w_n=\frac{(2n-1)\pi}{2}\ ,\ n=1,2,\ldots \ \to\ \boxed{\lambda_n=1+w_n^2\,,\ X_n=\{e^{-x}\operatorname{sen} w_nx\}}\ .$$

ii] $\begin{cases} u_t-u_{xx}-2u_x=0\,,\ x\in(0,1)\,,\ t>0 \\ u(x,0)=e^{-x},\ u(0,t)=u(1,t)+u_x(1,t)=0 \end{cases}$ $\quad u=XT \to \frac{X''+2X'}{X}=\frac{T'}{T}=-\lambda \to \begin{cases} X''+2X'+\lambda X=0 \\ X(0)=X(1)+X'(1)=0 \end{cases}\uparrow$ y

$T'+\lambda T=0 \overset{\lambda=\lambda_n}{\longrightarrow} T_n=\{e^{-\lambda_n t}\}$

$$\to u(x,t)=\sum_{n=1}^{\infty}c_n\,e^{-\lambda_n t}\,e^{-x}\operatorname{sen} w_nx \to u(x,0)=\sum_{n=1}^{\infty}c_n\,e^{-x}\operatorname{sen} w_nx=e^{-x}\to$$

$$1=\sum_{n=1}^{\infty}c_n\operatorname{sen}\frac{(2n-1)\pi x}{2}\ ,\ c_n=2\int_0^1\operatorname{sen}\frac{(2n-1)\pi x}{2}dx=\frac{4}{\pi(2n-1)}\ ,\ \boxed{u=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-x-t-(2n-1)^2\pi^2t/4}}{2n-1}\operatorname{sen}\frac{(2n-1)\pi x}{2}}\ .$$

Sin simplificar el e^{-x}: $\ e^{-x}=\sum_{n=1}^{\infty}c_nX_n(x) \to c_n=\frac{\langle e^{-x},X_n\rangle}{\langle X_n,X_n\rangle}=2\int_0^1\underset{\text{peso}}{e^{2x}}\,e^{-x}\,e^{-x}\operatorname{sen}\frac{(2n-1)\pi x}{2}dx$,

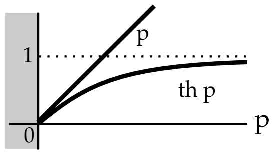
pues $\langle X_n\,,X_n\rangle=2\int_0^1\underset{\text{peso}}{e^{2x}}\,e^{-2x}\operatorname{sen}^2\frac{(2n-1)\pi x}{2}dx=\frac{1}{2}$.

$u=e^{pt+qx}w \to w_t-w_{xx}-(2q+2)w_x+(p-q^2-2q)w=0 \to q=p=-1$ lleva al calor. Así pues:

$w=e^{t+x}u\ \left[w_x=(u+u_x)e^{t+x}\right] \to \begin{cases} w_t-w_{xx}=0 \\ w(x,0)=1,\ w(0,t)=w_x(1,t)=0 \end{cases} \to X_n=\{\operatorname{sen} w_nx\},\ T_n=\{e^{-w_n^2t}\}$.

$\sum c_nT_nX_n$ lleva al desarrollo de antes y haciendo $u=e^{-t-x}w$ llegamos a la solución de arriba.

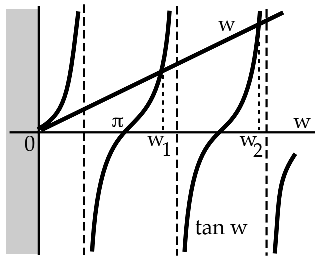
9 **i]** $\boxed{\begin{array}{l} X''+\lambda X=0 \\ X(0)=X(1)-X'(1)=0 \end{array}}$ Como $\beta\beta'<0$ el problema puede tener autovalores negativos:



$\lambda<0$, $\sqrt{-\lambda}=p \rightarrow \begin{array}{l} X=c_1 e^{px}+c_2 e^{-px} \\ X'=p(c_1 e^{px}-c_2 e^{-px}) \end{array} \rightarrow \begin{array}{l} c_1+c_2=0 \searrow \\ c_1\left[(e^p-e^{-p})-p(e^p+e^{-p})\right]=0 \end{array}$

$\rightarrow c_1=c_2=0$, pues $p\neq \frac{e^p-e^{-p}}{e^p+e^{-p}}=\text{th}\,p$, si $p>0$ $[\,\text{th}'(0)=1\,]$.

$\lambda=0 \rightarrow \begin{array}{l} X=c_1+c_2x \\ X'=c_2 \end{array} \rightarrow \begin{array}{l} c_1=0 \\ c_1+c_2-c_2=c_1=0 \end{array} \rightarrow \lambda=0$ autovalor, $X_0=\{x\}$.



$\lambda>0$, $\sqrt{\lambda}=w \rightarrow X=c_1\cos wx+c_2\,\text{sen}\,wx \rightarrow \begin{array}{l} c_1=0 \searrow \\ c_2[\text{sen}\,w-w\cos w]=0 \end{array}$

$\rightarrow$ infinitos w_n con $\tan w_n=w_n \rightarrow \lambda_n=w_n^2$, $X_n=\{\text{sen}\,w_nx\}$.

Todas las X_n deben ser ortogonales. En particular:

$\int_0^1 x\,\text{sen}\,w_nx\,dx=-\frac{x\cos w_nx}{w_n}\Big]_0^1+\int_0^1\frac{\cos w_nx}{w_n}\,dx=\frac{\text{sen}\,w_n-w_n\cos w_n}{w_n^2}=0$.

ii] $\boxed{\begin{cases} u_t-u_{xx}+2u=2,\ x\in(0,1),\ t>0 \\ u(x,0)=1-x,\ u(0,t)=u(1,t)-u_x(1,t)=1 \end{cases}}$ Para hacer las condiciones de contorno homogéneas necesitamos una v que las cumpla.

A simple vista: $v=1 \xrightarrow{w=u-1} \begin{cases} w_t-w_{xx}+2w=0 \\ w(x,0)=-x,\ w(0,t)=w(1,t)-w_x(1,t)=0 \end{cases}$

$w=XT \rightarrow XT'-X''T+2XT=0 \rightarrow \frac{X''}{X}=\frac{T'}{T}+2=-\lambda \rightarrow \begin{cases} X''+\lambda X=0 \\ X(0)=X(1)-X'(1)=0 \end{cases}$ y $T'+(\lambda+2)T=0$.

Las autofunciones del problema en x son las de arriba: $\{x\}$ y $\{\text{sen}\,w_nx\}$. Probamos entonces:

$$w(x,t)=c_0\,e^{-2t}x+\sum_{n=1}^{\infty}c_n\,e^{-\left(w_n^2+2\right)t}\,\text{sen}\,w_nx \rightarrow w(x,0)=c_0x+\sum_{n=0}^{\infty}c_n\,\text{sen}\,w_nx=-x$$

$\rightarrow c_0=-1$ y los demás c_n son cero. Así pues, $u(x,t)=1-e^{-2t}x$.

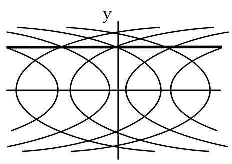
[Para estas condiciones de contorno 'no físicas' no sabemos probar la unicidad (ni para el calor ni para esta ecuación similar), con lo que tal vez (o tal vez no) pudieran existir otras soluciones no calculables por separación de variables].

10 $\boxed{\begin{cases} u_{tt}-u_{xx}=0,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=0 \\ u(0,t)=\text{sen}\,wt,\ u(\pi,t)=0 \end{cases}}$ $v=(1-\frac{x}{\pi})\,\text{sen}\,wt \underset{u=v+u^*}{\rightarrow} \begin{cases} u^*_{tt}-u^*_{xx}=w^2(1-\frac{x}{\pi})\,\text{sen}\,wt \\ u^*(x,0)=0,\ u^*_t(x,0)=-w(1-\frac{x}{\pi}) \\ u^*(0,t)=u^*(\pi,t)=0 \end{cases}$

$\sum_{n=1}^{\infty}T_n(t)\,\text{sen}\,nx$, $1-\frac{x}{\pi}=\sum_{n=1}^{\infty}\frac{2}{n\pi}\,\text{sen}\,nx \rightarrow \begin{cases} T_n''+n^2T_n=\frac{2w^2}{n\pi}\,\text{sen}\,wt \rightarrow T_n=c_1\cos nt+c_2\,\text{sen}\,nt+T_{np} \\ T_n(0)=0,\ T_n'(0)=-\frac{2w}{n\pi} \end{cases}$

Si $w^2\neq n^2$, $T_{np}=A\,\text{sen}\,wt\ \forall n$. Si $w^2=n^2$, una T_{np} debe engordarse con una $t \Rightarrow u$ no acotada.

11 $\boxed{tu_{tt}-4t^3u_{xx}-u_t=0}$ a] $B^2-4AC=16t^4$, hiperbólica. $\frac{dx}{dt}=\frac{\pm 4t^2}{2t} \rightarrow x\pm t^2=C$.



$\begin{cases} \xi=x+t^2 \\ \eta=x-t^2 \end{cases} \rightarrow \begin{cases} u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta} \\ u_t=2t[u_\xi-u_\eta] \\ u_{tt}=4t^2[u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}]+2[u_\xi-u_\eta] \end{cases} \rightarrow u_{\xi\eta}=0$, $u=p(x+t^2)+q(x-t^2)$

$\begin{cases} p(x+1)+q(x-1)=x \rightarrow p'(x+1)+q'(x-1)=1 \\ 2p'(x+1)-2q'(x-1)=2x \rightarrow p'(x+1)-q'(x-1)=x \end{cases} \rightarrow p'(x+1)=\frac{x+1}{2} \xrightarrow{v=x+1} p'(v)=\frac{v}{2} \rightarrow p(v)=\frac{v^2}{4}+K$.

$q(x-1)=x-p(x+1)=\frac{-(x-1)^2}{4}-K \rightarrow q(v)=-\frac{v^2}{4}-K \rightarrow u=\frac{(x+t^2)^2}{4}-\frac{(x-t)^2}{4}=\boxed{xt^2}$ [única].

b] $\frac{X''}{X}=\frac{tT''-T'}{4t^3T}=-\lambda \rightarrow \begin{cases} X''+\lambda X=0 \\ tT''-T'+4\lambda t^3T=0 \end{cases} \xrightarrow{\lambda=0} \begin{array}{l} X''=0 \rightarrow X=c_1+c_2x \\ tT''-T'=0 \rightarrow T=k_1+k_2t^2 \end{array} \rightarrow \{1\},\{x\},\{t^2\},\{xt^2\}$.

Soluciones de problemas 3 (L y 3) de EDII(r) (2011)

1 a) $\Delta u=2x\cos^2 y,\ (x,y)\in(0,\pi)\times(0,\pi)$; $u(\pi,y)=5+\cos y$; $u(0,y)=u_y(x,0)=u_y(x,\pi)=0$

$Y''+\lambda Y=0,\ Y'(0)=Y'(\pi)=0 \to Y_n=\{\cos ny\},\ n=0,1,\dots$
$\to u=X_0(x)+\sum_{n=1}^{\infty}X_n(x)\cos ny \to$

$$X_0''+\sum_{n=1}^{\infty}[X_n''-n^2X_n]\cos ny=x+x\cos 2y\,,\quad X_n(0)=0\,,\ X_0(\pi)+\sum_{n=1}^{\infty}X_n(\pi)\cos ny=5+\cos y \to$$

$$\begin{cases}X_0''=x\ [X_0=\frac{x^3}{6}+c_1+c_2x]\\ X_0(0)=0,X_0(\pi)=5\end{cases},\ \begin{cases}X_1''-X_1=0\ [X_1=c_1e^x+c_2e^{-x}]\\ X_1(0)=0,X_1(\pi)=1\end{cases},\ \begin{cases}X_2''-4X_2=x\ [X_2=c_1e^{2x}+c_2e^{-2x}-\frac{x}{4}]\\ X_2(0)=X_2(\pi)=0\end{cases}$$

$$\to u(x,y)=\frac{x^3}{6}+\frac{5x}{\pi}-\frac{\pi^2x}{6}+\frac{\operatorname{sh}x}{\operatorname{sh}\pi}\cos y+\Big[\frac{\pi\operatorname{sh}2x}{4\operatorname{sh}2\pi}-\frac{x}{4}\Big]\cos 2y\,.$$

b) $\Delta u=y\cos x,\ (x,y)\in(0,\pi)\times(0,1)$; $u_x(0,y)=u_x(\pi,y)=u_y(x,0)=u_y(x,1)=0$

$u=\sum_{n=0}^{\infty}Y_n(y)\cos nx \to \begin{matrix}Y_1''-Y_1=y\\ Y_1'(0)=Y_1'(1)=0\end{matrix} \to Y_1=\frac{e^y-e^{1-y}}{1+e}-y$

Como es de Neumann aparece (al resolver $Y_0''=0+$c.c.) una C arbitraria: $u=C+\big[\frac{e^y-e^{1-y}}{1+e}-y\big]\cos x$

c) $\Delta u=-1,\ (x,y)\in(0,\pi)\times(0,\pi)$; $u=0$ en $x=0,x=\pi,y=0,y=\pi$

Convirtiéndolo en homogéneo con una $v(x)$ que cumpla las c.c.:
$v_{xx}=-1,\ v=c_1+c_2x-\frac{x^2}{2}\ \xrightarrow{v(0)=v(\pi)=0}\ v=\frac{1}{2}x(\pi-x),\ w=u-v \to$

$\begin{cases}\Delta w=0,\ w(0,y)=w(\pi,y)=0\\ w(x,0)=w(x,\pi)=\frac{1}{2}x(x-\pi)\end{cases}$ Resolviendo se llega (dice Weimberger) a: $u=\frac{1}{2}x(\pi-x)+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\operatorname{ch}[(2k-1)(y-\frac{\pi}{2})]}{(2k-1)^3\operatorname{ch}[(2k-1)\frac{\pi}{2}]}\operatorname{sen}(2k-1)x$

O bien: $u=\sum_{k=1}^{\infty}Y_n(y)\operatorname{sen}nx \to Y_n''-n^2Y_n=-\frac{2}{\pi}\int_0^{\pi}\operatorname{sen}nx\,dx\,,\ Y_n=c_1e^{ny}+c_2e^{-ny}-\frac{2[(-1)^n-1]}{\pi n^3}\ \xrightarrow[Y_n(0)=Y_n(\pi)=0]{}$

$u=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}\Big[\frac{(1-e^{-n\pi})e^{nx}-(1-e^{n\pi})e^{-nx}}{e^{n\pi}-e^{-n\pi}}-1\Big]\operatorname{sen}nx$ (ambas series deben poder hacerse coincidir).

