1. Resolver $\begin{cases} u_t-4u_{xx}=0,\ x\in(0,\pi),\ t>0 \\ u(x,0)=f(x),\ u_x(0,t)=u(\pi,t)=0 \end{cases}$ si: i) $f(x)=\frac{\pi}{4}$, ii) $f(x)=\cos\frac{x}{2}$.

2. Resolver y hallar para cada $x\in(0,\pi)$ el límite de la solución $u(x,t)$ si $t\to\infty$: $\begin{cases} u_t-u_{xx}=0,\ x\in(0,\pi),\ t>0 \\ u(x,0)=0,\ u_x(0,t)=u_x(\pi,t)=t \end{cases}$.

ad4 y **4.** Resolver por separación de variables:

c) $\begin{cases} u_t-\frac{1}{t}u_{xx}=2\cos x,\ x\in\left(0,\frac{\pi}{2}\right),\ t>1 \\ u(x,1)=\cos 3x,\ u_x(0,t)=u\left(\frac{\pi}{2},t\right)=0 \end{cases}$ f) $\begin{cases} u_t-u_{xx}=0,\ x\in[0,1],\ t>0 \\ u(x,0)=2-x^2 \\ u_x(0,t)=u_x(1,t)+2u(1,t)=0 \end{cases}$

ad21. Desarrollar $g(x)=\begin{cases} 1,\ 0\leq x\leq\pi/2 \\ 0,\ \pi/2<x\leq\pi \end{cases}$ en $\{\operatorname{sen} nx\}$. ¿Cuánto suma la serie si $x=\frac{\pi}{2}$?

Sea $\begin{cases} u_{tt}-u_{xx}=0,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u_t(x,0)=g(x) \\ u(x,0)=u(0,t)=u(\pi,t)=0 \end{cases}$. Hallar $u\left(\frac{\pi}{3},\frac{\pi}{2}\right)$ con D'Alembert. Separar variables y aproximar el valor con 2 términos de la serie.

10. Resolver $\begin{cases} u_{tt}-u_{xx}=t\operatorname{sen} x,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=u(0,t)=u(\pi,t)=0 \end{cases}$ a) Separando variables.
b) Con extensiones y D'Alembert: i) directamente, ii) tras $w=u-t\operatorname{sen} x$.

ad26. a) Resolver $\begin{cases} u_{tt}-4u_{xx}=0,\ x\in\left[0,\frac{1}{2}\right],\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=0,\ u(0,t)=t,\ u_x\left(\frac{1}{2},t\right)=0 \end{cases}$.

12. Resolver $\begin{cases} u_{tt}+2u_t-5u_{xx}=0,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u_t(x,0)=g(x),\ u(x,0)=u(0,t)=u(\pi,t)=0 \end{cases}$ i) para cualquier g ii) si $g(x)=2\operatorname{sen} x$.

1. $\boxed{\begin{array}{l} u_t-4u_{xx}=0\,,\ x\in(0,\pi)\,,\ t>0 \\ u(x,0)=f(x)\,, u_x(0,t)=u(\pi,t)=0 \end{array}}$ $\begin{cases} X''+\lambda X=0 \\ X'(0)=X(\pi)=0 \end{cases}$, $\lambda_n=\frac{(2n-1)^2}{4}$, $n=1,2,\dots$

$X_n=\left\{\cos\frac{(2n-1)x}{2}\right\}$, y además $T'=-4\lambda_n T=-(2n-1)^2T \to T_n=\left\{e^{-(2n-1)^2t}\right\}$.

Probamos pues $u(x,t)=\sum_{n=1}^{\infty} c_n\, e^{-(2n-1)^2t}\cos\frac{(2n-1)x}{2}$.

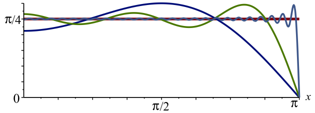
Y por el dato incial debe ser: $u(x,0)=\sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)x}{2}=f(x)$.

Para i) $f(x)=\frac{\pi}{4}$, son los $c_n=\frac{1}{2}\int_0^{\pi}\cos\frac{(2n-1)x}{2}\,dx=\frac{1}{2n-1}\,\mathrm{sen}\frac{(2n-1)x}{2}\Big]_0^{\pi}=\frac{(-1)^{n+1}}{2n-1}$.

Y por tanto la solución es:

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\,e^{-(2n-1)^2t}\cos\frac{(2n-1)x}{2}}$$

$$=e^{-t}\cos\frac{x}{2}-\frac{1}{3}e^{-9t}\cos\frac{3x}{2}+\cdots$$



2,5,50 términos de la serie de $\pi/4$

Para ii) $f(x)=\cos\frac{x}{2}$, a simple vista se ve que $c_n=1$ y los demás $c_n=0$,

con lo que la solución tiene ahora un único término: $\boxed{u(x,t)=e^{-t}\cos\frac{x}{2}}$.

2. $\boxed{\begin{array}{l} u_t - u_{xx} = 0\,,\ x\in(0,\pi),\ t>0 \\ u(x,0)=0,\ u_x(0,t)=u_x(\pi,t)=t \end{array}}$ A ojo se ve que $v=xt$ cumple los datos de contorno. $w=u-xt \to$

$$\begin{cases} w_t - w_{xx} = -x \\ w(x,0)=0, w_x(0,t)=w_x(\pi,t)=0 \end{cases} \to \begin{cases} X''+\lambda X=0 \\ X'(0)=X'(\pi)=0 \end{cases},\ X_n = \{\cos nx\}_{n=0,1,\dots},$$

$$w = T_0(T) + \sum_{n=0}^{\infty} T_n(t)\cos nx \to T_0' + \sum_{n=1}^{\infty}[T_n' + n^2 T_n]\cos nx = -x = \tfrac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos nx\,,$$

con $b_n = -\frac{2}{\pi}\int_0^\pi x\cos nx\,dx$: $\ b_0 = -\pi\,,\ b_n = \frac{2[1-\cos n\pi]}{n^2\pi} = \begin{cases} \frac{4}{n^2\pi}\,, & n \text{ impar} \\ 0\,, & n \text{ par} \end{cases}$.

$$\begin{cases} T_0' = -\frac{\pi}{2} \\ T_0(0)=0 \end{cases} \to T_0(t) = -\tfrac{\pi}{2}t\,,\quad \begin{cases} T_n' + n^2 T_n = b_n \\ T_n(0)=0 \end{cases} \to T_n(t) = \tfrac{b_n}{n^2}[1-e^{-n^2 t}]\,.$$

$$\boxed{u(x,t) = t\big(x-\tfrac{\pi}{2}\big) + \sum_{m=1}^{\infty} \frac{4}{\pi(2m-1)^4}\,[1-e^{-(2m-1)^2 t}]\cos(2m-1)x}$$

