Soluciones de problemas 1 de MII(C) (2023-24)

1 a) $\boxed{\begin{array}{l}(y-2e^x)u_y-u_x=u\\ u(0,y)=y\end{array}}$ $\frac{dy}{dx}=-y+2e^x$ lineal, $y=Ce^{-x}+e^x$, $e^xy-e^{2x}=C$ características. Mucho mejor:

$\begin{cases}\xi=e^xy-e^{2x}\\ \eta=x\end{cases}$, $\begin{cases}u_y=e^xu_\xi\\ u_x=(e^xy-2e^{2x})u_\xi+u_\eta\end{cases}$, $u_\eta=-u$, $u(x,y)=p(\xi)\,e^{-\eta}=p\big(e^xy-e^{2x}\big)\,e^{-x}$.

$u(0,y)=p(y-1)=y$, $p(v)=v+1$, $\boxed{u(x,y)=y-e^x+e^{-x}}$. [Única por no ser tangente $x=0$ a las características (y' siempre finita), o $T(y)=0\cdot(y-2)-1\cdot(-1)=1\neq 0\ \forall y$].

b) $\boxed{\begin{array}{l}yu_y+u_x=u-ye^{-x}\\ u(x,-1)=1\end{array}}$ $\frac{dy}{dx}=y\to y=Ce^x$. O bien $\begin{cases}\xi=ye^{-x}\\ \eta=y\end{cases}\to u_\eta=\frac{u-\xi}{\eta}$, $u=p(\xi)\eta+\xi=p\big(ye^{-x}\big)y+ye^{-x}$

[a ojo o $u_p \underset{\downarrow}{=} -e^{\eta}\int \xi e^{-\eta}\,d\eta$] [$u_p=\xi$ a ojo o $u_p \overset{\uparrow}{=} \eta\int\frac{-\xi}{\eta^2}\,d\eta$].

O bien $\begin{cases}\xi=ye^{-x}\\ \eta=x\end{cases}$, $u_\eta=u-\xi$, $u=q(\xi)\,e^{\eta}+\xi=q(ye^{-x})\,e^x+ye^{-x}$.

$u(x,-1)=1\to p(v)=v-1$ o $q(v)=v^2-v\to\boxed{u(x,y)=(y^2+y)e^{-x}-y}$.

[Única: $T(x)=1\cdot(-1)-0\cdot 1\neq 0$. $p(v)$ fijada sólo para $v<0$, pero se evalúa en valores negativos cerca de la recta].

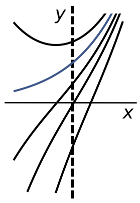
c) $\boxed{\begin{array}{l}yu_y+(2y-x)u_x=x\\ u(x,1)=0\end{array}}$ $\frac{dy}{dx}=\frac{y}{2y-x}$ (exacta u homogénea) o mejor $\frac{dx}{dy}=-\frac{x}{y}+2$ lineal $\to xy-y^2=C$, $\begin{cases}\xi=xy-y^2\\ \eta=y\end{cases}$

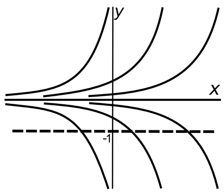
$\to \eta u_\eta=\frac{\xi+\eta^2}{\eta}$, $u=p(\xi)-\frac{\xi}{\eta}+\eta=p(xy-y^2)+2y-x\to p(v)=v-1\ \forall v\to\boxed{u=xy-y^2+2y-x-1}$ [$T=1$].

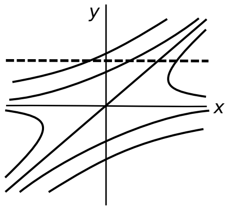
d) $\boxed{\begin{array}{l}3x^2u_y-u_x=4yu\\ u(1,y)=1\end{array}}$ $\frac{dy}{dx}=-3x^2\to y=C-x^3$. $\begin{cases}\xi=y+x^3\\ \eta=x\end{cases}\to u_\eta=-4yu=(4\eta^3-4\xi)u\to u=p(\xi)\,e^{\eta^4-4\xi\eta}$

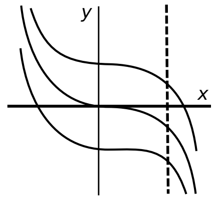
$\to u=p(y+x^3)\,e^{-4xy-3x^4}$, $u(1,y)=p(y+1)\,e^{-4y-3}=1$, $p(v)=e^{4v-1}$, $\boxed{u(x,y)=e^{4y-4xy-3x^4+4x^3-1}}$.

Solución única pues $x=1$ no tangente a las características, o porque $T=0\cdot 3-1\cdot(-1)=1\neq 0$.

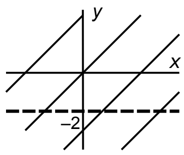








2 a) $\boxed{\begin{array}{l}u_y+u_x=u-x-y\\ u(x,-2)=x\end{array}}$ $\frac{dy}{dx}=1$. $\int dy=\int dx+C$. Características: $\boxed{y-x=C}$.