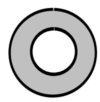
d) $\Delta u=r^2\cos 3\theta,\ r<2,0<\theta<\frac{\pi}{6}$; $u(2,\theta)=u_\theta(r,0)=u(r,\frac{\pi}{6})=0$

$\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta'(0)=\Theta(\frac{\pi}{6})=0\end{cases} \to \begin{matrix}\lambda_n=9(2n-1)^2,n=1,2,\dots\\ \Theta_n=\{\cos 3(2n-1)\theta\}\end{matrix}$ $[$y $r^2R''+rR'-\lambda R=0\,]$.

Probamos: $u(r,\theta)=\sum_{n=1}^{\infty}R_n(r)\cos 3(2n-1)\theta \to \sum_{n=1}^{\infty}\Big[R_n''+\frac{1}{r}R_n'-\frac{9(2n-1)^2}{r^2}R_n\Big]\cos 3(2n-1)\theta=r^2\cos 3\theta \to$

$\begin{cases}r^2R_1''+rR_1'-9R_1=r^4\\ R_1 \text{ acotada en } 0,\ R_1(2)=0\end{cases}\xrightarrow{R_p=Ar^4}R_1=c_1r^3+c_2r^{-3}+\frac{1}{7}r^4\xrightarrow{\text{c.c.}}R_1=\frac{1}{7}[r^4-2r^3]\,,\ u(r,\theta)=\frac{1}{7}[r^4-2r^3]\cos 3\theta\,.$

e) $\Delta u=r^2\cos 2\theta,\ 1<r<2$; $u(1,\theta)=u(2,\theta)=0$



$\Theta''+\lambda\Theta=0$ y 2π-periódica $\to\Theta_n=\{\cos n\theta,\operatorname{sen}n\theta\}$,
$n=0,1,\dots\to u=a_0(r)+\sum_{n=1}^{\infty}[a_n(r)\cos n\theta+b_n(r)\operatorname{sen}n\theta] \to$

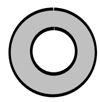
$$a_0''+\frac{1}{r}a_0'+\sum_{n=1}^{\infty}\Big[(a_n''+\frac{1}{r}a_n'-\frac{n^2}{r^2}a_n)\cos n\theta+(b_n''+\frac{1}{r}b_n'-\frac{n^2}{r^2}b_n)\operatorname{sen}n\theta\Big]=r^2\cos 2\theta$$

De los datos de contorno: $a_n(1)=a_n(2)=b_n(1)=b_n(2)=0\ \ \forall n$. El único problema con solución no trivial:

$\begin{cases}r^2a_2''+ra_2'-4a_2=r^4\\ a_2(1)=a_2(2)=0\end{cases}\xrightarrow{a_{2p}=Ar^4}a_2=c_1r^2+c_2r^{-2}+\frac{1}{12}r^4\xrightarrow{\text{c.c.}}\begin{matrix}c_1+c_1+\frac{1}{12}=0\\ 4c_1+\frac{1}{4}c_2+\frac{4}{3}=0\end{matrix}\to u=\Big[\frac{r^4}{12}-\frac{7r^2}{20}+\frac{4}{15r^2}\Big]\cos 2\theta\,.$

$\Big[$Más largo es hallar a_{2p} con la fvc: $\begin{vmatrix}r^2 & r^{-2}\\ 2r & -2r^{-3}\end{vmatrix}=-\frac{4}{r}\,,\ a_{2p}=r^{-2}\int\frac{r^4}{-4/r}-r^2\int\frac{1}{-4/r}=-\frac{r^4}{24}+\frac{r^4}{8}=\frac{r^4}{12}\Big]$.

f) $u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}=\cos\theta,\ 1<r<2$; $u_r(1,\theta)=0,\ u_r(2,\theta)=\cos 2\theta$

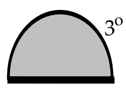


$\Theta''+\lambda\Theta=0$ y 2π-periódica $\to\Theta_n=\{\cos n\theta,\operatorname{sen}n\theta\}$,
$n=0,1,\dots\to u=a_0(r)+\sum_{n=1}^{\infty}[a_n(r)\cos n\theta+b_n(r)\operatorname{sen}n\theta]$

$\begin{cases}a_0''+\frac{a_0'}{r}=0\ [a_0=c_1+c_2\ln r]\\ a_0'(1)=a_0'(2)=0\end{cases},\ \begin{cases}a_1''+\frac{a_1'}{r}-\frac{a_1}{r^2}=1\ [a_{1p}=Ar^2]\\ a_1'(1)=a_1'(2)=0\end{cases},\ \begin{cases}a_2''+\frac{a_2'}{r}-\frac{4a_2}{r^2}=0\ [a_2=c_1r^2+c_2r^{-2}]\\ a_2'(1)=0,a_2'(2)=1\end{cases}$

(y $a_{n\geq3},b_n\equiv0$) $\to u=C+\big(\frac{r^2}{3}-\frac{14r}{9}-\frac{8}{9r}\big)\cos\theta+\big(\frac{4r^2}{15}-\frac{4}{15r^2}\big)\cos 2\theta$ (era de Neumann).

g) $\Delta u=r,\ r<2,0<\theta<\pi$; $u_\theta(r,0)=u_\theta(r,\pi)=0$; $u(2,\theta)=3$

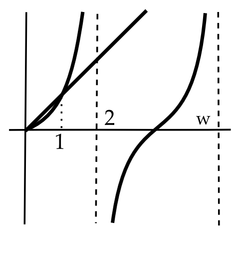


$\Theta''+\lambda\Theta=0,\ \Theta'(0)=\Theta'(\pi)=0\to\Theta_n=\{\cos n\theta\},\ n=0,1,\dots\to$
$u=R_0(r)+\sum_{n=1}^{\infty}R_n(r)\cos n\theta \to \begin{matrix}R_0''+\frac{1}{r}R_0'=r\\ R_0(2)=3,R_0 \text{ acotada}\end{matrix}$ $(R_{n\geq1}\equiv0)$.

$R_0=\frac{r^3}{9}+c_1+c_2\ln r\xrightarrow{R_0\ ac.}R_0=\frac{r^3}{9}+c_1\xrightarrow{R_0(2)=3}u=\frac{r^3}{9}+\frac{19}{r}$ [Se podía haber buscado directamente una $u(r)$ pues $u_\theta\equiv0$ y los datos no dependen de θ].

h) $\Delta u=0,\ 1<r<2,\ 0<\theta<\frac{\pi}{4}$; $u(1,\theta)=0,\ u_r(2,\theta)=\operatorname{sen}\theta$; $u(r,0)=u(r,\frac{\pi}{4})-u_\theta(r,\frac{\pi}{4})=0$

$\Theta''+\lambda\Theta=0,\ \Theta'(0)=\Theta(\frac{\pi}{4})-\Theta'(\frac{\pi}{4})=0\to$
$\lambda_n=w_n^2$, con $\tan\frac{\pi w_n}{4}=w_n$, $\Theta_n=\{\operatorname{sen}w_n\theta\}$.
Casualmente es $\lambda_1=1$ y $\Theta_1=\{\operatorname{sen}\theta\}$.



$r^2R''+rR'-\lambda_nR=0\to R=c_1r^{w_n}+c_2r^{-w_n}\xrightarrow{R(1)=0}R_n=\{r^{w_n}-r^{-w_n}\}$.

$u=\sum_{n=1}^{\infty}c_nR_n(r)\operatorname{sen}w_n\theta\xrightarrow{u_r(2,\theta)=\operatorname{sen}\theta}u(r,\theta)=\frac{4}{5}\big(r-\frac{1}{r}\big)\operatorname{sen}\theta\,.$

2 $\begin{cases} u_{xx}+u_{yy}+6u_x=0 \text{ en } (0,\pi)\times(0,\pi) \\ u_y(x,0)=0,\ u_y(x,\pi)=0 \\ u_x(0,y)=0,\ u(\pi,y)=2\cos^2 2y \end{cases}$ $\quad u=XY \to \frac{X''+6X'}{X}=-\frac{Y''}{Y}=\lambda \to \begin{cases} Y''+\lambda Y=0,\ Y'(0)=Y'(\pi)=0 \\ X''+6X'-\lambda X=0,\ X'(0)=0 \end{cases}$

$\to \lambda_n=n^2,\ Y_n=\{\cos ny\}\ n=0,1,\dots \to\ X''+6X'-n^2X=0,\ X=c_1 e^{(\sqrt{9+n^2}-3)x}+c_2 e^{-(\sqrt{9+n^2}+3)x} \underset{X'(0)=0}{\longrightarrow}$

$$X_0=\{1\}\ ;\ \ X_n=\Big\{(\sqrt{9+n^2}+3)e^{(\sqrt{9+n^2}-3)x}+(\sqrt{9+n^2}-3)e^{-(\sqrt{9+n^2}+3)x}\Big\},\ n\ge 1 .$$

$u=\sum_{n=0}^{\infty} c_nX_n(x)\cos ny \to\ u(\pi,y)=\sum_{n=0}^{\infty} c_nX_n(\pi)\cos ny=1+\cos 4y \to\ c_0=1,\ c_4=\frac{1}{X_4(\pi)}$ y los demás cero

$$\to\ u=1+\frac{4e^{2x}+e^{-8x}}{4e^{2\pi}+e^{-8\pi}}\cos 4y$$

3 $\begin{cases} u_{xx}+u_{yy}-9u=0 \text{ en } (0,\pi)\times(0,\pi) \\ u_x(0,y)=u_x(\pi,y)=0 \\ u_y(x,0)=0,\ u_y(x,\pi)=f(x) \end{cases}$ $\quad u(x,y)=X(x)Y(y)\to\frac{X''}{X}=\frac{9Y-Y''}{Y}=-\lambda\to$

$\begin{cases} X''+\lambda X=0,\ X'(0)=X'(\pi)=0 \to \lambda_n=n^2,\ X_n=\{\cos nx\},\ n=0,1,\dots \\ Y''-(\lambda+9)Y=0,\ Y'(0)=0 \qquad\qquad \downarrow\ Y_n=\{\operatorname{ch}\sqrt{n^2+9}\,y\},\ n=0,1,\dots \end{cases}$

$$u(x,y)=\sum_{n=0}^{\infty} c_n \operatorname{ch}\sqrt{n^2+9}\,y\,\cos nx\ \to\ u_y(x,\pi)=\sum_{n=0}^{\infty}\sqrt{n^2+9}\,c_n\operatorname{sh}\sqrt{n^2+9}\,y\,\cos nx=f(x)=\frac{a_o}{2}+\sum_{n=0}^{\infty}a_n\cos nx\ \to$$

$$c_0=\frac{1}{\pi\operatorname{sh}\sqrt{3}\,\pi}\int_0^{\pi}f(x)\,dx,\quad c_n=\frac{2}{\pi\operatorname{sh}\sqrt{9+n^2}\,\pi}\int_0^{\pi}f(x)\cos nx\,dx .$$

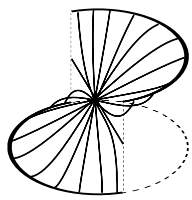
Si $f(x)=\cos 4x$ es $c_4\sqrt{25}\operatorname{sh}\sqrt{25}\,\pi=1$ y los demás cero $\to\ u=\frac{\operatorname{ch}5y}{5\operatorname{sh}5\pi}\cos 4x$

La solución es única. Diferencia de soluciones $u=u_1-u_2$ satisface el problema con todo 0 y por tanto:

$$\iint_D u\Delta u=\oint_{\partial D}u\,u_{\mathbf n}-\iint_D\|\nabla u\|^2=\iint_D 9u^2\ \Rightarrow\ \iint_D\Big[9u^2+\|\nabla u\|^2\Big]=0\ \ (u_{\mathbf n}=0 \text{ en } \partial D)\ \Rightarrow\ u\equiv 0 .$$

[O lo que es lo mismo: $\int_0^{\pi}\!\int_0^{\pi}(uu_{xx}+uu_{yy})=\int_0^{\pi}[uu_x]_0^{\pi}dy+\int_0^{\pi}[uu_y]_0^{\pi}dx-\int_0^{\pi}\!\int_0^{\pi}(u_x^2+u_y^2)=9\int_0^{\pi}\!\int_0^{\pi}u^2\ \cdots$].

4 $\begin{cases} \Delta u=0,\ r<1 \\ u(1,\theta)=f(\theta) \end{cases}$ con $f(\theta)=\begin{cases} 1,\ 0\le\theta\le\pi \\ 0,\ \pi<\theta<2\pi \end{cases}$ El principio del máximo ya da la cota superior: $0\le \boxed{u\le 1}$.



Necesitamos la solución para la otra. Lo más corto es utilizar la fórmula de **Poisson**:

$$u(r,\theta)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\phi)\,d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}\to\ u(\tfrac12,\tfrac{\pi}{2})=\frac{3}{8\pi}\int_0^{\pi}\frac{d\phi}{\frac54-\cos(\frac{\pi}{2}-\phi)}=\frac{3}{2\pi}\int_0^{\pi}\frac{d\phi}{5-4\sin\phi}\equiv I$$

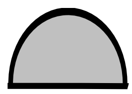
$$s=\tan\frac{\phi}{2}\to I=\frac{3}{5\pi}\int_0^{\infty}\frac{ds}{u^2-\frac85 s+1}=\frac{1}{\pi}\int_0^{\infty}\frac{5/3\,ds}{1+(\frac{5u-4}{3})^2}=\frac12+\frac{1}{\pi}\arctan\frac43>\frac12+\frac14>\frac32 .$$

[Acotar el integrando $\frac15\le\frac{1}{5-4\sin\phi}\le 1\to\frac{3}{10}\le I\le\frac32$ no basta, pero era un buen intento].