$u \to \infty$, si $x\in(\pi/2,\pi)$
$u \to 0$, si $x=\pi/2$
$u \to -\infty$, si $x\in(0,\pi/2)$

Estamos todo el rato (y cada vez más) metiendo calor por el extremo derecho y sacándolo por el izquierdo].

ad4. c) $\boxed{\begin{array}{l} u_t-\frac{1}{t}u_{xx}=2\cos x\,,\ x\in\left(0,\frac{\pi}{2}\right),\ t>1 \\ u(x,1)=\cos 3x\,, u_x(0,t)=u\left(\frac{\pi}{2},t\right)=0 \end{array}}$ $\xrightarrow{u=XT} \frac{X''}{X}=\frac{tT'}{T}=-\lambda \to$

$$\begin{cases} X''+\lambda X=0 \\ X'(0)=X\left(\frac{\pi}{2}\right)=0 \end{cases},\ X_n=\left\{\cos\left(2n-1\right)x\right\},\ n=1,2,\dots \quad \left[T'=-\tfrac{\lambda}{t}T \text{ no se usa aquí}\right].$$

Llevamos $u=\sum_{n=1}^{\infty} T_n(t)\cos(2n-1)x$ a la EDP y al dato inicial:

$$\sum_{n=1}^{\infty}\left[T_n'+\frac{(2n-1)^2}{t}T_n\right]\cos(2n-1)x=2\cos x$$

$$u(x,1)=\sum_{n=1}^{\infty}T_n(1)\cos(2n-1)x=\cos 3x$$

(ya desarrolladas ambas en cosenos)

$$\begin{cases} T_1'=-\frac{1}{t}T_1+2 \\ T_1(1)=0 \end{cases} \to T_1=\frac{C}{t}+t \xrightarrow{d.i.} C=-1\,.$$

$$\begin{cases} T_2'=-\frac{9}{t}T_2 \\ T_2(1)=1 \end{cases} \to T_2=Ct^{-9} \xrightarrow{d.i.} C=1\,.$$

(Únicas no nulas)

La solución es entonces: $\boxed{u(x,t)=\left(t-t^{-1}\right)\cos x+t^{-9}\cos 3x}$.

4. f) $\boxed{\begin{array}{l} u_t - u_{xx} = 0,\ x\in[0,1],\ t>0 \\ u(x,0)=2-x^2,\ u_x(0,t)=u_x(1,t)+2u(1,t)=0 \end{array}}$

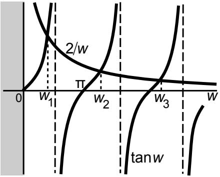
$\begin{cases} X''+\lambda X=0 \\ X'(0)=X'(1)+2X(1)=0 \end{cases}$ nuevo, y además $T'=-\lambda_n T$, $T_n=\{e^{-\lambda_n t}\}$.

Como $\alpha\alpha'=0$, $\beta\beta'=2$ y $q\equiv 0$, no hay $\lambda<0$.

$\lambda=0$: $X=c_1+c_2x \xrightarrow{c.c.} \begin{array}{l} c_2=0 \\ 2c_1+3c_2=0 \end{array} \to c_1=c_2=0$.

$\lambda>0$: $X=c_1\cos wx+c_2\,\text{sen}\,ww$, $w=\sqrt{\lambda} \xrightarrow{c.c.}$

$\begin{cases} c_2=0 \searrow \\ c_1[-w\,\text{sen}\,w+2\cos w]=0 \to \end{cases}$



w_n son las infinitas raíces de $\tan w=\frac{2}{w}$, $\lambda_n=w_n^2$ y $X_n=\{\cos w_n x\}$.

Será $u(x,t)=\sum_{n=1}^{\infty} c_n\, e^{-w_n^2 t}\cos w_n x$, con $u(x,0)=\sum_{n=1}^{\infty} c_n \cos w_n x = f(x) \to$

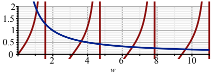
$c_n=\frac{\langle y_n,f\rangle}{\langle y_n,y_n\rangle}$. $\langle y_n,y_n\rangle=\int_0^1\cos^2 w_n x\,dx=\frac{1}{2}+\frac{\text{sen}\,2w_n}{4w_n}=\frac{2+\text{sen}^2 w_n}{4}$ $\left[\frac{\cos w_n}{w_n}=\frac{\text{sen}\,w_n}{2}\right]$.

$\langle y_n,f\rangle=\int_0^1(2-x^2)\cos w_n x=\frac{2-x^2}{w_n}\,\text{sen}\,w_n x\Big]_0^1+\frac{2}{w_n}\int_0^1 x\,\text{sen}\,w_n x=\cdots=\frac{2\,\text{sen}\,w_n}{w_n^3}$.

La solución única es: $\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{8\,\text{sen}\,w_n}{w_n^3(2+\text{sen}^2 w_n)}\,e^{-w_n^2 t}\cos w_n x}$.

Empezamos con un dibujo para decirle a Maple donde están las wn :

```
> plot([tan(w),2/w],w=0..11,0..2,gridlines=true);
```



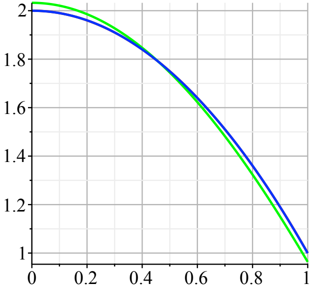

```
> g:=w*sin(w)-2*cos(w):
  w1:=fsolve(g,w=1..2):w2:=fsolve(g,w=3..4):
  w3:=fsolve(g,w=6..7):w4:=fsolve(g,w=9..10):
  evalf([w1,w2,w3,w4]);
  y1:=cos(w1*x):y2:=cos(w2*x):y3:=cos(w3*x):y4:=cos(w4*x):
```

$$[1.076873986,\ 3.643597167,\ 6.578333733,\ 9.629560343]$$

```
> c:=normal(subs(cos(w)=w*sin(w)/2,
   int((2-x^2)*cos(w*x),x=0..1)/int(cos(w*x)^2,x=0..1))):
  c1:=subs(w=w1,c):c2:=subs(w=w2,c):c3:=subs(w=w3,c):
  c4:=subs(w=w4,c):[c,evalf([c1,c2,c3,c4],4)];
```

$$\left[\frac{8\sin(w)}{w^3\left(\sin(w)^2+2\right)},\ [2.033,\ -0.03568,\ 0.003917,\ -0.0008941]\right]$$

```
> plot([2-x^2,c1*y1,c1*y1+c2*y2+c3*y3+c4*y4],x=0..1,
  thickness=2,color=[red,green,blue],gridlines=true);
```



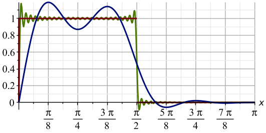
El primer término ya se parece y con 4 ya no se distinguen.

[La convergencia es tan buena porque $f(x)$ cumple los datos de contorno].