Haciendo $\begin{cases}\xi=y-x\\ \eta=y\end{cases}\to\begin{cases}u_y=u_\xi+u_\eta\\ u_x=-u_\xi\end{cases}$, $u_\eta=u-x-y=u+\xi-2\eta$. $u=p(\xi)\,e^{\eta}+2\eta-\xi+2=p(y-x)\,e^y+x+y+2$.

O bien: $\begin{cases}\xi=y-x\\ \eta=x\end{cases}\to\begin{cases}u_y=u_\xi\\ u_x=-u_\xi+u_\eta\end{cases}$, $u_\eta=u-x-y=u-\xi-2\eta$. $u=p(\xi)\,e^{\eta}+2\eta+\xi+2=q(y-x)\,e^x+x+y+2$.

[Para las u_p, mejor que integrar, probamos $u_p=A\eta+B\to A=2, B=2-\xi$, $A=2, B=2+\xi$ respectivamente].

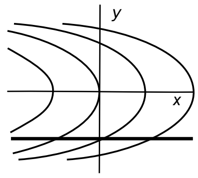
Imponiendo el dato inicial: $\begin{array}{l}u(x,-2)=p(-x-2)\,e^{-2}+x=x\to p(v)\equiv 0\\ u(x,-2)=q(-x-2)\,e^x+x=x\to q(v)\equiv 0\end{array}$, $\boxed{u(x,y)=x+y+2}$.

Como los datos se dan sobre una recta no característica, la solución debía ser única.

b) $\boxed{\begin{array}{l}u_y-2yu_x=4xy\\ u(x,-1)=2x+1\end{array}}$ $\frac{dy}{dx}=-\frac{1}{2y}\to\xi=x+y^2$ $\begin{array}{l}\nearrow \eta=y\to u_\eta=4\eta\xi-4\eta^3,\ u=2\eta^2\xi-\eta^4+p(\xi)=2y^2x+y^4+p(x+y^2)\\ \searrow \eta=x\to u_\eta=-2\eta,\ u=q(\xi)-\eta^2=q(x+y^2)-x^2\end{array}$

$u(x,-1)=2x+1$ $\begin{array}{l}\nearrow p(x+1)+2x+1=2x+1\to p(v)=0\ \searrow\\ \searrow q(x+1)-x^2=2x+1\to q(v)=v^2\ \nearrow\end{array}$ $\boxed{u(x,y)=2y^2x+y^4}$

Solución única, pues $y=1$ no tangente a las características o $T=1\cdot 1-0(-2)=1\neq 0$.



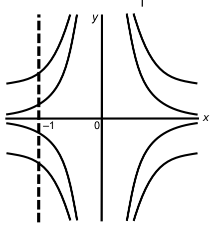
c) $\boxed{\begin{array}{l}3yu_y-xu_x=2xyu\\ u(-1,y)=1\end{array}}$ $\frac{dy}{dx}=-\frac{3y}{x}\overset{\text{lineal}}{\longrightarrow}y=\frac{C}{x^3}$. O bien $\begin{cases}\xi=x^3y\\ \eta=x\end{cases}\to u_\eta=-2yu=-2\xi\eta^{-3}u$,

$u=p(\xi)e^{\xi\eta^{-2}}$, $u(x,y)=p(x^3y)\,e^{xy}$ solución general

O bien $\begin{cases}\xi=x^3y\\ \eta=y\end{cases}$, $u_\eta=\frac{2}{3}xu=\frac{2}{3}\xi^{1/3}\eta^{-1/3}u$, $u=p(\xi)e^{\xi^{1/3}\eta^{2/3}}$, $u(x,y)=p(x^3y)\,e^{xy}$.

Imponiendo el dato: $u(-1,y)=p(-y)e^{-y}=1$, $p(v)=e^{-v}\to\boxed{u(x,y)=e^{(x-x^3)y}}$.

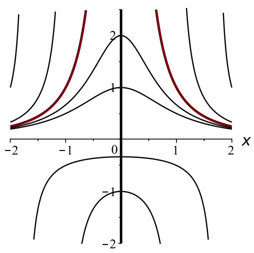
Única por no ser $x=-1$ tangente a las características. O porque en el cálculo quedó $p(v)$ determinada de forma única $\forall v$. O porque $T(y)=0\cdot 3y-1\cdot(+1)=-1\neq 0\ \forall y$.



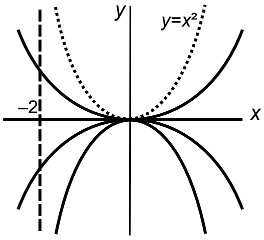
d) $\boxed{\begin{array}{l}2xy^2u_y-u_x=2xyu\\ u(0,y)=1\end{array}}$ $\frac{dy}{dx}=-2xy^2$, $\boxed{\frac{1}{y}-x^2=C}$. $\begin{cases}\xi=\frac{1}{y}-x^2\\ \eta=y\end{cases}\to u_\eta=\frac{1}{\eta}u$,

$u=p(\xi)\,\eta=p\big(\frac{1}{y}-x^2\big)\,y$. Peor: $\begin{cases}\xi=\frac{1}{y}-x^2\\ \eta=x\end{cases}\to u_\eta=-\frac{2\eta}{\xi+\eta^2}u$, $u=\frac{p(\xi)}{\xi+\eta^2}=p\big(\frac{1}{y}-x^2\big)\,y$.