Con la serie de 3.2 es más largo: $u=\frac{a_o}{2}+\sum_{n=1}^{\infty}r^n[a_n\cos n\theta+b_n\operatorname{sen}n\theta]=\cdots=\frac12+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n-1}}{2n-1}\operatorname{sen}(2n-1)\theta$

$\to I=\frac12+\frac{2}{\pi}\big[\frac12-\frac13\frac{1}{2^3}+\frac15\frac{1}{2^5}-\cdots\big]>\frac12+\frac{1}{\pi}\big[1-\frac{1}{12}\big]>\frac12+\frac{11}{48}=\frac{35}{48}>\frac23\ \ \big[I=\frac12+\frac{2}{\pi}\arctan\frac12$ otro valor exacto de I].

5 $\begin{cases} u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}=\cos^2\theta,\ r<1,\ 0<\theta<\pi \\ u_r(1,\theta)=a,\ u_\theta(r,0)=u_\theta(r,\pi)=0 \end{cases}$ **Neumann**. Para que tenga solución $\int_0^{\pi}\!\int_0^1 r\cos^2\theta\,dr\,d\theta=\int_0^{\pi}a\,d\theta\Leftrightarrow a=\frac14$.



$$u=\sum_{n=0}^{\infty}R_n(r)\cos n\theta\ \to\ r^2R_0''+rR_0=\frac{r^2}{2}\ \underset{R_0'(1)=1/4}{\xrightarrow{R_0 \text{ acot.}}}\ R_0=C+\frac{r^2}{8},\quad r^2R_2''+rR_2'-4R_2=\frac{r^2}{2}\ \underset{R_2'(1)=0}{\xrightarrow{R_2 \text{ acot.}}}\ R_2=\frac{r^2}{8}\ln r-\frac{r^2}{16}$$

y las demás $R_n\equiv 0$. $\ u=C+\frac{r^2}{8}+r^2\Big(\frac{\ln r}{8}-\frac{1}{16}\Big)\cos 2\theta$.

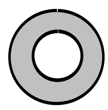
6 a] $\begin{cases} r^2y''+ry'-y=r^2 \\ y'(1)+ay(1)=y(2)=0 \end{cases}$ $\quad r^2y''+ry'-y=0\to y=c_1r+\frac{c_2}{r}\underset{\text{datos}}{\to}\ \begin{matrix} c_1-c_2+ac_1+ac_2=0 & c_1[5-3a]=0 \\ 2c_1+\frac{c_2}{2}=0\to c_2=-4c_1 \nearrow & \end{matrix}\ \Rightarrow$

Si $a\ne\frac53$ el no homogéneo tiene solución única (el homogéneo tiene sólo la $y\equiv 0$).

Si $a=\frac53$ el homogéneo tiene ∞: $y_h=\{r-\frac4r\}$ y el no homogéneo cero: $(ry')'-\frac{y}{r}=r\to\int_1^2(r-\frac4r)r\,dr=-\frac53\ne 0$.

[Directamente: $y=c_1r+\frac{c_2}{r}+\frac{r^2}{3}\ \to\ \begin{matrix} c_1-c_2+\frac23+\frac53c_1+\frac53c_2+\frac59=\frac83c_1+\frac23c_2+\frac{11}{9}=0 \to 4c_1+c_2=-\frac{11}{6} \\ 2c_1+\frac{c_2}{2}+\frac43=0 \qquad\qquad\qquad\qquad\qquad\qquad \to 4c_1+c_2=-\frac83 \end{matrix}\Big\}$ Imposible].

b] $\begin{cases} \Delta u=\operatorname{sen}\theta,\ 1<r<2 \\ u_r(1,\theta)=u(2,\theta)=0 \end{cases}$ Las autofunciones de $\begin{matrix}\Theta''+\lambda\Theta=0 \\ \Theta\ 2\pi\text{-periódica}\end{matrix}$ llevan a probar:



$$u=a_0(r)+\sum_{n=1}^{\infty}[a_n(r)\cos n\theta+b_n(r)\operatorname{sen}n\theta]\to\frac{ra_0''+a_0'}{r}+\sum_{n=1}^{\infty}\Big[\frac{r^2a_n''+ra_n'-n^2a_n}{r^2}\cos n\theta+\frac{r^2b_n''+rb_n'-n^2b_n}{r^2}\operatorname{sen}n\theta\Big]=\operatorname{sen}\theta .$$

Datos de contorno $\to a_n'(1)=b_n'(1)=0,\ a_n(2)=b_n(2)=0\ \to\ a_n, b_{n\ne1}\equiv 0$ (es solución y hay unicidad).

Además: $r^2b_1''+rb_1'-b_1=r^2$ con $b_1'(1)=b_1(2)=0$. Ecuación resuelta arriba. Imponiendo los datos:

$\begin{matrix} c_1-c_2+\frac23=0\ \to\ c_2=c_1+\frac23\downarrow & c_2=0 \\ 2c_1+\frac{c_2}{2}+\frac43=0 & \frac52c_1+\frac53=0,\ c_1=-\frac23\uparrow \end{matrix}\ \to\ \boxed{u=\frac{r^2-2r}{3}\operatorname{sen}\theta}$.

7 $\begin{cases}\Delta u=0\,,\ r<1,\ \theta\in(0,\pi)\\ u(1,\theta)+2u_r(1,\theta)=4\,\text{sen}\,\frac{3\theta}{2}\\ u(r,0)=u_\theta(r,\pi)=0\end{cases}$ $\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta'(\pi)=0\end{cases}\to\lambda_n=\frac{(2n-1)^2}{4}\,,\ \Theta_n=\{\text{sen}\,\frac{2n-1}{2}\theta\},\ n=1,2,\ldots\to$

$r^2R''+rR-\lambda_nR=0\to R=c_1r^{n-\frac12}+c_2r^{-n+\frac12}\xrightarrow{R\text{ acotada en }0}R_n=\{r^{n-\frac12}\}.$

$\to\ u(r,\theta)=\sum_{n=1}^{\infty}c_n\,r^{n-\frac12}\,\text{sen}\,\frac{2n-1}{2}\theta\,,\quad u_r(r,\theta)=\sum_{n=1}^{\infty}c_n(n-\tfrac12)r^{n-\frac32}\,\text{sen}\,\frac{2n-1}{2}\theta\ \xrightarrow{\text{dato que falta}}$

$\sum_{n=1}^{\infty}c_n\big[1+2n-1\big]\,\text{sen}\,\frac{2n-1}{2}\theta=4\,\text{sen}\,\frac{3\theta}{2}\to c_2=1$ y todos los demás $c_n=0\to$ $\boxed{u=r^{3/2}\,\text{sen}\,\frac{3\theta}{2}}$

Si $u(1,\theta)-2u_r(1,\theta)=4\,\text{sen}\,\frac{3\theta}{2}$ todo igual hasta: $\sum_{n=1}^{\infty}2c_n\big[1-n\big]\,\text{sen}\,\frac{2n-1}{2}\theta=4\,\text{sen}\,\frac{3\theta}{2}\to$ $c_2=-2$, c_1 cualquiera y los demás $c_n=0$.

Hay infinitas soluciones de la forma $\boxed{u=C\,r^{1/2}\,\text{sen}\,\frac{\theta}{2}-2\,r^{3/2}\,\text{sen}\,\frac{3\theta}{2}}$ [En este caso, la fórmula de Green no permitía probar la unicidad].

8 $\begin{cases}u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}+4u=0\,,\ r<1\,,\ 0<\theta<\pi\\ u(1,\theta)=\text{sen}\,\frac{\theta}{2}\,,\ u(r,0)=u_\theta(r,\pi)=0\end{cases}$ $u=R\Theta\to R''\Theta+\frac{R'\Theta}{r}+\frac{R\Theta''}{r^2}+4R\Theta=0\to\frac{r^2R''}{R}+\frac{rR'}{R}+4r^2=-\frac{\Theta''}{\Theta}=\lambda$

$\to\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta'(\pi)=0\end{cases}\to\lambda_n=\frac{(2n-1)^2}{4},\ \Theta_n=\{\text{sen}\,\frac{2n-1}{2}\theta\},\ n=1,2,\ldots$ y $r^2R''+rR'+(4r^2-\lambda)R=0$.

Esta ecuación se parece mucho a Bessel. Para quitar el 4 que sobra, como se hace habitualmente:

$$s=\sqrt{4}\,r=2r\to R'=2\frac{dR}{ds}\,,\ R''=4\frac{d^2R}{ds^2}\to s^2\frac{d^2R}{ds^2}+s\frac{dR}{ds}+(s^2-\lambda)R=0\,,$$

que para los λ_n de arriba es Bessel con $p=n-\frac12$, cuyas soluciones acotadas en $r=0$ son las

$\{J_{n-\frac12}(s)\}=\{J_{n-\frac12}(2r)\}=R_n$ (todas se pueden escribir en términos de funciones elementales).

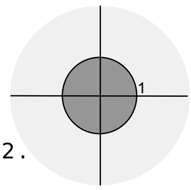
Probamos pues: $u=\sum_{n=1}^{\infty}c_nJ_{n-\frac12}(2r)\,\text{sen}\,\frac{2n-1}{2}\theta$, a la que sólo le falta satisfacer:

$$\sum_{n=1}^{\infty}c_nJ_{n-\frac12}(2)\,\text{sen}\,\frac{2n-1}{2}\theta=\text{sen}\,\frac{\theta}{2}\to c_1=\frac{1}{J_{\frac12}(2)}\ \text{y los demás}\ c_n=0\to u=\frac{1}{J_{\frac12}(2)}J_{\frac12}(2r)\,\text{sen}\,\frac{\theta}{2}\,.$$

Podemos escribir la solución anterior en términos de funciones elementales. Como (salvo constante)

$J_{\frac12}(2r)=\frac{\text{sen}\,2r}{\sqrt{2r}}$ $\Big[\frac{\cos 2r}{\sqrt{2r}}$ no está acotada en $r=0\Big]$ y $J_{\frac12}(2)=\frac{\text{sen}\,2}{\sqrt2}$, $\boxed{u(r,\theta)=\frac{\text{sen}\,2r}{\text{sen}\,2\,\sqrt r}\,\text{sen}\,\frac{\theta}{2}}$

9 $\begin{cases}u_{rr}+\frac1r u_r+\frac{1}{r^2}u_{\theta\theta}=\frac{2\,\text{sen}\,\theta}{1+r^2}\\ u(1,\theta)=1\,,\ u\ \text{acotada}\end{cases}$ Separando variables en la homogénea: $\Theta''+\lambda\Theta=0$ y $r^2R''+rR'-\lambda R=0$. Las autofunciones son $\Theta_n=\{\cos n\theta,\text{sen}\,n\theta\}$, $n=0,1,\ldots$ [Θ 2π-periódica].



Probamos en ambos casos: $u(r,\theta)=a_0(r)+\sum_{n=1}^{\infty}\big[a_n(r)\cos n\theta+b_n(r)\,\text{sen}\,n\theta\big]\ \to$

$a_0''+\frac1r a_0'+\sum_{n=1}^{\infty}\Big[(a_n''+\frac1r a_n'-\frac{n^2}{r^2}a_n)\cos n\theta+(b_n''+\frac1r b_n'-\frac{n^2}{r^2}b_n)\,\text{sen}\,n\theta\Big]=\frac{2}{1+r^2}\,\text{sen}\,\theta\ \to$

$r^2a_0''+ra_0'=0$; $r^2a_n''+ra_n'-n^2a_n=0$, $n\geq1$; $r^2b_1''+rb_1'-b_1=\frac{2r^2}{1+r^2}$; $r^2b_n''+rb_n'-n^2b_n=0$, $n\geq2$.

$u(1,\theta)=1\Rightarrow a_0(1)=1$ y que las demás se anulan en 1. Sólo tendrán solución no trivial:

$$\begin{cases}r^2a_0''+ra_0'=0\\ a_0(1)=1,\ a_0\ \text{acotada}\end{cases}\to a_0=c_1+c_2\ln r\xrightarrow{\text{c.c.}}a_0=1\text{, para **i]** y para **ii]**.}$$

$\begin{cases}r^2b_1''+rb_1'-b_1=\frac{2r^2}{1+r^2}\\ b_1(1)=0,\ b_1\ \text{acotada}\end{cases}\to b_1=c_1r+c_2r^{-1}+b_{1p}$. Necesitamos la fvc para la particular:

$$\begin{vmatrix}r & r^{-1}\\ 1 & -r^{-2}\end{vmatrix}=-2r^{-1}\,,\ f(r)=\frac{2}{1+r^2}\,,\quad b_{1p}=-r^{-1}\int\frac{r^2+1-1}{1+r^2}+r\int\frac{1}{1+r^2}=\Big(r+\frac1r\Big)\arctan r-1\,.$$

No es trivial imponer la condición de acotación. En $r<1$, como $\frac{\arctan r}{r}\xrightarrow{r\to0}1$, debe ser $c_2=0$.

Imponiendo la otra: $c_1+2\arctan 1-1=0\ \to\ b_1=(1-\frac{\pi}{2})r+(r+\frac1r)\arctan r-1$, para **i]**.