Ondas homogéneo $\left[\text{se pide } u(\frac{\pi}{3},\frac{\pi}{2}) \text{ por dos caminos}\right]$ (de examen jl21)

ad21. $g(x)=\begin{cases}1\,,\ 0\le x\le \pi/2\\ 0\,,\ \pi/2<x\le\pi\end{cases} \to b_n=\frac{2}{\pi}\int_0^{\pi/2}\operatorname{sen} nx\,dx=-\frac{2\cos nx}{n\pi}\Big]_0^{\pi/2}$

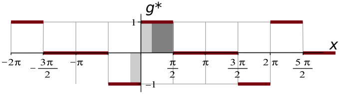
$$\to g(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big(1-\cos\frac{n\pi}{2}\Big)\operatorname{sen} nx\,.$$



Debe sumar $\frac{1}{2}$ en $\frac{\pi}{2}$. $\left[\text{Lo hace: } \frac{2}{\pi}\big(1-\frac{1}{3}+\frac{1}{5}-\cdots\big)=\frac{1}{2}\right]$.

g^* extensión impar y 2π-periódica de g:

$$u\big(\tfrac{\pi}{3},\tfrac{\pi}{2}\big)=\frac{1}{2}\int_{-\pi/6}^{5\pi/6}g^*(s)\,ds=\frac{1}{2}\int_{\pi/6}^{\pi/2}ds=\boxed{\frac{\pi}{6}}\,.$$



$$\boxed{\begin{array}{l} u_{tt}-u_{xx}=0\,,\ x\in[0,4\pi]\,,\ t\in\mathbf{R}\\ u_t(x,0)=\begin{cases}\pi\,,\ x\in[\pi,3\pi]\\ 0\,,\ \text{resto de } [0,4\pi]\end{cases}\\ u(x,0)=u(0,t)=u(4\pi,t)=0\end{array}}$$

$\begin{cases}X''+\lambda X=0\\ X(0)=X(\pi)=0\end{cases}$, $\lambda_n=n^2,\ X_n=\{\operatorname{sen} nx\}$, $n=1,2,\dots$

$\begin{cases}T''+\lambda T=0\\ T(0)=0\end{cases} \to T_n=\{\operatorname{sen} nt\}$.

$$u=\sum_{n=1}^{\infty}c_n\operatorname{sen} nt\operatorname{sen} nx\,.\ \ u_t(x,0)=\sum_{n=1}^{\infty}nc_n\operatorname{sen} nx=g(x)\to c_n=\frac{2}{\pi n^2}\Big[1-\cos\frac{n\pi}{2}\Big]$$

$$\Big[=\frac{2}{\pi},\frac{1}{\pi},\frac{2}{9\pi},0,\frac{2}{25\pi},\dots\Big]$$

$$u\big(\tfrac{\pi}{3},\tfrac{\pi}{2}\big)=\frac{2}{\pi}\Big[\operatorname{sen}\frac{\pi}{3}+\frac{1}{25}\operatorname{sen}\frac{5\pi}{3}+\cdots\Big]=\frac{\sqrt{3}}{\pi}\Big[1-\frac{1}{25}+\cdots\Big]\approx\boxed{\frac{24\sqrt{3}}{25\pi}}\approx 0{,}53\,.$$

$\left[\text{El exacto } \frac{\pi}{6}\approx 0.5236\right]$.

10. $\boxed{\begin{array}{l} u_{tt}-u_{xx}=t\,\text{sen}\,x\,,\ x\in[0,\pi],\, t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=u(0,t)=u(\pi,t)=0 \end{array}}$ $\begin{cases} X''+\lambda X=0 \\ X(0)=X(\pi)=0 \end{cases}$, $X_n=\{\text{sen}\,nx\}_{n=1,2,\dots}$

$$u=\sum_{n=1}^{\infty}T_n(t)\,\text{sen}\,nx \;\to\; \sum_{n=1}^{\infty}\big[T_n''+n^2T_n\big]\text{sen}\,nx=t\,\text{sen}\,x\,,$$

$$\sum_{n=1}^{\infty}T_n(0)\,\text{sen}\,nx=0 \;\text{ y }\; \sum_{n=1}^{\infty}T_n'(0)\,\text{sen}\,nx=0 \;\Rightarrow\; T_n(0)=T_n'(0)=0\;\;\forall n\,.$$

$$\to\begin{cases} T_1''+T_1=t \;\to\; T_1=c_1\cos t+c_2\,\text{sen}\,t+t\;\;\big(T_{1p}=At+B\;\text{ o a ojo}\big) \\ T_1(0)=T_1'(0)=0 \end{cases} \xrightarrow{d.i.}$$

$$T_1(0)=c_1=0,\; T_1'(0)=-c_2+1=0\,.\;\; \boxed{u(x,t)=(t-\text{sen}\,t)\,\text{sen}\,x} \text{ solución.}$$

$F(x,t)=t\,\text{sen}\,x$ está definida sólo en $[0,\pi]$ y para utilizar D'Alembert se debe extender de forma impar y 2π-periódica en x a todo $\mathbf{R}$. Pero F ya lo es, con lo que $F^*=F$. Así pues:

$$u=\tfrac{1}{2}\int_0^t\!\int_{x-(t-\tau)}^{x+(t-\tau)}\tau\,\text{sen}\,s\,ds\,d\tau=\tfrac{1}{2}\int_0^t\tau\,\overbrace{\big(\cos[x-(t-\tau)]-\cos[x+(t-\tau)]\big)}^{=\text{sen}\,x\ \text{sen}(t-\tau)}d\tau$$

$$=\text{sen}\,x\,(t-\text{sen}\,t)\,,\text{ pues } \int_0^t\tau\,\text{sen}(t-\tau)\,d\tau=t-\text{sen}\,t\,.$$

Haciendo $w=u-t\,\text{sen}\,x$ se llega a problema con $f=F=0$, $g(x)=-\text{sen}\,x$ (ya impar y 2π-periódica). Por tanto es:

$$w=-\tfrac{1}{2}\int_{x-t}^{x+t}\text{sen}\,s\,ds=\tfrac{1}{2}\big[\cos(x+t)-\cos(x-t)\big]=-\,\text{sen}\,x\,\text{sen}\,t\;\uparrow$$

ad26. a) $\boxed{\begin{array}{l} u_{tt}-4u_{xx}=0\,,\ x\in\left[0,\frac{1}{2}\right],\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=0\,,\ u(0,t)=t\,,\,u_x\!\left(\frac{1}{2},t\right)=0 \end{array}}$ $v=t$ clara. Con $w=u-t$:

$$\begin{cases} w_{tt}-4w_{xx}=0 \\ w(x,0)=0, w_t(x,0)=-1\,,\ w(0,t)=w_x(1/2,t)=0 \end{cases} \to$$

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X'(1/2)=0 \end{cases} \to X_n=\{\text{sen}(2n-1)\pi x\}$$

$$n=1,2,\ldots$$

$$\begin{cases} T''+4\lambda T=0 \\ T(0)=0 \end{cases} \to T_n=\{\text{sen}(4n-2)\pi t\}$$

Probamos: $w=\sum_{n=1}^{\infty} c_n \,\text{sen}(4n-2)\pi t\,\text{sen}(2n-1)\pi x\,.$

$$w_t(x,0)=\sum_{n=1}^{\infty}(4n-2)\pi c_n\,\text{sen}(2n-1)\pi x=-1 \to$$

$$(4n-2)\pi c_n=\frac{2}{1/2}\int_0^{1/2}-\text{sen}(2n-1)\pi x\,dx=\frac{-4}{\pi(2n-1)}\,.$$

$$u(x,t)=t-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,\text{sen}(4n-2)\pi t\,\text{sen}(2n-1)\pi x\,.$$

[Para resolverlo con D'Alembert se debería extender impar respecto a 0 [así $w(0,t)=0$] y par respecto a L. Aparece una g^* de periodo $4L$, que es precisamente el periodo de estos senos impares. Análoga situación (con cosenos) se da si hay w_x en la izquierda y w en la derecha].

12. $\boxed{\begin{array}{l} u_{tt}+2u_t-5u_{xx}=0\,,\ x\in[0,\pi]\,,\ t\in\mathbf{R} \\ u_t(x,0)=g(x)\,,\ u(x,0)=u(0,t)=u(\pi,t)=0 \end{array}}$ a) $\forall g(x)$, b) si $g=2\,\text{sen}\,x$.

$$u=X(x)T(t)\ \to\ \frac{T''+2T}{5T}=\frac{X''}{X}=-\lambda\ \to\ \begin{cases} X''+\lambda X=0 \\ T''+2T'+5\lambda T=0 \end{cases}$$

$$u(0,t)=u(\pi,t)=0\to x(0)=X(\pi)=0\ \to\ \underset{n=1,2\dots}{\lambda_n=n^2},\ X_n=\{\text{sen}\,nx\}$$

Para esos λ: $\mu^2+2\mu+5n^2=0$, $\mu=-1\pm i\sqrt{5n^2-1}\ \to$

$$T=e^{-t}\left[c_1\cos(\sqrt{5n^2-1}\,t)+c_2\,\text{sen}(\sqrt{5n^2-1}\,t)\right]\xrightarrow{T(0)=0}T_n=\left\{e^{-t}\,\text{sen}\left(\sqrt{\cdot}\,t\right)\right\}$$

$$\to\ u(x,t)=\sum_{n=1}^{\infty}c_n\,e^{-t}\,\text{sen}\left(\sqrt{5n^2-1}\,t\right)\text{sen}\,nx\ .$$

Falta sólo imponer un dato: $u_t(x,0)=\sum_{n=1}^{\infty}c_n\sqrt{5n^2-1}\ \text{sen}\,nx=g(x)$

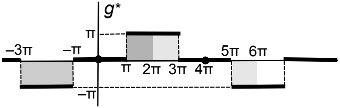
a) En general, será $c_n=\dfrac{2}{\pi\sqrt{5n^2-1}}\int_0^{\pi}g(x)\,\text{sen}\,nx\,dx$.

b) Si $g(x)=2\,\text{sen}\,x$, todos los $c_n=0$ excepto $c_1\sqrt{4}=2\ \to$

$$\boxed{u(x,t)=e^{-t}\,\text{sen}\,2t\ \text{sen}\,x}\ .$$

ad19. $\begin{cases} u_{tt}-u_{xx}=0\,,\, x\in[0,4\pi],\, t\in\mathbf{R} \\ u_t(x,0)=\begin{cases}\pi\,,\, x\in[\pi,3\pi] \\ 0\,,\text{ resto de }[0,4\pi]\end{cases} \\ u(x,0)=u(0,t)=u(4\pi,t)=0 \end{cases}$ Dar $u(\pi,4\pi)$ y $u(3\pi,3\pi)$ con D'Alembert. Separando variables comprobar $u(\pi,4\pi)$ y aproximar $u(3\pi,3\pi)$ con 2 términos $\left[32\sqrt{2}/9\approx 5.03\right]$.

Se extiende g a g^* impar y 8π-periódica definida $\forall x$ y es $u(x,t)=\frac{1}{2}\int_{x-t}^{x+t} g^*(s)\,ds \rightarrow$



i) $u(\pi,4\pi)=\frac{1}{2}\int_{-3\pi}^{5\pi} g^* = \boxed{0}$.

ii) $u(3\pi,3\pi)=\frac{1}{2}\int_{0}^{6\pi} g^* = \boxed{\frac{1}{2}\pi^2}$.

Separando variables: $\begin{cases} X''+\lambda X=0 \\ X(0)=X(4\pi)=0\end{cases}$, $\lambda_n=\frac{n^2}{16}$, $X_n=\left\{\operatorname{sen}\frac{nx}{4}\right\}$, $n=1,2,\dots$,

$T''+\lambda T=0\,,\; T(0)=0 \rightarrow T_n=\left\{\operatorname{sen}\frac{nt}{4}\right\}$. $u(x,t)=\sum_{n=1}^{\infty} c_n \operatorname{sen}\frac{nt}{4}\operatorname{sen}\frac{nx}{4}$.

$u_t(x,0)=\sum_{n=1}^{\infty}\frac{n}{4}c_n\operatorname{sen}\frac{nx}{4}=g(x) \rightarrow c_n=\frac{2}{n\pi}\int_{\pi}^{3\pi}\pi\operatorname{sen}\frac{nx}{4}\,dx=\frac{8}{n^2}\left[\cos\frac{n\pi}{4}-\cos\frac{3n\pi}{4}\right]$.

Es $u(\pi,4\pi)=\sum_{n=1}^{\infty} c_n \underbrace{\operatorname{sen} n\pi}_{=0}\operatorname{sen}\frac{n\pi}{4}=\boxed{0}$. Y con 2 términos de la serie:

$$u(3\pi,3\pi)=8\Big[\underbrace{\cos\tfrac{\pi}{4}}_{\sqrt{2}/2}-\underbrace{\cos\tfrac{3\pi}{4}}_{-\sqrt{2}/2}\Big]\underbrace{\operatorname{sen}^2\tfrac{3\pi}{4}}_{1/2}+\frac{8}{9}\Big[\underbrace{\cos\tfrac{3\pi}{4}}_{-\sqrt{2}/2}-\underbrace{\cos\tfrac{9\pi}{4}}_{\sqrt{2}/2}\Big]\underbrace{\operatorname{sen}^2\tfrac{9\pi}{4}}_{1/2}+\cdots$$

$$=4\sqrt{2}-\frac{4}{9}\sqrt{2}+\cdots\approx\frac{32}{9}\sqrt{2}\;\left[\approx 5.03\,,\text{ cerca del exacto }\tfrac{1}{2}\pi^2\approx 4.93\right].$$

14. Resolver: a) $\begin{cases} \Delta u = 0\,,\ (0,\pi)\times(0,\pi) \\ u(0,y)=0, u(\pi,y)=5+\cos y \\ u_y(x,0)=u_y(x,\pi)=0 \end{cases}$ b) $\begin{cases} \Delta u = y\cos x\,,\ (0,\pi)\times(0,1) \\ u_x(0,y)=u_x(\pi,y)=0 \\ u_y(x,0)=u_y(x,1)=0 \end{cases}$

15. $\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0\,,\ r<1\,,\ 0<\theta<\frac{\pi}{2} \\ u(1,\theta)=f(\theta),\ u(r,0)=u\big(r,\frac{\pi}{2}\big)=0 \end{cases}$. Dar la solución si $f(\theta)=\operatorname{sen}2\theta$ y un término de la serie si $f(\theta)=\cos\theta$.