En el infinito $b_{1p}\sim\frac{\pi}{2}r-1$. Para que b_1 pueda estar acotada debemos tomar $c_1=-\frac{\pi}{2}$. Además:

$-\frac{\pi}{2}+c_2+\frac{\pi}{2}-1=0\ \to\ b_1=\frac1r-\frac{\pi}{2}r+(r+\frac1r)\arctan r-1$, para **ii]**. $\Big[\frac{\arctan r-\pi/2}{1/r}\xrightarrow[r\to\infty]{}-1\Big]$.

i] $u=1+\Big[r-\frac{\pi}{2}r+(r+\frac1r)\arctan r-1\Big]\,\text{sen}\,\theta$; **ii]** $u=1+\Big[\frac1r-\frac{\pi}{2}r+(r+\frac1r)\arctan r-1\Big]\,\text{sen}\,\theta$.

10 $\begin{cases}\Delta u=r\cos^2\theta,\ \ r<1\\ u(1,\theta)=0,\ 0\leq\theta<2\pi\end{cases}$ Con la fórmula obtenida a través de la función de Green en los apuntes:

$u(r,\theta)=\frac{1}{4\pi}\int_0^1\!\int_0^{2\pi}\sigma\ln\big([\sigma^2+r^2-2r\sigma\cos(\theta-\phi)]-\ln[1+r^2\sigma^2-2r\sigma\cos(\theta-\phi)]\big)\sigma\cos^2\phi\,d\phi\,d\sigma\ \to$

$$u(0,0)=\frac{1}{4\pi}\int_0^1\!\int_0^{2\pi}2\sigma^2\ln\sigma\cos^2\phi\,d\phi\,d\sigma=\frac12\int_0^1\sigma^2\ln\sigma\,d\sigma=\boxed{-\frac{1}{18}}$$

[Más largo: $u=a_0(r)+\sum_{n=1}^{\infty}[a_n(r)\cos n\theta+b_n(r)\text{sen}\,n\theta]\to$ $\begin{matrix}a_0''+\frac{a_0'}{r}=\frac r2\\ a_2''+\frac{a_2'}{r}-\frac{4a_2}{r^2}=\frac r2\end{matrix}$ $\xrightarrow[\text{acotado}]{a_n(1)=0}$ $u=\frac{r^3-1}{18}+\frac{r^3-r^2}{10}\cos 2\theta$]. ($\nwarrow r=0$)

11 $\boxed{\begin{array}{l}\Delta u=0\,,\ r<3\\ u_r(3,\theta)+u(3,\theta)=\operatorname{sen}^2\theta\end{array}}$ La serie de los apuntes satisface todo excepto el nuevo dato inicial:

$$u(r,\theta)=\sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta) \;\to\; u_r(3,\theta)+u(3,\theta)=\sum_{n=0}^{\infty} 3^{n-1}(n+3)a_n P_n(\cos\theta)=\operatorname{sen}^2\theta$$

$$\to\; a_n=\frac{2n+1}{3^{n-1}2(n+3)}\int_0^{\pi}\operatorname{sen}^2\theta\, P_n(\cos\theta)\operatorname{sen}\theta\, d\theta=\frac{2n+1}{3^{n-1}2(n+3)}\int_{-1}^{1}(1-t^2)P_n(t)\,dt\,.$$

Para calcular los a_n una posibilidad (la más larga y general) es hacer un par de integrales:

$$a_0=\tfrac{1}{2}\int_{-1}^{1}(1-t^2)dt=\tfrac{2}{3}\quad,\quad a_2=\tfrac{1}{6}\int_{-1}^{1}(1-t^2)(\tfrac{3}{2}t^2-\tfrac{1}{2})dt=\tfrac{1}{3}\int_0^1(-\tfrac{1}{2}+2t^2-\tfrac{3}{2}t^4)dt=-\tfrac{2}{45}\,.$$

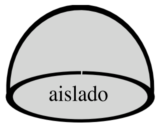
Los demás $a_n=0$ pues $\int_{-1}^{1}=0$ si n impar, y para desarrollar un Q_k bastan los k primeros P_n.

Pero para esta $f(\theta)$ mejor tanteamos: $1-\cos^2\theta=-\frac{2}{3}\underbrace{(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2})}_{P_2(\cos\theta)}+\frac{2}{3}\cdot\underbrace{1}_{P_0(\cos\theta)} \;\to\; a_0=\frac{2}{3}\,,\ 15a_2=-\frac{2}{3}\,.$

Por tanto, $u=\frac{2}{3}-\frac{2}{45}r^2(\frac{3}{2}\cos^2\theta-\frac{1}{2})=\frac{2}{3}+\frac{1}{45}r^2-\frac{1}{15}r^2\cos^2\theta \;=\frac{2}{3}+\frac{1}{45}[x^2+y^2-2z^2]\,.$

12 $\boxed{\begin{cases}u_{rr}+\frac{2u_r}{r}+\frac{u_{\theta\theta}}{r^2}+\frac{\cos\theta\, u_\theta}{r^2\operatorname{sen}\theta}=0\,,\ r<1,\ 0<\theta<\frac{\pi}{2}\\ u_r(1,\theta)=f(\theta)\,,\ u_\theta(r,\frac{\pi}{2})=0\end{cases}}$ Como en apuntes hasta nuevo problema de contorno:

$$\begin{cases}[1-t^2]\,\Theta''-2t\,\Theta'+\lambda\Theta=0\\ \Theta'(0)=0\,,\ \Theta \text{ acotada en } 1\end{cases}$$



$\to\ \lambda_n=2n(2n+1)\,,\ \Theta_n=\{P_{2n}(\cos\theta)\}$ [Legendre pares] , $u=a_0+\sum_{n=1}^{\infty}a_{2n}r^{2n}P_{2n}(\cos\theta)\to$

$\sum_{n=1}^{\infty}2na_{2n}P_{2n}(\cos\theta)=f(\theta)\to\ a_0$ indet. y $a_{2n}=\frac{4n+1}{2n}\int_0^{\pi/2}P_{2n}(\cos\theta)f(\theta)\operatorname{sen}\theta\,d\theta\ \left[\,2\int_0^1 P_{2n}^2=\frac{2}{4n+1}\right]$,

siempre que el primer término del desarrollo de f sea 0: $\int_0^{\pi/2}f(\theta)\operatorname{sen}\theta\,d\theta=0$.

$$\text{Si } f(\theta)=\cos^2\theta-a\,,\ \int_0^1(t^2-a)\,dt=0\ \to\ a=\tfrac{1}{3}\,.$$

$$\cos^2\theta-\tfrac{1}{3}=2a_2\left(\frac{3\cos^2\theta-1}{2}\right)+4a_4P_4(\cos\theta)+\cdots\to\ u=C+\frac{r^2}{2}\cos^2\theta-\frac{r^2}{6}\,.$$

13 Problemas exteriores resueltos en los apuntes.

Plano

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^{-n}[a_n\cos n\theta+b_n\operatorname{sen}n\theta]$$

$$a_n=\frac{R^n}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta\,,\ n=0,1,\dots$$

$$b_n=\frac{R^n}{\pi}\int_0^{2\pi}f(\theta)\operatorname{sen}n\theta\,d\theta\,,\ n=1,2,\dots$$

$\boxed{f(\theta)=\cos^3\theta}$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\frac{a_n}{R^n}\cos n\theta+\frac{b_n}{R^n}\operatorname{sen}n\theta\right]=\frac{\cos 3\theta}{4}+\frac{3\cos\theta}{4}$$

$$\to\ \boxed{u=\frac{3R}{4r}\cos\theta+\frac{R^3}{4r^3}\cos 3\theta}$$

Espacio

$$u(r,\theta)=\frac{a_0}{r}+\sum_{n=1}^{\infty}a_n r^{-(n+1)}P_n(\cos\theta)$$

$$a_n=\frac{(2n+1)R^{n+1}}{2}\int_0^{\pi}f(\theta)P_n(\cos\theta)\operatorname{sen}\theta\,d\theta$$

$$a_n=\frac{(2n+1)R^{n+1}}{2}\int_{-1}^{1}t^3P_n(t)\,dt$$

$$a_1=3R^2\int_0^1 t^4dt=\frac{3R^2}{5}$$

$$a_3=7R^4\int_0^1\left(\frac{5t^6}{2}-\frac{3t^4}{2}\right)dt=\frac{2R^4}{5}$$

$$\to\ \boxed{u=\frac{3R^2}{5r^2}\cos\theta+\frac{2R^4}{5r^4}\left(\tfrac{5}{2}\cos^3\theta-\tfrac{3}{2}\cos\theta\right)}$$

[O tanteando, como en el plano].

15 En los apuntes los tenemos escritos en esféricas:

$\boxed{Y_0^0}=\{P_0\}=\boxed{\{1\}}$.

$\boxed{rY_1^0}=\{rP_1\}=\{r\cos\theta\}=\boxed{\{z\}}$.

$\boxed{rY_1^1}=\{rP_1^1\cos\phi,rP_1^1\operatorname{sen}\phi\}=\{r\operatorname{sen}\theta\cos\phi,r\operatorname{sen}\theta\operatorname{sen}\phi\}=\boxed{\{x\,,\,y\}}$.

$\boxed{r^2Y_2^0}=\{r^2P_2\}=\{\tfrac{1}{2}[3r^2\cos^2\theta-r^2]\}=\boxed{\{z^2-\tfrac{1}{2}(x^2+y^2)\}}$.

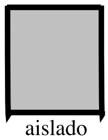
$\boxed{r^2Y_2^1}=\{3r^2\operatorname{sen}\theta\cos\theta\cos\phi,3r^2\operatorname{sen}\theta\cos\theta\operatorname{sen}\phi\}=\boxed{\{3xz\,,\,3yz\}}$.

$\boxed{r^2Y_2^2}==\{3r^2\operatorname{sen}^2\theta\cos 2\phi,3r^2\operatorname{sen}^2\theta\operatorname{sen}2\phi\}=\boxed{\{3[x^2-y^2]\,,\,6xy\}}$.

14

$$\begin{cases} u_t-\Delta u=0,\ (x,y)\in(0,\pi)\times(0,\pi),\ t>0 \\ u(x,y,0)=1+\cos x\cos 2y \\ u_x(0,y,t)=u_x(\pi,y,t)=0 \\ u_y(x,0,t)=u_y(x,\pi,t)=0 \end{cases}$$

$$u=XYT\to\begin{cases} \left.\begin{array}{l} X''+\lambda X=0 \\ X'(0)=X'(\pi)=0 \end{array}\right\} X_n=\{\cos nx\} \\ \left.\begin{array}{l} Y''+\mu Y=0 \\ Y'(0)=Y'(\pi)=0 \end{array}\right\} Y_n=\{\cos my\} \\ T'+(\lambda+\mu)T=0,\ T_{nm}=\left\{e^{-(n^2+m^2)t}\right\},\ n,m=0,1,\dots \end{cases}$$



aislado

$$u=\frac{a_{00}}{4}+\sum_{n=1}^{\infty}\frac{a_{n0}}{2}e^{-n^2t}\cos nx+\sum_{m=1}^{\infty}\frac{a_{0m}}{2}e^{-m^2t}\cos my+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{nm}e^{-(n^2+m^2)t}\cos nx\cos my\Big|_{t=0}=1+\cos x\cos 2y$$

$$\boxed{u(x,y,t)=1+e^{-5t}\cos x\cos 2y}\ \overset{t\to\infty}{\to}\ 1\ ,$$ valor medio de las temperaturas iniciales.

$$\begin{cases} u_{tt}-\Delta u=0\ ,\ (x,y)\in(0,\pi)\times(0,\pi),\ t\in\mathbf{R} \\ u(x,y,0)=0\ ,\ u_t(x,y,0)=\operatorname{sen}3x\ \operatorname{sen}^2 2y \\ u(0,y,t)=u(\pi,y,t)=0 \\ u_y(x,0,t)=u_y(x,\pi,t)=0 \end{cases}$$

$$u=XYT\to\begin{cases} \left.\begin{array}{l} X''+\lambda X=0 \\ X(0)=X(\pi)=0 \end{array}\right\} X_n=\{\operatorname{sen}nx\},\ n=1,2,\cdots \\ \left.\begin{array}{l} Y''+\mu Y=0 \\ Y'(0)=Y'(\pi)=0 \end{array}\right\} Y_n=\{\cos my\},\ m=1,2,\cdots \\ T''+(\lambda+\mu)T=0,\ T(0)=0,\ T_{nm}=\{\operatorname{sen}\sqrt{n^2+m^2}\,t\} \end{cases}$$

$$u=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\operatorname{sen}\sqrt{n^2+m^2}\,t\ \operatorname{sen}nx\ \cos my\ ,\ u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\sqrt{n^2+m^2}\ \operatorname{sen}nx\cos my=\frac{1-\cos 4y}{2}\operatorname{sen}3x$$

$$\to\ \begin{array}{l} c_{30}\sqrt{9}=\frac{1}{2} \\ c_{34}\sqrt{25}=-\frac{1}{2} \end{array}\quad [\text{los demás } c_{nm}=0]\ \to\ \boxed{u(x,y,t)=\frac{1}{6}\operatorname{sen}3t\operatorname{sen}3x-\frac{1}{10}\operatorname{sen}5t\operatorname{sen}3x\cos 4y}$$

$$\begin{cases} \Delta u=z\ ,\ x^2+y^2+z^2<1 \\ u=z^3\ \text{ si }\ x^2+y^2+z^2=1 \end{cases}\quad (\mathrm{P}_1)\begin{cases} \Delta u=0\ ,\ r<1 \\ u(1,\theta)=\cos^3\theta \end{cases}\ \overset{\text{apuntes}}{\longrightarrow}\ u_1=\sum_{n=0}^{\infty}a_nr^nP_n(\cos\theta)\ \to$$

$$u_1(1,\theta)=\frac{2}{5}\left[\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right]+\frac{3}{5}\cos\theta\ \to\ u_1=\frac{3r}{5}\cos\theta+\frac{r^3}{5}\left[5\cos^3\theta-3\cos\theta\right]=\frac{3z+2z^3-3x^2z-3y^2z}{5}\ .$$

$$\left[\ \operatorname{sen}^2\theta P_n''-2\cos\theta P_n'=n(n+1)P_n\ \right]$$

$$(\mathrm{P}_2)\begin{cases} u_{rr}+\frac{2u_r}{r}+\frac{u_{\theta\theta}}{r^2}+\frac{\cos\theta\, u_\theta}{r^2\operatorname{sen}\theta}=r\cos\theta \\ u(1,\theta)=0 \end{cases}\ \to\ u_2=\sum_{n=0}^{\infty}a_n(r)P_n(\cos\theta)\to\sum_{n=0}^{\infty}\left[a_n''+\frac{2a_n'}{r}-\frac{n(n+1)a_n}{r^2}\right]P_n(\cos\theta)=r\cos\theta$$

$$\to\ r^2a_1''+2ra_1'-2a_1=r^3\ \to\ a_1=c_1r+\frac{c_2}{r^2}+\frac{r^3}{10}\ \underset{a_1(1)=0}{\overset{a_1\text{ acot.}}{\longrightarrow}}\ a_1=\frac{r^3-r}{10}\ \to\ u_2=\frac{r^3-r}{10}\cos\theta=\frac{z^3+x^2z+y^2z-z}{10}\ .$$