18. Resolver por separación de variables estos problemas planos:

e) $\begin{cases} \Delta u = 3\operatorname{sen}\frac{\theta}{2}\,,\ r<1\,,\ \theta\in(0,\pi) \\ u(1,\theta)=u(r,0)=u_\theta(r,\pi)=0 \end{cases}$ a) $\begin{cases} \Delta u = 0\,,\ 1<r<2 \\ u(1,\theta)=1+\operatorname{sen}2\theta\,,\ u_r(2,\theta)=0 \end{cases}$

19. a) Resover $\begin{cases} \Delta u = 0\,,\ r<2\,,\ \theta\in(0,\pi/2) \\ u_r(2,\theta)+ku(2,\theta)=8\cos 2\theta\,,\ u_\theta(r,0)=u_\theta\big(r,\frac{\pi}{2}\big)=0 \end{cases}$, i) $k=1$, ii) $k=0$.

22. Resolver $\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} = \frac{2\operatorname{sen}\theta}{1+r^2} \\ u(1,\theta)=1\,,\ u \text{ acotada} \end{cases}$, i] en el círculo $r<1$, ii] en la región infinita $r>1$.

20. Resolver en el espacio: a) $\begin{cases} \Delta u=0\,,\ 1<r<2 \\ u(1,\theta)=\cos\theta, u(2,\theta)=0 \end{cases}$, b) $\begin{cases} \Delta u=0\,,\ r<1 \\ u_r(1,\theta)=\cos^3\theta \end{cases}$

25. b) Resolver $\begin{cases} u_{tt}-[u_{xx}+u_{yy}]=0\,,\ (x,y)\in(0,\pi)\times(0,\pi),\ t\in\mathbf{R} \\ u(x,y,0)=0\,,\ u_t(x,y,0)=\operatorname{sen}3x\,\operatorname{sen}^2 2y \\ u(0,y,t)=u(\pi,y,t)=0\,,\ u_y(x,0,t)=u_y(x,\pi,t)=0 \end{cases}$.

ad40c. Resolver $\begin{cases} \Delta u=0\,,\ x^2+y^2+z^2<1 \\ u=x^3 \text{ si } x^2+y^2+z^2=1 \end{cases}$.

14. a) $\boxed{\begin{array}{l}\Delta u=0,\ (x,y)\in(0,\pi)\times(0,\pi)\\ u(\pi,y)=5+\cos y,\ u(0,y)=u_y(x,0)=u_y(x,\pi)=0\end{array}}$

$Y''+\lambda Y=0\,,\ Y'(0)=Y'(\pi)=0 \to\ Y_n=\{\cos ny\}\,,\ n=0,1,\ldots$

$X_n''-n^2X_n=0,\ X(0)=0\ \to\ X_0=\{x\}\,,\ X_n=\{\operatorname{sh} nx\}\,,\ n\geq 1\,.$

$$u(x,y)=c_0\,x+\sum_{n=1}^{\infty}c_n\operatorname{sh} nx\cos ny\ \to$$

$$u(x,\pi)=c_0\pi+\sum_{n=1}^{\infty}c_n\operatorname{sh} n\pi\cos ny=5+\cos y\ \to\ u=\frac{5x}{\pi}+\frac{\operatorname{sh} x}{\operatorname{sh}\pi}\cos y\ .$$

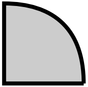
b) $\boxed{\begin{array}{l}\Delta u=y\cos x,\ (x,y)\in(0,\pi)\times(0,1)\\ u_x(0,y)=u_x(\pi,y)=u_y(x,0)=u_y(x,1)=0\end{array}}$ $\quad u=\sum_{n=0}^{\infty}Y_n(y)\cos nx \to$

$$\sum_{n=0}^{\infty}\left[Y_n''-n^2Y_n\right]\cos nx=y\cos x\ \to\ \begin{array}{l}Y_1''-Y_1=y\\ Y_1'(0)=Y_1'(1)=0\end{array}\ \to\ Y_1=\frac{e^y-e^{1-y}}{1+e}-y$$

Es de Neumann. Aparece (al resolver $Y_0''=0+$c.c.) una C arbitraria:

$$u(x,y)=C+\left[\frac{e^y-e^{1-y}}{1+e}-y\right]\cos x\,.$$

15. $\boxed{\begin{array}{l} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0\,,\ r<1\,,\ 0<\theta<\frac{\pi}{2} \\ u(1,\theta)=f(\theta)\,,\ u(r,0)=u\big(r,\frac{\pi}{2}\big)=0 \end{array}}$ $u=R\Theta \to \begin{cases}\Theta''+\lambda\Theta=0 \\ \Theta(0)=\Theta\big(\frac{\pi}{2}\big)=0\end{cases}$, $\lambda_n=4n^2\,,\ \Theta_n=\big\{\text{sen}\,2n\theta\big\}$. $n=1,2,\dots$



Además: $r^2R''+rR'-\lambda R=0 \to \mu^2=4n^2\,,\ R=c_1r^{2n}+c_2r^{-2n} \xrightarrow{\text{acotada}} R_n=\{r^{2n}\}$.

Probamos, pues, la serie $u(r,\theta)=\sum_{n=1}^{\infty} c_n\, r^{2n}\,\text{sen}\,2n\theta$ que debe cumplir el dato

de contorno final: $u(1,\theta)=\sum_{n=1}^{\infty} c_n\,\text{sen}\,2n\theta=f(\theta) \to c_n=\frac{4}{\pi}\int_0^{\pi/2} f(\theta)\,\text{sen}\,2n\theta\,d\theta$.

Si $f(\theta)=\text{sen}\,2\theta$ no hay que integrar: $c_1=0$ y resto 0 . $\boxed{u=r^2\,\text{sen}\,2\theta}$ $[=2xy]$.

Si $f(\theta)=\cos\theta$ sí se integra. El primer término (único que se pide) lo da:

$$c_1=\frac{4}{\pi}\int_0^{\pi/2}\cos\theta\,\text{sen}\,2\theta\,d\theta=-\frac{8}{3\pi}\cos^3\theta\Big]_0^{\pi/2}=\frac{8}{3\pi} \to \boxed{u=\frac{8}{3\pi}\,r^2\,\text{sen}\,2\theta+\cdots}\,.$$

[No sería demasiado largo dar la serie solución no pedida con todos sus términos].

18. b) $\boxed{\begin{array}{l}\Delta u=3\,\text{sen}\,\frac{\theta}{2}\,,\ r<1\,,\ \theta\in(0,\pi)\\ u(1,\theta)=u(r,0)=u_\theta(r,\pi)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta'(\pi)=0\end{array}\right.\to$

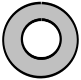
ya desarrollada

$$u(r,\theta)=\sum_{n=1}^{\infty}R_n(r)\,\text{sen}\,\frac{(2n-1)\theta}{2}\to\sum_{n=1}^{\infty}\Big[R_n''+\frac{R_n'}{r}-\frac{(2n-1)^2R_n}{4r^2}\Big]\,\text{sen}\,\frac{(2n-1)\theta}{2}=3\,\text{sen}\,\frac{\theta}{2}$$

Además $R_n(1)=0\ \forall n$ y la acotación. La única $R_n\not\equiv 0$ sale de:

$$\left\{\begin{array}{l}r^2R_1''+rR_1-\frac{1}{4}R_1=3r^2\\ R_1\text{ acotada},\ R_1(1)=0\end{array}\right.\to R_1=c_1r^{1/2}+\frac{c_2}{r^{1/2}}+\frac{4r^2}{5}\overset{c.c.}{\to}u(r,\theta)=\frac{4}{5}\big[r^2-r^{1/2}\big]\,\text{sen}\,\frac{\theta}{2}\,.$$

a) $\boxed{\begin{array}{l}\Delta u=0\,,\ 1<r<2\\ u(1,\theta)=1+\text{sen}\,2\theta\,,\ u_r(2,\theta)=0\end{array}}$ $u=R\Theta\to\Theta''+\lambda\Theta=0$ y $r^2R''+rR'-\lambda R=0$.



$$\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta\ 2\pi\text{-periódica}\end{array}\right.\to\lambda_n=n^2,\ \Theta_n=\{\text{sen}\,n\theta,\cos n\theta\}\,,\ n=0,1,\dots$$

$$r^2R''+rR'-n^2R=0\to\begin{array}{l}R_0=c_1+c_2\ln r\\ R_n=c_1r^n+c_2r^{-n}\end{array}\xrightarrow{R'(2)=0}\begin{array}{l}R_0=\{1\}\\ R_n=\{r^n+2^{2n}r^{-n}\}\end{array}$$

Probamos $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\big[r^n+2^{2n}r^{-n}\big]\big[a_n\cos n\theta+b_n\,\text{sen}\,n\theta\big]$.

Falta imponer $u(1,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\big[1+2^{2n}\big]\big[a_n\cos n\theta+b_n\,\text{sen}\,n\theta\big]=1+\text{sen}\,2\theta\to$

$\frac{a_0}{2}=1$, $17b_2=1$, resto 0. La solución única es: $u=1+\frac{1}{17}\big[r^2+\frac{16}{r^2}\big]\,\text{sen}\,2\theta$.

Comprobemos al menos los datos: el de u es claro y además $u_r(2,\theta)=\frac{1}{17}\big[2\cdot 2-\frac{32}{8}\big]\,\text{sen}\,2\theta=0$.

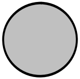
ad34d. $\boxed{\begin{array}{l}\Delta u=0\,,\ r<1\,,\ 0<\theta<\pi\\ u_r(1,\theta)=\cos\theta, u(r,0)=u(r,\pi)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta(\pi)=0\end{array}\right.\to\lambda_n=n^2,\ \Theta_n=\{\operatorname{sen} n\theta\}\,,\ n=1,2,\dots$

$r^2R''+rR'-n^2R=0$ de soluciones acotadas $R_n=\{r^n\}$.

$$u(r,\theta)=\sum_{n=1}^{\infty}c_nr^n\operatorname{sen} n\theta\to u_r(1,\theta)=\sum_{n=1}^{\infty}nc_n\operatorname{sen} n\theta=\cos\theta\to$$

$$c_n=\frac{2}{n\pi}\int_0^{\pi}\cos\theta\operatorname{sen} n\theta\,d\theta=\frac{2[1+(-1)^n]}{\pi(n^2-1)}\,.\quad u(r,\theta)=\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{4}{4m^2-1}r^{2m}\operatorname{sen} 2m\theta\,.$$

18. g) $\boxed{\begin{array}{l}\Delta u=\cos\theta\,,\ r<2\\ u(2,\theta)=\operatorname{sen} 2\theta\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta\ \ 2\pi\text{-per.}\end{array}\right.\to\Theta_n=\{\cos n\theta,\operatorname{sen} n\theta\}\,,\ n=0,1,\dots$



$$\to u=a_0(r)+\sum_{n=1}^{\infty}[a_n(r)\cos n\theta+b_n(r)\operatorname{sen} n\theta]\to$$

$$\frac{ra_0''+a_0'}{r}+\sum_{n=1}^{\infty}\Big[\frac{r^2a_n''+ra_n'-n^2a_n}{r^2}\cos n\theta+\frac{r^2b_n''+rb_n'-n^2b_n}{r^2}\operatorname{sen} n\theta\Big]=\cos\theta\,.$$

$$u(2,\theta)=a_0(2)+\sum_{n=1}^{\infty}[a_n(2)\cos n\theta+b_n(2)\operatorname{sen} n\theta]=\operatorname{sen} 2\theta\to$$

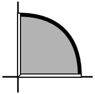
$$a_n(2)=0\,;\ b_{n\neq2}(2)=0\,;\ b_2(2)=1\ \text{y acotadas}\Rightarrow a_{n\neq1},b_{n\neq2}\equiv0\,.$$

$$\left\{\begin{array}{l}r^2a_1''+ra_1'-a_1=r^2\\ \text{acotada},\ a_1(2)=0\end{array}\right.\ \xrightarrow[3A=1]{a_{1p}=Ar^2}\ a_1=c_1r+c_2r^{-1}+\frac{r^2}{3}\ \xrightarrow{c.c.}\ a_1=\frac{r^2}{3}-\frac{2r}{3}$$

$$\left\{\begin{array}{l}r^2b_2''+rb_2'-4b_2=0\\ \text{acotada},\ b_2(2)=1\end{array}\right.\to b_2=c_1r^2+c_2r^{-2}\ \xrightarrow{c.c.}\ \begin{array}{l}c_2=0\\ 4c_1=1\end{array}\to b_2=\frac{r^2}{4}$$

$$\to u(r,\theta)=\frac{1}{3}r(r-2)\cos\theta+\frac{1}{4}r^2\operatorname{sen} 2\theta\,.$$