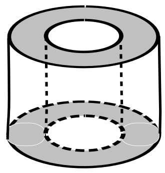
$$\boxed{u=u_1+u_2=\frac{1}{2}r\cos\theta\left[1-r^2+2r^2\cos^2\theta\right]=\frac{1}{2}z\left[1-x^2-y^2+z^2\right]}$$

$$\begin{cases} \Delta u=0\ ,\ x^2+y^2+z^2<1 \\ u=x^3\ \text{ si }\ x^2+y^2+z^2=1 \end{cases}\quad \begin{cases} \Delta u=0\ ,\ r<1 \\ u|_{r=1}=\operatorname{sen}^3\theta\cos^3\phi \end{cases}\quad P_3=\frac{5}{2}t^3-\frac{3}{2}t\ ,\quad \begin{array}{l} P_3'=\frac{3}{2}[5t^2-1]\ ,\ P_3^1=\frac{3}{2}\operatorname{sen}\theta[5\cos^2\theta-1] \\ P_3'''=15\ ,\ P_3^3=15\operatorname{sen}^3\theta \end{array}$$

$$\frac{1}{4}\operatorname{sen}^3\theta\ \cos 3\phi+\frac{3}{4}\operatorname{sen}^3\theta\cos\phi=\frac{1}{4}\operatorname{sen}^3\theta\cos 3\phi-\frac{3}{20}\operatorname{sen}\theta[5\cos^2\theta-1]\cos\phi+\frac{3}{5}\operatorname{sen}\theta\cos\phi\ \to$$

$$\boxed{u=\frac{r^3}{4}\operatorname{sen}^3\theta\cos 3\phi-\frac{3r^3}{20}\operatorname{sen}\theta\left[5\cos^2\theta-1\right]\cos\phi+\frac{3r}{5}\operatorname{sen}\theta\cos\phi=\frac{1}{5}x\left[3+2x^2-3y^2-3z^2\right]}$$

$$\begin{cases} u_t-\Delta u=0,\ 1<r<2,\ 0<z<1,\ t>0 \\ u(r,\theta,0)=\operatorname{sen}\pi z \\ u(1,z,t)=u(2,z,t)=0 \\ u(r,0,t)=u(r,1,t)=0 \end{cases}\quad u=RZT\to\begin{cases} \left.\begin{array}{l} Z''+\mu Z=0 \\ Z(0)=Z(1)=0 \end{array}\right\} Z_n=\{\operatorname{sen}n\pi z\} \\ \left.\begin{array}{l} rR''+R''+\lambda rR=0 \\ R(1)=R(2)=0 \end{array}\right\} \\ T'+(\lambda+\mu)T=0 \end{cases}$$



Haciendo $t=r\sqrt{\lambda}$ en la ecuación de R se transforma en la de Bessel de orden cero: $(tR')'+\lambda tR=0$.

$$\to\ R=c_1J_0(t)+c_2K_0(t)=c_1J_0(r\sqrt{\lambda})+c_2K_0(r\sqrt{\lambda})\ \overset{c.c.}{\to}\ \begin{cases} c_1J_0(\sqrt{\lambda})+c_2K_0(\sqrt{\lambda})=0 \\ c_1J_0(2\sqrt{\lambda})+c_2K_0(2\sqrt{\lambda})=0 \end{cases}\ \to$$

$\mu_m=c_m^2$, donde c_m son las infinitas raíces de $J_0(c_m)K_0(2c_m)-J_0(2c_m)K_0(c_m)$ [Problema S-L regular $\Rightarrow$ existen].

Las autofunciones correspondientes son: $R_m(r)=\{K_0(c_m)J_0(c_mr)-J_0(c_m)K_0(c_mr)\}$.

$$T_{nm}=\left\{e^{-(n^2\pi^2+\lambda_m)t}\right\}\ \to\ u(r,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{nm}\,e^{-(n^2\pi^2+\lambda_m)t}R_m(r)\operatorname{sen}n\pi z\ \to$$

$$u(r,z,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{nm}R_m(r)\operatorname{sen}n\pi z=\operatorname{sen}\pi x\ \Leftrightarrow\ \sum_{m=1}^{\infty}c_{1m}R_m(r)=1\ \to\ c_{1m}=\frac{\langle R_m,1\rangle}{\langle R_m,R_m\rangle}=\boxed{\frac{\int_1^2 rR_m\,dr}{\int_1^2 rR_m^2\,dr}}$$

$$\boxed{u(r,z,t)=\operatorname{sen}\pi z\sum_{m=1}^{\infty}c_{1m}\,e^{-(\pi^2+c_m^2)t}\left[K_0(c_m)J_0(c_mr)-J_0(c_m)K_0(c_mr)\right]}$$

soluciones de problemas 4 de EDII(r) (2011)

1 a] $\left\{\begin{array}{l} u_{tt}-4u_{xx}=e^{-t}\,,\ x,t\in\mathbf{R} \\ u(x,0)=x^2,\ u_t(x,0)=-1 \end{array}\right.$ $\quad u=\frac{1}{2}\big[(x+2t)^2+(x-2t)^2\big]+\frac{1}{4}\int_{x-2t}^{x+2t}ds+\frac{1}{4}\int_0^t\!\!\int_{x-2[t-\tau]}^{x+2[t-\tau]}e^{-\tau}\,ds\,d\tau$

$\qquad = x^2+4t^2+e^{-t}-1\ .$

Una solución particular que sólo depende de t es: $v_{tt}=e^{-t}\to v=e^{-t}$. Con $w=u-e^{-t}$ se tiene:

$\left\{\begin{array}{l} w_{tt}-w_{xx}=0 \\ w(x,0)=x^2-1,\ w_t(x,0)=0 \end{array}\right. \to w=\frac{1}{2}\big[(x+2t)^2-1+(x-2t)^2-1\big]=x^2+4t^2-1$, como antes.

b] $\left\{\begin{array}{l} u_{tt}-4u_{xx}=16\,,\ x,t\in\mathbf{R} \\ u(0,t)=t\,,\ u_x(0,t)=0 \end{array}\right.$ Lo más sencillo es cambiar papeles de x y t aplicar D'Alembert: $\xrightarrow{x\leftrightarrow t} \left\{\begin{array}{l} u_{tt}-\frac{1}{4}u_{xx}=-4\,,\ x,t\in\mathbf{R} \\ u(x,0)=x\,,\ u_t(x,0)=0 \end{array}\right. \to$

$u=\frac{1}{2}\big[(x+\frac{t}{2})+(x-\frac{t}{2})\big]-4\int_0^t\!\!\int_{x-\frac{1}{2}(t-\tau)}^{x+\frac{1}{2}(t-\tau)}ds\,d\tau=x-4\int_0^t(t-\tau)d\tau=x-2t^2 \xrightarrow{x\leftrightarrow t} \boxed{u=t-2x^2}$

Podríamos ahorrarnos esta integral doble con una solución v que sólo dependiese de una variable:

$v''(t)=-4\,,\ v=-2t^2 \underset{w=u-v}{\to} \left\{\begin{array}{l} w_{tt}-\frac{1}{4}w_{xx}=0 \\ w(x,0)=x\,,\ w_t(x,0)=0 \end{array}\right.\ ,\ w=\frac{1}{2}\big[(x+\frac{t}{2})+(x-\frac{t}{2})\big]=x\to u=x-2t^2$

$v''(x)=16\,,\ v=8x^2 \underset{w=u-v}{\to} \left\{\begin{array}{l} w_{tt}-\frac{1}{4}w_{xx}=0 \\ w(x,0)=x-8x^2,\ w_t(x,0)=0 \end{array}\right.\ ,\ w=x-4\big[(x+\frac{t}{2})^2+(x-\frac{t}{2})^2\big]=x-8x^2-2t^2\ldots$

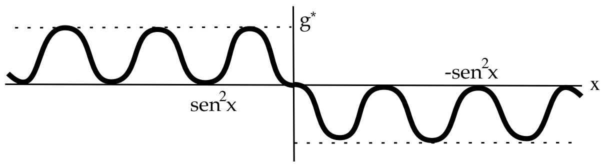
Sin atajos: $\left\{\begin{array}{l} \xi=x+2t \\ \eta=x-2t \end{array}\right. \underset{\text{forma canónica}}{\to u_{\xi\eta}=-1\to}\ u=p(\xi)+q(\eta)-\xi\eta=p(x+2t)+q(x-2t)+4t^2-x^2$

$\left\{\begin{array}{l} u(0,t)=p(2t)+q(-2t)+4t^2=t\to 2p'(2t)-2q'(-2t)=1-8t \\ u_x(0,t)=p'(2t)+q'(-2t)=0\ \to q'(-2t)=-p'(2t) \end{array}\right. \to p'(2t)=\frac{1}{4}-2t\,,\ p'(v)=\frac{1}{4}-v\,,\ p(v)=\frac{v}{4}-\frac{v^2}{2}+K$

$\to q(-v)=\frac{v}{2}-v^2-p(v)=\frac{v}{4}-\frac{v^2}{2}-K\,,\ q(v)=-\frac{v}{4}-\frac{v^2}{2}-K\to u=\frac{x+2t}{4}-\frac{x-2t}{4}-\frac{(x+2t)^2}{2}-\frac{(x-2t)^2}{2}+4t^2-x^2\uparrow$

2 $\left\{\begin{array}{l} u_{tt}-u_{xx}=0\,,\ x\geq 0,\ t\in\mathbf{R} \\ u(x,0)=0\,,\ u_t(x,0)=\cos^2 x \\ u(0,t)=t \end{array}\right.$ Una v evidente que cumple la condición de contorno no homogénea es $v=t$. Haciendo $w=u-v$:

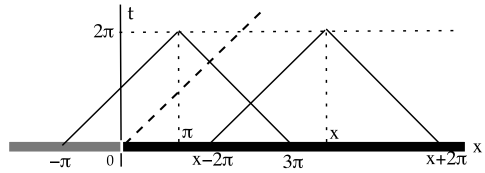
$\left\{\begin{array}{l} w_{tt}-w_{xx}=0\,,\ x\geq 0 \\ w_t(x,0)=\cos^2 x-1 \\ w(x,0)=0\,,\ w(0,t)=0 \end{array}\right. \to \left\{\begin{array}{l} w_{tt}-w_{xx}=0\,,\ x\in\mathbf{R} \\ w(x,0)=0 \\ w_t(x,0)=g^*(x) \end{array}\right.$,



siendo $g^*(x)$ la extensión impar respecto a $x=0$ de

$g(x)=\cos^2 x-1=-\sin^2 x=\frac{1}{2}(\cos 2x-1)$.

La solución del problema inicial es $u=t+\frac{1}{2}\int_{x-t}^{x+t}g^*(s)\,ds$.



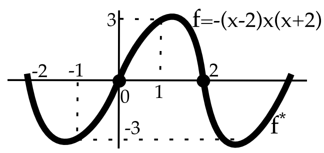
a] $u(\pi,2\pi)=2\pi+\frac{1}{2}\int_{-\pi}^{3\pi}g^* \underset{g^*\text{ impar}}{=} 2\pi+\frac{1}{4}\int_{\pi}^{3\pi}(\cos 2s-1)ds=\boxed{\frac{3\pi}{2}}$.

b] Para hallar $u(x,2\pi)$ si $x\geq 2\pi$ sólo necesitamos la expresión de la g inicial:

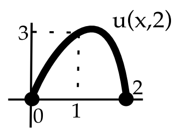
$w(x,2\pi)=\frac{1}{2}\int_{x-2\pi}^{x+2\pi}g^*=\frac{1}{4}\int_{x-2\pi}^{x+2\pi}(\cos 2s-1)ds=\frac{1}{8}\big[\,\mathrm{sen}\,2s\big]_{x-2\pi}^{x+2\pi}-\pi=-\pi\to \boxed{u(x,2\pi)=\pi}\ ,\ x\geq 2\pi\,.$

3 $\left\{\begin{array}{l} u_{tt}-4u_{xx}=0\,,\ x\in[0,2],t\in\mathbf{R} \\ u(x,0)=4x-x^3,\ u_t(x,0)=0 \\ u(0,t)=u(2,t)=0 \end{array}\right.$ $\quad \left\{\begin{array}{l} u_{tt}-u_{xx}=0\,,x\in\mathbf{R} \\ u(x,0)=f^*(x),\ u_t(x,0)=0 \end{array}\right.$

$u=\frac{1}{2}\big[f^*(x+2t)+f^*(x-2t)\big]$.



$u(\frac{3}{2},\frac{3}{4})=\frac{1}{2}\big[f^*(3)+f^*(0)\big] \underset{\text{4-per.}}{=} \frac{1}{2}f^*(-1) \underset{\text{impar}}{=} -\frac{1}{2}f^*(1)=-\frac{3}{2}$.

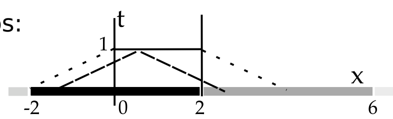


$u(x,2)=\frac{1}{2}\big[f^*(x+4)+f^*(x-4)\big] \underset{\text{4-per.}}{=} \frac{1}{2}\big[f^*(x)+f^*(x)\big]=f(x)=u(x,0)$.

[Sabíamos que era $\frac{2L}{c}$-periódica. Trasladando y sumando sale lo mismo].

Para hallar $u(x,1)$ necesitamos la expresión de f^* en más intervalos:

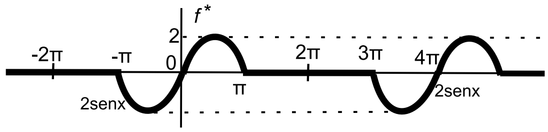
$\left.\begin{array}{l} f^*(x)=-(x-2)x(x+2)=f(x)\ \text{si}\ x\in[-2,2] \\ f^*(x)=-(x-6)(x-4)(x-2)\ \text{si}\ x\in[2,6] \end{array}\right. \to$



$u(x,1)=\frac{1}{2}\big[f^*(x+2)+f(x-2)\big]=\frac{1}{2}\big[-(x-4)(x-2)x-(x-4)(x-2)x\big]=-(x-4)(x-2)x$.

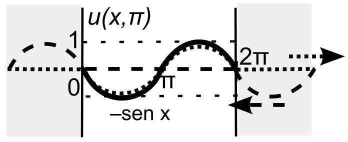
4 $\begin{cases} u_{tt}-u_{xx}=0\,,\ x\in[0,2\pi]\,,\ t\in\mathbf{R} \\ u(x,0)=\begin{cases} 2\,\mathrm{sen}\,x\,,\ x\in[0,\pi] \\ 0\ ,\ x\in[\pi,2\pi] \end{cases},\ u_t(x,0)=0 \\ u(0,t)=u(2\pi,t)=0 \end{cases}$

La solución es $u(x,t)=\frac{1}{2}\big[f^*(x+t)+f^*(x-t)\big]$, con f^* extensión impar y 4π-periódica de la f inicial.



Para dibujar $u(x,\pi)$ basta trasladar $\frac{1}{2}f(x)$ a la izquierda y a la derecha π unidades y sumar las gráficas en $[0,2\pi]$.

[Sólo queda lo que va a la derecha con la mitad de altura].



Como para $x\in[0,2\pi]$ siempre $x+\pi\in[\pi,3\pi]$ y $x-\pi\in[-\pi,\pi]$ y en todo este intervalo es $f^*(x)=2\,\mathrm{sen}\,x$ ($\mathrm{sen}\,x$ impar), es:

$$u(x,\pi)=\tfrac{1}{2}\big[f^*(x+\pi)+f^*(x-\pi)\big]=\tfrac{1}{2}\big[0+2\,\mathrm{sen}(x-\pi)\big]=\boxed{-\,\mathrm{sen}\,x}\quad \forall x\in[0,2\pi]\,.$$

$$u=XT\ \to \begin{cases} X''+\lambda X=0 \\ X(0)=X(2\pi)=0 \end{cases} \to\ \lambda_n=\tfrac{n^2}{4}\,,\ X_n=\{\mathrm{sen}\,\tfrac{nx}{2}\}\,,\ n=1,2,\dots\ \ \mathrm{y}\ \begin{cases} T'+\lambda T=0 \\ T'(0)=0 \end{cases} \to\ T_n=\{\cos\tfrac{nt}{2}\}\,.$$

$$u(x,t)=\sum_{n=1}^{\infty}c_n\cos\tfrac{nt}{2}\,\mathrm{sen}\,\tfrac{nx}{2}\ \to\ u(x,0)=\sum_{n=1}^{\infty}c_n\,\mathrm{sen}\,\tfrac{nx}{2}=\begin{cases} 2\,\mathrm{sen}\,x\,,\ x\in[0,\pi] \\ 0\ ,\ x\in[\pi,2\pi] \end{cases}\ \to$$

$$c_n=\tfrac{2}{2\pi}\int_0^{\pi}2\,\mathrm{sen}\,x\,\mathrm{sen}\,\tfrac{nx}{2}\,dx=\tfrac{1}{\pi}\int_0^{\pi}\Big[\cos\big(\tfrac{n}{2}-1\big)x-\cos\big(\tfrac{n}{2}+1\big)x\Big]dx\underset{n\neq 2}{=}\tfrac{1}{\pi}\Big[\frac{\mathrm{sen}(\frac{n}{2}-1)\pi}{\frac{n}{2}-1}-\frac{\mathrm{sen}(\frac{n}{2}+1)\pi}{\frac{n}{2}+1}\Big]$$

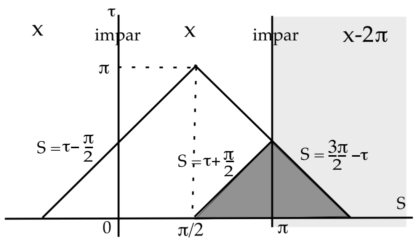
$$=\tfrac{2}{\pi}\Big[-\frac{\mathrm{sen}\,\frac{n\pi}{2}}{n-2}+\frac{\mathrm{sen}\,\frac{n\pi}{2}}{n+2}\Big]=-\frac{8\,\mathrm{sen}\,\frac{n\pi}{2}}{\pi(n^2-4)}=\begin{cases} 0\,,\ n=2m \\ \frac{8}{\pi}\frac{(-1)^m}{(2m-1)^2-4}\,,\ n=2m-1 \end{cases}.\ \text{Además } c_2=\tfrac{1}{\pi}\int_0^{\pi}[1-\cos 2x]dx=1\,.$$

$$\boxed{u(x,t)=\cos t\ \mathrm{sen}\,x+\tfrac{8}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2-4}\cos\frac{(2m-1)t}{2}\,\mathrm{sen}\,\frac{(2m-1)x}{2}}\ \to\ u(x,\pi)=-\,\mathrm{sen}\,x\,,\ \text{pues}\ \begin{matrix}\cos\pi=-1\\ \cos\frac{(2m-1)\pi}{2}=0\end{matrix}$$

5 $\begin{cases} u_{tt}-u_{xx}=x\,,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u(x,0)=x,\ u_t(x,0)=0 \\ u(0,t)=0,\ u(\pi,t)=\pi \end{cases}$

Hay que hacer las condiciones de contorno homogéneas. Usemos primero la v de la página 60 de los apuntes: $v=x$. Haciendo $u=v+w$ se obtiene:

$$\begin{cases} w_{tt}-w_{xx}=x\,,\ x\in[0,\pi] \\ w(x,0)=w_t(x,0)=0 \\ w(0,t)=w(\pi,t)=0 \end{cases} \to \begin{cases} w_{tt}-w_{xx}=F^*(x)\,,\ x\in\mathbf{R} \\ w(x,0)=0 \\ w_t(x,0)=0 \end{cases},$$



con F^* impar y 2π-periódica. $\displaystyle w(x,t)=\frac{1}{2}\int_0^t\int_{x-[t-\tau]}^{x+[t-\tau]}F^*(s,\tau)\,ds\,d\tau$

Para hallar $w(\frac{\pi}{2},\pi)$, en principio, hay que hacer 3 integrales dobles, pero como (es F^* impar respecto a $x=\pi$) la integral de F^* sobre el triángulo oscuro se anula, basta:

$$w(\tfrac{\pi}{2},\pi)=\tfrac{1}{2}\int_0^{\pi/2}\int_{\tau-\frac{\pi}{2}}^{\tau+\frac{\pi}{2}}s\,ds\,d\tau+\tfrac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\int_{\tau-\frac{\pi}{2}}^{\frac{3\pi}{2}-\tau}s\,ds\,d\tau=\tfrac{\pi^3}{8}\ \to\ \boxed{u(\tfrac{\pi}{2},\pi)=\tfrac{\pi}{2}+\tfrac{\pi^3}{8}}$$

Pero mejor se busca una $v(x)$ que cumpla la ecuación y las condiciones de contorno:

$$v=c_1+c_2x-\tfrac{x^3}{6}\ \overset{c.c.}{\to}\ v=(1+\tfrac{\pi^2}{6})x-\tfrac{x^3}{6}\ \overset{u=v+w}{\longrightarrow}\ \begin{cases} w_{tt}-w_{xx}=0\,,\ w(x,0)=\frac{x^3-\pi^2x}{6} \\ w_t(x,0)=w(0,t)=w(\pi,t)=0 \end{cases}$$

Extendiendo la última f podemos aplicar D'Alembert (mucho más corto que antes):

$$w(\tfrac{\pi}{2},\pi)=\tfrac{1}{2}\Big[f^*(\tfrac{3\pi}{2})+f^*(-\tfrac{\pi}{2})\Big]\underset{\uparrow}{=}-f(\tfrac{\pi}{2})=\tfrac{\pi^3}{16}\,,\ v(\tfrac{\pi}{2})=\tfrac{\pi}{2}+\tfrac{\pi^3}{16}\ \dots$$

impar y 2π-periódica

Por separación de variables es necesario, también, que las condiciones de contorno sean homogéneas. De los dos problemas para w de arriba es más sencillo el segundo cuya solución, según 3.1, es:

$$w=\sum_{n=1}^{\infty}c_n\cos nt\,\mathrm{sen}\,nx\,,\ c_n=\tfrac{2}{\pi}\int_0^{\pi}\tfrac{x^3-\pi^2x}{6}\,\mathrm{sen}\,nx=\tfrac{2(-1)^n}{n^3}\ \to\ w(\tfrac{\pi}{2},\pi)=2\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+1)^3}=\tfrac{\pi^3}{32}\ \text{[ver apuntes].}$$

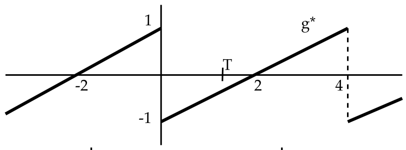
Para resolver el otro problema (no homogéneo) probamos una serie de autofunciones:

$$w=\sum_{n=1}^{\infty}T_n(t)\,\mathrm{sen}\,nx\ \to\ T_n''+n^2T_n=b_n=\tfrac{2}{\pi}\int_0^{\pi}x\,\mathrm{sen}\,nx\,dx=\tfrac{2(-1)^{n+1}}{n}\ \to\ T_n=c_1\cos nt+c_2\,\mathrm{sen}\,nt+\tfrac{b_n}{n^2}$$

$$\underset{T_n(0)=T_n'(0)=0}{\longrightarrow}\ T_n=\tfrac{b_n}{n^3}[1-\cos nt]\,,\ w=\sum_{n=1}^{\infty}\tfrac{2(-1)^{n+1}}{n^3}[1-\cos nt]\,\mathrm{sen}\,nx\,\Big|_{(\frac{\pi}{2},\pi)}=\sum_{k=0}^{\infty}\tfrac{4(-1)^k}{(2k+1)^3}=\tfrac{\pi^3}{8}\,.$$

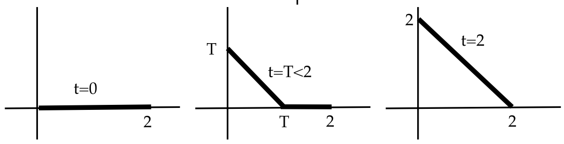
6 $\begin{cases} u_{tt}-u_{xx}=0\,,\ x\in[0,2],\ t\in\mathbf{R}\\ u(x,0)=u_t(x,0)=0\\ u(0,t)=t\,,\ u(2,t)=0\end{cases}$ a] $v=t(1-\frac{x}{2}) \overset{u=w+v}{\to} \begin{cases} w_{tt}-w_{xx}=0\,,\ x\in[0,2]\\ w(x,0)=0,\ w_t(x,0)=\frac{x}{2}-1\\ w(0,t)=w(2,t)=0\end{cases} \to \begin{cases} w_{tt}-w_{xx}=0\,,\ x\in\mathbf{R}\\ w(x,0)=0\\ w_t(x,0)=g^*(x)\end{cases}$

$[x-T,x+T]$ no contiene valores negativos a partir de $x=T$. Por tanto:



$$w(x,T)=\begin{cases} \frac{1}{2}\int_{x-T}^{0}(\frac{s}{2}+1)ds+\frac{1}{2}\int_0^{x+T}(\frac{s}{2}-1)ds=x(\frac{T}{2}-1),\ x\in[0,T]\\ \frac{1}{2}\int_{x-T}^{x+T}(\frac{s}{2}-1)ds=T(\frac{x}{2}-1),\ x\in[T,2]\end{cases}$$

$$\to\ u(x,T)=\begin{cases} T-x\,,\ x\in[0,T]\\ 0\,,\ x\in[T,2]\end{cases}$$

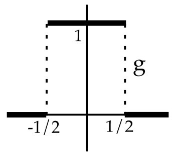


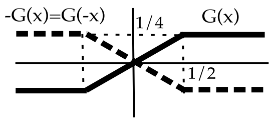
La perturbación viaja a velocidad 1. La cuerda debía estar en reposo para $x\geq T$ en el instante T.

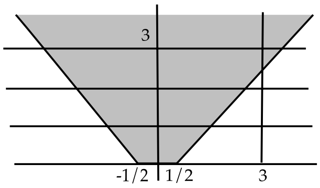
b] $u(x,2k)=w(x,2k)+v(x,2k)=\boxed{k(2-x)}$, pues $w(x,2k)=\frac{1}{2}\int_{x-2k}^{x+2k}g^*=0$, por ser g^* impar y 4-periódica, o porque $w(x,2k)=\sum c_n \operatorname{sen} nk\pi \operatorname{sen}\frac{n\pi x}{2}=0$.