19. b) $\boxed{\begin{array}{l} \Delta u = 0\,,\ r<2\,,\ \theta\in\left(0,\frac{\pi}{2}\right) \\ u_r(2,\theta)+ku(2,\theta)=8\cos 2\theta \\ u_\theta(r,0)=u_\theta\left(r,\frac{\pi}{2}\right)=0 \end{array}}$ $\quad \begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta'(0)=\Theta'\left(\frac{\pi}{2}\right)=0 \end{cases}$, $\Theta_n = \{\cos 2n\theta\}$,



$\lambda_n = 4n^2$, $n=0,1,2,\ldots$ Y además: $r^2R''+rR'-\lambda R=0 \rightarrow$

$$\begin{array}{l} R_0=c_1+c_2\ln r \xrightarrow{R\text{ acotado}} R_0=\{1\} \\ R_{2n}=c_1r^{2n}+c_2r^{-2n} \rightarrow R_{2n}=\{r^{2n}\} \end{array} \rightarrow u = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n}\, r^{2n} \cos 2n\theta\ .$$

Imponiendo el último dato de contorno:

$$k\frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n}\left[2n2^{2n-1}+k2^{2n}\right]\cos 2n\theta = 8\cos 2\theta\,.$$

i) Para $k=1$, todos los $a_{2n}=0$, excepto $a_2[4+4]=8$.

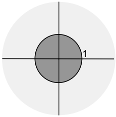
La solución (única) es: $\boxed{u(r,\theta) = r^2\cos 2\theta}$.

ii) Para $k=0$ (es de Neumann), a_0 queda libre, $a_2[4+0]=8$, y resto de $a_{2n}=0$. En este caso existen infinitas soluciones:

$$\boxed{u(r,\theta) = C + 2r^2\cos 2\theta}\ .$$

Interior y exterior en un círculo (27ab)

22. $\boxed{\begin{array}{l} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=\frac{2\operatorname{sen}\theta}{1+r^2} \\ u(1,\theta)=1\,,\ u \text{ acotada}\end{array}}$ $\Theta''+\lambda\Theta=0\,,\ \Theta\ 2\pi\text{-periódica} \to$ $\Theta_n=\{\cos n\theta, \operatorname{sen} n\theta\},\ n\geq 0$



Para ambos es: $u=a_0(r)+\sum_{n=1}^{\infty}\big[a_n(r)\cos n\theta+b_n(r)\operatorname{sen} n\theta\big] \to$

$$a_0''+\frac{1}{r}a_0'+\sum_{n=1}^{\infty}\Big[\big(a_n''+\frac{a_n'}{r}-\frac{n^2a_n}{r^2}\big)\cos n\theta+\big(b_n''+\frac{b_n'}{r}-\frac{n^2b_n}{r^2}\big)\operatorname{sen} n\theta\Big]=\frac{2r^2}{1+r^2}\operatorname{sen}\theta$$

$$r^2a_n''+ra_n'-n^2a_n=0\,,\ n\geq 0\,;\ r^2b_1''+rb_1'-b_1=\frac{2r^2}{1+r^2}\ ;\ r^2b_n''+rb_n'-n^2b_n=0\,,\ n\geq 2\,.$$

$u(1,\theta)=1 \Rightarrow a_0(1)=1$ y el resto valen 0 en 1. De solución no nula:

$$\begin{cases} r^2a_0''+ra_0'=0 \\ a_0(1)=1,\ a_0 \text{ acotada}\end{cases} \to a_0=c_1+c_2\ln r \xrightarrow{\text{c.c.}} a_0=1\,, \text{ para i] y para ii].}$$

$$\begin{cases} r^2b_1''+rb_1'-b_1=\frac{2r^2}{1+r^2} \\ b_1(1)=0,\ b_1 \text{ acotada}\end{cases} \xrightarrow{f.v.c.} b_1=c_1r+c_2r^{-1}+\big(r+\tfrac{1}{r}\big)\arctan r-1\,.$$

i] En $r<1$, como $\frac{\arctan r}{r}\xrightarrow{r\to 0}1$, debe ser $c_2=0$. Imponiendo la otra:

$c_1+2\arctan 1-1=0 \ \to\ u=1+\big[r-\frac{\pi}{2}r+\big(r+\frac{1}{r}\big)\arctan r-1\big]\operatorname{sen}\theta$.

ii] En infinito $b_{1p}\sim\frac{\pi}{2}r-1$. Para que b_1 esté acotada será $c_1=-\frac{\pi}{2}$.

Además: $-\frac{\pi}{2}+c_2+\frac{\pi}{2}-1=0 \to b_1=\frac{1}{r}-\frac{\pi}{2}r+\big(r+\frac{1}{r}\big)\arctan r-1$.

$$u=1+\big[\tfrac{1}{r}-\tfrac{\pi}{2}r+\big(r+\tfrac{1}{r}\big)\arctan r-1\big]\operatorname{sen}\theta\quad\Big[\frac{\arctan r-\pi/2}{1/r}\xrightarrow{r\to\infty}-1\Big].$$

20. a) $\boxed{\begin{array}{l}\Delta u=0\,,\ 1<r<2\\ u(1,\theta)=\cos\theta\,,\ u(2,\theta)=0\end{array}}$ $u=R\Theta\ \rightarrow$ $\big(\operatorname{sen}\theta\,\Theta'\big)'+\big(\lambda\operatorname{sen}\theta\big)\Theta=0$ y $r^2R''+2rR'-\lambda R=0$.

La acotación en $\theta=0,\pi$ da los $\lambda_n=n(n+1)$ y $\Theta_n=\big\{P_n(\cos\theta)\big\}$, $n=0,1,\dots$

Para esos λ_n $R=c_1r^n+c_2r^{-n-1}$. Imponiendo $u(r,2)=R(2)\Theta(\theta)=0$, $R(2)=0$

$$\rightarrow R_n=\big\{r^n-2^{2n+1}r^{-n-1}\big\}\ \rightarrow\ u(r,\theta)=\sum_{n=0}^{\infty}a_n\big[r^n-2^{2n+1}r^{-n-1}\big]P_n(\cos\theta)\,.$$

Falta $u(1,\theta)=\sum_{n=0}^{\infty}a_n\big[1-2^{2n+1}\big]P_n(\cos\theta)=\cos\theta=P_1(\cos\theta)\rightarrow\ a_1=-\frac{1}{7}$, resto 0.