7 $\begin{cases} u_{tt}-u_{xx}=0\,,\ x,t\in\mathbf{R}\\ u(x,0)=0\\ u_t(x,0)=\begin{cases}1,\ |x|\leq 1/2\\ 0\ \ |x|>1/2\end{cases}\end{cases}$

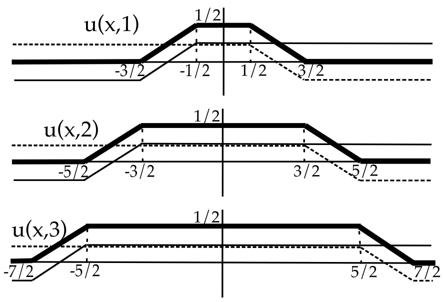
$u=G(\overleftarrow{x+t})-G(\overrightarrow{x-t})$, con $G(x)=\frac{1}{2}\int_0^x g$.



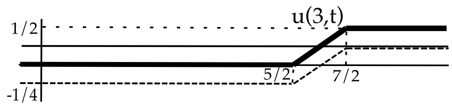




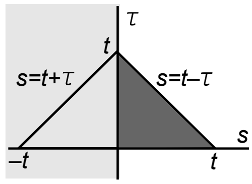
dominio de influencia



$$u(3,t)=G(3+t)-G(3-t)$$
$$=G(t+3)+G(t-3)$$



8 $\begin{cases} u_{tt}-u_{xx}=6x\,,\ x\geq 0,\ t\in\mathbf{R}\\ u(x,0)=u_t(x,0)=u_x(0,t)=0\end{cases}$ Hay que extender F par respecto a x ($-6x$ si $x\geq 0$).



$$u(0,t)=\frac{1}{2}\int_0^t\int_{-(t-\tau)}^{(t-\tau)}F^*\,ds\,d\tau=\int_0^t\int_0^{t-\tau}6s\,ds\,d\tau=t^3\,.$$

Otra posibilidad: $v=-x^3$ es solución y cumple el dato de contorno.

$$w=u+x^3\ \to \begin{cases} w_{tt}-w_{xx}=0\,,\ x\geq 0\\ w(x,0)=x^3\\ w_t(x,0)=w_x(0,t)=0\end{cases}\ ,\ \begin{cases} w_{tt}-w_{xx}=0\,,\ x\in\mathbf{R}\\ w(x,0)=\begin{cases}x^3\,,\ x\geq 0\\ -x^3\,,\ x\leq 0\end{cases}\\ w_t(x,0)=0\end{cases}\ \to\ w(0,t)=\frac{1}{2}\left[t^3+(-(-t)^3)\right]=t^3=u(0,t)\,.$$

9 $\begin{cases} u_{tt}-(u_{rr}+\frac{2}{r}u_r)=0\,,\ r\geq 0,\ t\in\mathbf{R}\\ u(r,0)=r\,,\ u_t(r,0)=-2\end{cases}$ $\overset{v=ru}{\longrightarrow} \begin{cases} v_{tt}-v_{rr}=0\,,\ r\geq 0\\ v(r,0)=r^2,\ v_t(r,0)=-2r\ \text{[impar]}\\ v(0,t)=0\end{cases} \to \begin{cases} v_{tt}-v_{rr}=0\,,\ r\in\mathbf{R}\\ v(r,0)=f^*(r)=\begin{cases}r^2\,,\ r\geq 0\\ -r^2\,,\ r\leq 0\end{cases}\\ v_t(r,0)=-2r\end{cases}$

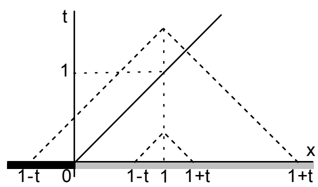
$$v(r,t)=\frac{1}{2}\left[f^*(r+t)+f^*(r-t)\right]-\int_{r-t}^{r+t}s\,ds\ \to\ u(r,t)=\frac{1}{2r}\left[f^*(r+t)+f^*(r-t)\right]-2t\,.$$

i] En particular, $u(1,2)=\frac{1}{2}\left[f^*(3)+f^*(-1)\right]-4\overset{\text{impar}}{=}\frac{1}{2}\left[f(3)-f(1)\right]-4=\boxed{0}$.

ii] Para $u(1,t)$ hay que distingir dos casos:

Si $t\leq 1$, $u(1,t)=\frac{1}{2}\left[(1+t)^2+(1-t)^2\right]-2t=1+t^2-2t=\boxed{(1-t)^2}$.

Si $t\geq 1$, $u(1,t)=\frac{1}{2}\left[(1+t)^2-(1-t)^2\right]-2t=2t-2t=\boxed{0}$.



10 $\begin{cases} u_{tt}-u_{rr}-\frac{2u_r}{r}=0,\ 1\leq r\leq 2,\ t\geq 0\\ u(r,0)=0,\ u_t(r,0)=\frac{1}{r}\operatorname{sen}\pi r\\ u(1,t)=u(2,t)=0\end{cases}$

i) $rR''+2R'+\lambda rR=0\,,\ R(1)=R(2)=0\to\lambda_n=n^2\pi^2,\ R_n=\{\frac{\operatorname{sen}n\pi r}{r}\}$

$T''+\lambda T=0\,,\ T(0)=0\to T_n=\{\operatorname{sen}n\pi t\}\,,\ u=\sum_{n=1}^{\infty}b_n\operatorname{sen}n\pi t\,\frac{\operatorname{sen}n\pi r}{r}$.

$$u_t(r,0)=\sum_{n=1}^{\infty}n\pi b_n\frac{\operatorname{sen}n\pi r}{r}=\frac{\operatorname{sen}\pi r}{r}\ \to\ u=\frac{\operatorname{sen}\pi t\operatorname{sen}\pi r}{\pi r}\ .$$

ii) $v=ur\to\begin{cases} v_{tt}-v_{rr}=0,\ 1\leq r\leq 2\\ v_t(r,0)=\operatorname{sen}n\pi r\equiv G(r)\\ v(r,0)=v(1,t)=v(2,t)=0\end{cases}\ \to\ u=\frac{1}{2r}\int_{r-t}^{r+t}G^*(s)\,ds$ $\quad G^*$ extensión impar de G respecto a 1 y 2.

Como $\operatorname{sen}\pi r$ es impar respecto a esos puntos, $G^*(r)=\operatorname{sen}\pi r$, $u=\frac{1}{2r}\int_{r-t}^{r+t}\operatorname{sen}\pi s\,ds=\frac{\operatorname{sen}\pi t\operatorname{sen}\pi r}{\pi r}$.

11 $I(a,x)=\int_0^\infty e^{-ak^2}\cos kx\,dk$ $\quad \frac{dI}{dx}\underset{\uparrow}{=}\int_0^\infty \frac{d}{dx}e^{-ak^2}\cos kx\,dk=-\int_0^\infty k e^{-ak^2}\operatorname{sen}kx\,dk\underset{(\bullet)}{=}\frac{e^{-ak^2}\operatorname{sen}kx}{2a}\Big]_0^\infty-\frac{x}{2a}I=-\frac{x}{2a}I.$

continuidad y convergencia

Además: $I(a,0)=\int_0^\infty e^{-ak^2}dk \underset{u=k\sqrt{a}}{=} \frac{1}{\sqrt{a}}\int_0^\infty e^{-u^2}du=\frac{\sqrt{\pi}}{2\sqrt{a}} \to I(a,x)=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-x^2/4a}=\frac{\sqrt{\pi}}{\sqrt{2}}\mathcal{F}_c^{-1}(e^{-ak^2})$.

$\mathcal{F}^{-1}(e^{-ak^2})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ak^2}(\cos kx+i\operatorname{sen}kx)\,dk=\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty e^{-ak^2}\cos kx\,dk=\frac{1}{\sqrt{2a}}e^{-x^2/4a}$ [$\mathcal{F}(e^{-ax^2})$ cambiando papeles]

$\mathcal{F}_s^{-1}(k\,e^{-ak^2})=\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty k e^{-ak^2}\operatorname{sen}kx\,dk \underset{\text{por}(\bullet)}{=} \frac{\sqrt{2}}{\sqrt{\pi}}\frac{x}{2a}\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-x^2/4a}=\frac{x}{[2a]^{3/2}}e^{-x^2/4a}$.

12 $\begin{cases} u_t-u_{xx}=(x^2-1)e^{-x^2/2},\ x\in\mathbf{R},\ t>0\\ u(x,0)=0,\ u \text{ acotada}\end{cases}$ a] Como $\mathcal{F}[f'']=-k^2\hat{f}$ y $\mathcal{F}[e^{-ax^2}]=\frac{1}{\sqrt{2a}}e^{-k^2/4a}\xrightarrow{a=1/2}e^{-k^2/2}$:

$\begin{cases}\hat{u}_t+k^2\hat{u}=-k^2e^{-k^2/2}\\ \hat{u}(k,0)=0\end{cases} \underset{x_p \text{ a ojo}}{\to} \hat{u}(k,t)=p(k)e^{-k^2t}-e^{-k^2/2} \underset{d.i.}{\to} \hat{u}(k,t)=e^{-k^2/2}e^{-k^2t}-e^{-k^2/2}$

$\to u(x,t)=\mathcal{F}^{-1}[e^{-k^2(t+\frac{1}{2})}]-e^{-x^2/2}=\boxed{\frac{1}{\sqrt{1+2t}}e^{-x^2/(4t+2)}-e^{-x^2/2}} \underset{t\to\infty}{\to}-e^{-x^2/2}$ $(\bullet)$

b] $v=-e^{-\frac{x^2}{2}}$ satisface la ecuación, $\xrightarrow{w=u-v}\begin{cases}w_t-w_{xx}=0\\ u(x,0)=e^{-x^2/2}\end{cases}\xrightarrow{\text{formulario}} w=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^\infty e^{-s^2/2}e^{-(x-s)^2/4t}ds$.

Para evaluar la integral completamos cuadrados buscando $\int_{-\infty}^\infty e^{-p^2}dp=\sqrt{\pi}$:

$$-\frac{(2t+1)s^2-2xs+x^2}{4t}=-\frac{\left[\sqrt{2t+1}\,s-\frac{x}{\sqrt{2t+1}}\right]^2}{(2\sqrt{t})^2}-\frac{x^2}{4t}+\frac{x^2}{4t(2t+1)}=-\left[\frac{\sqrt{2t+1}\,s}{2\sqrt{t}}-\frac{x}{2\sqrt{t}\sqrt{2t+1}}\right]^2-\frac{x^2}{4t+2}$$

Llamando p al último corchete, con lo que $dp=\frac{\sqrt{2t+1}}{2\sqrt{t}}ds$, tenemos:

$$w=\frac{1}{2\sqrt{\pi t}}\frac{2\sqrt{t}}{\sqrt{2t+1}}e^{-x^2/(4t+2)}\int_{-\infty}^\infty e^{-p^2}dp=\frac{1}{\sqrt{2t+1}}e^{-x^2/(4t+2)} \quad \uparrow u=v+w$$

$(\bullet)$ [Estamos todo el rato sacando calor en $[-1,1]$ y dándolo (menos cantidad según nos alejamos) fuera de ese intervalo. Las temperaturas acaban siendo negativas y menores cerca del origen].

13 $\begin{cases}u_t-u_{xx}=e^{-x^2/4},\ x\in\mathbf{R},\ t>0\\ u(x,0)=0,\ u\text{ acotada}\end{cases}$ Su solución: $u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty \frac{e^{-k^2}-e^{-k^2(t+1)}}{k^2}e^{-ikx}dk$.

Como $\mathcal{F}[e^{-ax^2}]=\frac{1}{\sqrt{2a}}e^{-k^2/4a}\xrightarrow{a=1/4}\begin{cases}\hat{u}_t+k^2\hat{u}=\sqrt{2}\,e^{-k^2}\\ \hat{u}(k,0)=0\end{cases}\xrightarrow{\hat{u}_p\text{ a ojo}}\hat{u}(k,t)=p(k)e^{-k^2t}+\frac{\sqrt{2}}{k^2}e^{-k^2}\xrightarrow{d.i.}$

$p(k)=-\frac{\sqrt{2}}{k^2}e^{-k^2}$, $\hat{u}(k,t)=\frac{\sqrt{2}}{k^2}[e^{-k^2}-e^{-k^2(t+1)}]$. Y de $u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{u}(k,t)e^{-ikx}dk$, sale lo de arriba.

$$\to u(0,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty \frac{e^{-k^2}-e^{-k^2(t+1)}}{k^2}dk=-\left[\frac{e^{-k^2}-e^{-k^2(t+1)}}{\sqrt{\pi}\,k}\right]_{-\infty}^\infty-\frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-k^2}dk+\frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty (t+1)e^{-k^2(t+1)}dk.$$

El primer corchete se anula en $\pm\infty$, y haciendo en la última integral el cambio $s=k\sqrt{t+1}$,

se convierte en $\frac{2}{\sqrt{\pi}}\sqrt{t+1}\int_{-\infty}^\infty e^{-s^2}ds=2\sqrt{t+1}$, concluimos que: $\boxed{u(0,t)=2\sqrt{t+1}-2}$.

[Es normal que tienda a ∞. Estamos constantemente metiendo calor en toda la varilla].

14 $\begin{cases}u_t-u_{xx}+u_x=0,\ x\in\mathbf{R},\ t>0\\ u(x,0)=e^{-x^2/2}\end{cases}$ $\begin{cases}\hat{u}_t=(ik-k^2)\hat{u}\\ \hat{u}(k,0)=e^{-k^2/2}\end{cases}\to \hat{u}=e^{ikt}e^{-\frac{k^2(1+2t)}{2}}\to u=\frac{1}{\sqrt{1+2t}}e^{-\frac{(x-t)^2}{2(1+2t)}}$.

15 $\begin{cases}u_{tt}-c^2u_{xx}=0,\ x,t\in\mathbf{R}\\ u(x,0)=f(x),\ u_t(x,0)=g(x)\end{cases}$ $\begin{cases}\hat{u}_{tt}+c^2k^2\hat{u}=0\\ \hat{u}(k,0)=\hat{f}(k)\\ \hat{u}_t(k,0)=\hat{g}(k)\end{cases}\to\hat{u}=p(k)e^{ickt}+q(k)e^{-ickt}\to \begin{matrix}p(k)+q(k)=\hat{f}(k)\\ ick[p(k)-q(k)]=\hat{g}(k)\end{matrix}$

$p(k)=\frac{1}{2}[\hat{f}(k)+\frac{\hat{g}(k)}{ick}]$, $q(k)=\frac{1}{2}[\hat{f}(k)-\frac{\hat{g}(k)}{ick}]\to \hat{u}=\frac{1}{2}\hat{f}(k)[e^{ickt}+e^{-ickt}]+\frac{1}{2}\hat{g}(k)[\frac{e^{ickt}-e^{-ickt}}{ick}]$

$\to u=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}g(x)*\sqrt{2\pi}\,h(x)$, con $h(x)=\begin{cases}1\text{ si }x\in[-ct,ct]\\ 0\text{ si }x\notin[-ct,ct]\end{cases}$

$$\frac{1}{2c}g(x)*\sqrt{2\pi}\,h(x)=\frac{1}{2c}\int_{-\infty}^\infty g(x-s)\,h(s)\,ds=\frac{1}{2c}\int_{-ct}^{ct}g(x-s)\,ds\underset{u=x-s}{=}-\frac{1}{2c}\int_{x+ct}^{x-ct}g(u)\,du$$

16 $\begin{cases}u_{tt}-3u_{xt}+2u_{xx}=0,\ x,t\in\mathbf{R}\\ u(x,0)=f(x),\ u_t(x,0)=0\end{cases}$ $\begin{cases}\hat{u}_{tt}+3ik\hat{u}_t-2k^2\hat{u}=0\\ \hat{u}(k,0)=\hat{f}(k),\ \hat{u}_t(k,0)=0\end{cases}\to \mu^2+3ik\mu-2k^2=0,\ \mu=-ik,-2ik\to$

$\hat{u}(k,t)=p(k)e^{-ikt}+q(k)e^{-2ikt}\xrightarrow{c.i.}\begin{cases}p(k)+q(k)=\hat{f}(k)\\ -ikp(k)-2ikq(k)=0,\ p(k)=-2q(k)=2\hat{f}(k)\end{cases}$ $q(k)=-\hat{f}(k)$ $\to \hat{u}=2\hat{f}(k)e^{-ikt}-\hat{f}(k)e^{-2ikt}$.

Y como $\mathcal{F}^{-1}[\hat{f}(k)e^{ika}]=f(x-a)$, la solución (única) es $\boxed{u(x,t)=2f(x+t)-f(x+2t)}$.

Si $f(x)=x^2$ queda $u=2x^2+4xt+2t^2-x^2-4xt-4t^2=\boxed{x^2-2t^2}$ [Directamente no se podía usar la $\mathcal{F}$ por no tener x^2 transformada].

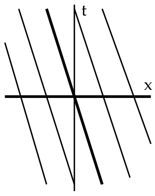
Comprobando: $u_{tt}-3u_{xt}+2u_{xx}=-4+2\cdot 2=0$, $u(x,0)=x^2$, $u_t(x,0)=-4\cdot 0=0$.

17 $\left\{\begin{array}{l} u_{tt}+2u_{tx}+u_{xx}=0\,,\ x,t\in\mathbf{R} \\ u(x,0)=0,\ u_t(x,0)=g(x)\end{array}\right.$ i) Parabólica. $\left\{\begin{array}{l}\xi=x-t\\ \eta=t\end{array}\right. \to u_{\eta\eta}=0\,,\ u=p(\xi)\eta+q(\xi)=tp(x-t)+q(x-t)$

$u(x,0)=q(x)=0 \to u_t(x,0)=p(x)=q(x) \to \boxed{u=t\,g(x-t)}$.

ii) Con la $\mathcal{F}$: $\left\{\begin{array}{l}\hat{u}_{tt}-2ik\hat{u}_t-k^2\hat{u}=0\\ \hat{u}(k,0)=0,\,\hat{u}_t(k,0)=\hat{g}(k)\end{array}\right. \to \lambda=ik$ doble, $\hat{u}=p(k)e^{ikt}+q(k)e^{ikt}t \underset{c.i.}{\to} \hat{u}=tg(k)e^{ikt}\uparrow$

18 $\boxed{3u_t-u_x=2}$ a] $\frac{dt}{dx}=\frac{3}{-1} \to t+3x=C$; $\left\{\begin{array}{l}\xi=t+3x\\ \eta=x\end{array}\right. \to u_\eta=-2$, $u=p(\xi)-2\eta=p(t+3x)-2x$.



O bien, $\left\{\begin{array}{l}\xi=t+3x\\ \eta=t\end{array}\right. \to 3u_\eta=2$, $u=p^*(\xi)+\frac{2}{3}\eta=p^*(t+3x)+\frac{2}{3}t$.

a$_1$] $u(x,0)=$ $\begin{array}{l} p(3x)-2x=x \to p(v)=v \to u=t+3x-2x \searrow \\ p^*(3x)=x \to p^*(v)=\frac{v}{3} \to u=\frac{t}{3}+x+\frac{2}{3}t \nearrow \end{array}$ $\boxed{u=t+x}$

Solución única pues $t=0$ no es tangente a las características [$\Delta=1\cdot3-0\cdot(-1)=3\neq0$].

a$_2$] $u(x,0)=$ $\begin{array}{l} p(0)-2x=-2x \to p(0)=0,\ \text{vale toda } p\in C^1 \text{ que lo cumpla, por ejemplo } p\equiv0 \to u=-2x \\ p^*(0)-2x=-2x \to p^*(0)=0,\ \text{vale toda } p^*\in C^1 \text{ que lo cumpla; } p^*\equiv0 \to u=\frac{2}{3}t \end{array}$

[Otra más: eligiendo $p(v)=v$ arriba o $p^*(v)=\frac{v}{3}$ abajo obtenemos la solución de a$_1$]].

Aquí el dato se da sobre una característica y no puede haber solución única [$\Delta=1\cdot3-(-3)\cdot(-1)\equiv0$].

b$_1$] Elegimos mejor $\eta=x$ y tenemos: $u_\eta=-g(\eta)$, $u=p(\xi)-\int_0^\eta g(s)\,ds=p(t+3x)-\int_0^x g(s)\,ds$.

$u(x,0)=p(3x)-\int_0^x g=f(x) \to p(v)=f(\frac{v}{3})+\int_0^{v/3} g \to u=f(x+\frac{t}{3})+\int_0^{x+\frac{t}{3}} g-\int_0^x g \to$

$\boxed{u=f(x+\frac{t}{3})+\int_x^{x+\frac{t}{3}} g(s)\,ds}$ (si $g(x)\equiv2$, $f(x)=x \to u=x+\frac{t}{3}+2\frac{t}{3}$, como arriba).

b$_2$] $\left\{\begin{array}{l}\hat{u}_t+ik\hat{u}=\hat{g}(k)\\ \hat{u}(k,0)=\hat{f}(k)\end{array}\right. \to \hat{u}(k,t)=p(k)\,e^{-ikt/3}+\frac{\hat{g}(k)}{ik}$, p arbitraria $\xrightarrow{\text{dato inicial}} p(k)=\hat{f}(k)-\frac{\hat{g}(k)}{ik} \to$

$\hat{u}(k,t)=\hat{f}(k)\,e^{-ikt/3}+\hat{g}(k)\big[\frac{1-e^{-ikt/3}}{ik}\big] \to u(x,t)=f(x+\frac{t}{3})+\sqrt{2\pi}\,g(x)*h(x)$, con $h(x)=\left\{\begin{array}{l}1 \text{ en } [-t/3,0]\\ 0 \text{ en el resto}\end{array}\right.$

$\sqrt{2\pi}\,g(x)*h(x)=\int_{-t/3}^0 g(x-u)\,du \underset{x-u=s}{=} -\int_{x+\frac{t}{3}}^x g(s)\,ds$ como antes.

19 a] $\boxed{\left\{\begin{array}{l}2u_t+u_x=tu\\ u(x,0)=e^{-x^2}\end{array}\right.}$ i) $\frac{dt}{dx}=2 \to \left\{\begin{array}{l}\xi=2x-t\\ \eta=t\end{array}\right. \to 2u_\eta=\eta u$, $u=p(\xi)\,e^{\eta^2/4}=p(2x-t)\,e^{t^2/4} \to$

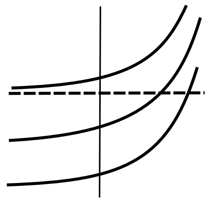
$u(x,0)=p(2x)=e^{-x^2} \to p(v)=e^{-v^2/4} \to u=e^{-(2x-t)^2/4}\,e^{t^2/4} \to \boxed{u=e^{xt-x^2}}$ [No hay problemas de unicidad: $\Delta=2\ \forall x$].

Haciendo $\eta=x$: $u_\eta=(2\eta-\xi)u$, $u=p(\xi)\,e^{\eta^2-\xi\eta}=p(2x-t)\,e^{xt-x^2} \to u(x,0)=p(2x)\,e^{-x^2}=e^{-x^2} \to p(v)\equiv1$.

ii) $\mathcal{F}(f')=-ik\hat{f}$, $\mathcal{F}(e^{-ax^2})=\frac{e^{-k^2/4a}}{\sqrt{2a}} \to \left\{\begin{array}{l}\hat{u}_t=\frac{ik}{2}\hat{u}+\frac{t}{2}\hat{u}\\ \hat{u}(k,0)=\frac{1}{\sqrt{2}}e^{-k^2/4}\end{array}\right. \to \hat{u}=p(k)e^{ikt/2}e^{t^2/4} \overset{\text{d. i.}}{\to} \hat{u}=\frac{1}{\sqrt{2}}\,e^{t^2/4}e^{-k^2/4}e^{ikt/2}$

$\to u=e^{t^2/4}\,\mathcal{F}^{-1}\big[\frac{e^{-k^2/4}}{\sqrt{2}}\,e^{ikt/2}\big]=e^{t^2/4}e^{-(x-\frac{t}{2})^2}=e^{xt-x^2}$, pues $\mathcal{F}^{-1}\big[\frac{e^{-k^2/4}}{\sqrt{2}}\big]=e^{-x^2}$ y $\mathcal{F}^{-1}\big[\hat{f}(k)\,e^{iak}\big]=f(x-a)$.

b] $\boxed{\left\{\begin{array}{l}u_t+e^t u_x+2tu=0\\ u(x,0)=f(x)\end{array}\right.}$ i) $\frac{dt}{dx}=\frac{1}{e^t}$, $x=\int e^t dt+C \to x-e^t=C$ características.



$\left\{\begin{array}{l}\xi=x-e^t\\ \eta=t \text{ (mejor)}\end{array}\right. \to u_\eta=-2tu \to u=p(\xi)\,e^{-\eta^2}=p(x-e^t)\,e^{-t^2}$.

[Con $\eta=x$ queda la ecuación más complicada $u_\eta=\frac{2\log(\eta-\xi)\,u}{\xi-\eta}$].

$u(x,0)=p(x-1)=f(x)$, $p(v)=f(v+1) \to \boxed{u(x,t)=f(x-e^t+1)\,e^{-t^2}}$.

[Solución única, pues $t=0$ no es tangente a las características, o porque: $\Delta=1\cdot1-0\cdot1=1\neq0\ \forall x$].

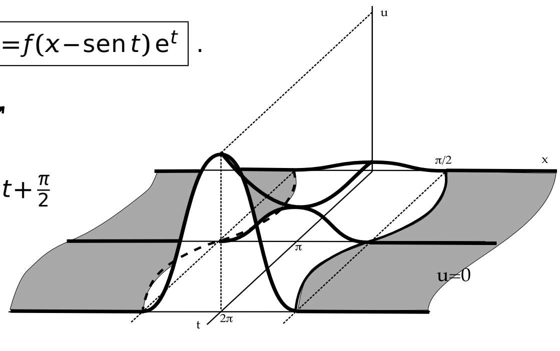
ii) $\left\{\begin{array}{l}\hat{u}_t-ike^t\hat{u}+2t\hat{u}=0\\ \hat{u}(k,0)=\hat{f}(k)\end{array}\right. \to \hat{u}(k,t)=p(k)\,e^{ike^t-t^2} \xrightarrow{c.i.} p(k)\,e^{ik}=\hat{f}(k) \to \hat{u}(k,t)=\hat{f}(k)\,e^{-t^2}e^{ik(e^t-1)}$.

Y como $\mathcal{F}^{-1}\big[\hat{f}(k)\,e^{ika}\big]=f(x-a)$, la solución es $\boxed{u(x,t)=e^{-t^2}f(x-e^t+1)}$, como antes.

20 $\left\{\begin{array}{l}u_t+(\cos t)u_x=u,\,x\in\mathbf{R},\,t\geq0\\ u(x,0)=f(x)\end{array}\right.$ $\left\{\begin{array}{l}\xi=x-\operatorname{sen}t\\ \eta=t\end{array}\right.$, $u_\eta=u$, $\boxed{u=f(x-\operatorname{sen}t)\,e^t}$.

$\left\{\begin{array}{l}\hat{u}_t-ik\cos t\,\hat{u}=\hat{u}\\ \hat{u}(k,0)=\hat{f}(k)\end{array}\right. \to \hat{u}=p(k)\,e^t\,e^{ik\operatorname{sen}t} \overset{c.i.}{=} \hat{f}(k)\,e^t\,e^{ik\operatorname{sen}t}\nearrow$

Si $f(x)=\left\{\begin{array}{l}\cos^2x,\ x\in[-\frac{\pi}{2},\frac{\pi}{2}]\\ 0 \text{ en el resto}\end{array}\right.$ $u\neq0$ si $\operatorname{sen}t-\frac{\pi}{2}\leq x\leq\operatorname{sen}t+\frac{\pi}{2}$
$u(x,n\pi)=e^{n\pi}f(x)$



La solución se contonea siguiendo las características, creciendo su altura exponencialmente con el tiempo.