La solución en la corona esférica $u(r,\theta)=\frac{1}{7}\big[8r^{-2}-r\big]\cos\theta$.

b) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<1\\ u_r(1,\theta)=\cos^3\theta\end{array}}$ A la $u=\sum_{n=0}^{\infty}a_nr^nP_n(\cos\theta)$ de los apuntes sólo le falta cumplir $u_r(1,\theta)=\sum_{n=1}^{\infty}na_nP_n(\cos\theta)=\cos^3\theta\ \rightarrow$

$$a_n=\frac{2n+1}{2n}\int_0^{\pi}\cos^3\theta\,P_n(\cos\theta)\operatorname{sen}\theta\,d\theta=\frac{2n+1}{2n}\int_{-1}^{1}t^3\,P_n(t)\,dt\ \text{ y }\ \forall a_0\ \text{(Neumann).}$$

(Desarrollo posible por ser 0 el primer término del desarrollo de $\cos^3\theta$).

Podemos integrar: $a_1=\frac{3}{2}\int_{-1}^{1}t^4\,dt=\frac{3}{5}$, $a_3=\frac{7}{6}\int_{-1}^{1}t^3\big[\frac{5}{2}t^3-\frac{3}{2}t\big]\,dt=\frac{2}{15}$.

Los demás $a_n=0$ pues $\int_{-1}^{1}=0$ si n par, y para desarrollar un Q_k bastan los k primeros P_n .

Pero mejor se tantea: $\cos^3\theta=\frac{2}{5}\big(\underbrace{\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta}_{P_3}\big)+\frac{3}{5}\underbrace{\cos\theta}_{P_1}\rightarrow\ 3a_3=\frac{2}{5}$, $a_1=\frac{3}{5}$.

$$\text{Por tanto,}\quad u=C+\frac{3}{5}r\cos\theta+\frac{2}{15}r^3\big(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\big)\,.$$

24. b) $\left\{\begin{array}{l} u_{tt}-[u_{xx}+u_{yy}]=0\,,\ (x,y)\in(0,\pi)\times(0,\pi),\ t\in\mathbf{R} \\ u(x,y,0)=0\,,\ u_t(x,y,0)=\operatorname{sen}3x\,\operatorname{sen}^2 2y \\ u(0,y,t)=u(\pi,y,t)=0\,,\ u_y(x,0,t)=u_y(x,\pi,t)=0 \end{array}\right.$

$$u=XYT\to\left\{\begin{array}{l} \left.\begin{array}{l} X''+\lambda X=0 \\ X(0)=X(\pi)=0 \end{array}\right\} X_n=\{\operatorname{sen}nx\},\ n=1,2,\cdots \\ \left.\begin{array}{l} Y''+\mu Y=0 \\ Y'(0)=Y'(\pi)=0 \end{array}\right\} Y_n=\{\cos my\},\ m=1,2,\cdots \\ T''+(\lambda+\mu)T=0,\ T(0)=0,\ T_{nm}=\{\operatorname{sen}\sqrt{n^2+m^2}\,t\} \end{array}\right.$$

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\operatorname{sen}\sqrt{n^2+m^2}\,t\,\operatorname{sen}nx\cos my\,,$$

$$u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\sqrt{n^2+m^2}\operatorname{sen}nx\cos my=\frac{1-\cos 4y}{2}\operatorname{sen}3x$$

$$\to c_{30}\sqrt{9}=\tfrac{1}{2}\,,\ c_{34}\sqrt{25}=-\tfrac{1}{2}\quad[\text{los demás }c_{nm}=0\,]\to$$

$$\boxed{u(x,y,t)=\tfrac{1}{6}\operatorname{sen}3t\operatorname{sen}3x-\tfrac{1}{10}\operatorname{sen}5t\operatorname{sen}3x\cos 4y}$$

ad40c. $\boxed{\begin{array}{l}\Delta u=0,\ x^2+y^2+z^2<1\\ u=x^3 \text{ si } x^2+y^2+z^2=1\end{array}}$ Es decir $\begin{cases}\Delta u=0,\ r<1\\ u|_{r=1}=\text{sen}^3\theta\cos^3\phi\end{cases}$.

Autofunciones en θ las da $P_n^m(s)=\left(1-s^2\right)^{m/2}\frac{d^m}{ds^m}P_n(s)$, con $m\leq n$, $s=\cos\theta$.

[En apuntes: $P_n^0=P_n$, $P_1^1=\text{sen}\,\theta$, $P_2^1=3\,\text{sen}\,\theta\cos\theta$, $P_2^2=3\,\text{sen}^2\theta$, ... (se precisan nuevas (impares))].

Son soluciones de la EDP: $u_n^m=r^nY_n^m(\theta,\phi)$ **(armónicos esféricos)**, siendo:

$$Y_n^m(\theta,\phi)=\left\{\cos m\phi\,P_n^m(\cos\theta),\,\text{sen}\,m\phi\,P_n^m(\cos\theta)\right\},\quad n=0,1,\dots,\ m=0\dots n.$$

[En apuntes: $Y_0^0=\{1\}$, $Y_1^0=\{\cos\theta\}$, $Y_1^1=\{\text{sen}\,\theta\cos\phi,\text{sen}\,\theta\,\text{sen}\,\phi\}$, $Y_2^0=\{\frac{3}{2}\cos^2\theta-\frac{1}{2}\}$,
$Y_2^1=\left\{3\,\text{sen}\,\theta\cos\theta\cos\phi,3\,\text{sen}\,\theta\cos\theta\,\text{sen}\,\phi\right\}$, $Y_2^2=\left\{3\,\text{sen}^2\theta\cos2\phi,3\,\text{sen}^2\theta\,\text{sen}\,2\phi\right\}$, ...]

$P_3=\frac{5}{2}t^3-\frac{3}{2}t$, $\quad P_3'=\frac{3}{2}[5t^2-1]$, $P_3^1=\frac{3}{2}\,\text{sen}\,\theta[5\cos^2\theta-1]$ (con $\cos\phi$ y $\text{sen}\,\phi$)
$\quad P_3'''=15$, $P_3^3=15\,\text{sen}^3\theta$ (con $\cos3\phi$ y $\text{sen}\,3\phi$)

Expresando (formulario) el $\cos^3\phi$ en función de $\cos3\phi$ y $\cos\phi$:

$$\text{sen}^3\theta\,\frac{\cos3\phi+3\cos\phi}{4}=\frac{1}{4}\,\text{sen}^3\theta\cos3\phi-\frac{3}{20}\,\text{sen}\,\theta[5\cos^2\theta-1]\cos\phi+\frac{3}{5}\,\text{sen}\,\theta\cos\phi$$

$$\to\ \boxed{u=\frac{r^3}{4}\,\text{sen}^3\theta\cos3\phi-\frac{3r^3}{20}\,\text{sen}\,\theta\,[5\cos^2\theta-1]\cos\phi+\frac{3r}{5}\,\text{sen}\,\theta\cos\phi}\,.$$

O en cartesianas: $\boxed{u=\frac{1}{5}x\left[3+2x^2-3y^2-3z^2\right]}$.