$u(0,y)=p\big(\frac{1}{y}\big)y=1$, $p(v)=v$, $\boxed{u(x,y)=1-x^2y}$. Única porque la recta $x=0$ no es tangente a las características. O porque $T(y)=0\cdot 0-1\cdot(-1)=1\neq 0\ \forall y$.



3 $\boxed{2yu_y+xu_x=2u-2y^2}$ $\frac{dy}{dx}=\frac{2y}{x} \to y=Cx^2$. $\begin{cases}\xi=y/x^2\\ \eta=y\end{cases} \to \begin{cases}u_y=\frac{1}{x^2}u_\xi+u_\eta\\ u_x=-\frac{2y}{x^3}u_\xi\end{cases} \to$



$2yu_\eta=2u-2y^2$, $u_\eta=\frac{u}{\eta}u-\eta \to u=p(\xi)\,\eta-\eta\int d\eta=p(\xi)\,\eta-\eta^2$. $u(x,y)=p\left(\frac{y}{x^2}\right)y-y^2$.

Peor $\begin{cases}\xi=y/x^2\\ \eta=x\end{cases} \to xu_\eta=2u-2y^2$, $u_\eta=\frac{2u}{\eta}-2\xi^2\eta^3 \to u=q(\xi)\,\eta^2-\xi^2\eta^4=q\left(\frac{y}{x^2}\right)x^2-y^2$.

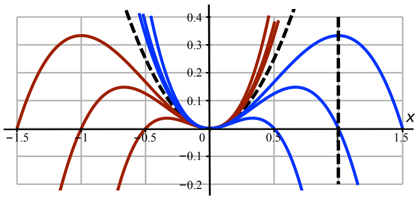
Imponiendo el dato: $p\left(\frac{y}{4}\right)y-y^2=4-y^2$, $p\left(\frac{y}{4}\right)=\frac{4}{y}$, $p(v)=\frac{1}{v}$, $\boxed{u(x,y)=x^2-y^2}$.

o bien: $q\left(\frac{y}{4}\right)4-y^2=4-y^2$, $q\left(\frac{y}{4}\right)=1$, $q(v)\equiv 1$, $u(x,y)=x^2-y^2$.

[Solución única por no ser tangente $x=-2$ a las características ó porque $T(y)=0\cdot(2y)-(-2)\cdot 1\equiv 2\neq 0$].

Para el otro dato (sobre característica) en principio puede haber infinitas o ninguna solución.
Imponemos el dato para saber lo que sucede: $p(1)x^2-x^4=0 \to p(1)=x^2$ [ó $q(1)=x^2$]. **Ninguna solución**.

4 $\boxed{(3y-x^2)u_y+xu_x=3u}$ $\frac{dy}{dx}=\frac{3y}{x}-x$. $y=Cx^3-x^3\int\frac{dx}{x^2}=Cx^3+x^2$, $\frac{y}{x^3}-\frac{1}{x}=C$.



$\begin{cases}\xi=yx^{-3}-x^{-1}\\ \eta=x\end{cases} \to \begin{cases}u_y=x^{-3}u_\xi\\ u_x=(-3x^{-4}y+x^{-2})u_\xi+u_\eta\end{cases}$, $xu_\eta=3u$, $u_\eta=\frac{3}{\eta}u$.

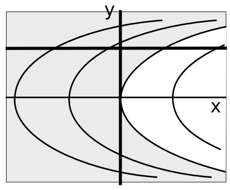
Su solución es $u=p(\xi)\eta^3$, o sea, $u(x,y)=p\left(\frac{y}{x^3}-\frac{1}{x}\right)x^3$.

Imponiendo el dato: $u(1,y)=p(y-1)=y$, $p(v)=v+1$, $\boxed{u(x,y)=y-x^2+x^3}$.

Solución única por no ser tangente a las características, ya que, por ejemplo, $T(y)=0-1\cdot 1=-1\neq 0\ \forall y$.

Datos sobre característica. $u(x,x^2)=p(0)x^3$ hace ii) imposible y proporciona infinitas soluciones para i) [pues se cumple $\forall p$ con $p(0)=0$]. Dos ejemplos cortos son $u=0$, $u=y-x^2$ [eligiendo $p(v)=0$, $p(v)=v$].

5 $\boxed{u_y+2yu_x=3xu}$ con: i) $u(x,1)=1$, ii) $u(0,y)=0$.



$\frac{dy}{dx}=\frac{1}{2y} \to x-y^2=K \to \begin{cases}\xi=x-y^2\\ \eta=y\end{cases} \to u_\eta=3xu=(3\xi+3\eta^2)u$

$\to u=p(\xi)e^{3\xi\eta+\eta^3}=p(x-y^2)\,e^{3xy-2y^3}$.

[Es bastante más largo con $\begin{cases}\xi=x-y^2\\ \eta=x\end{cases} \to 2yu_\eta=3xu$, $u_\eta=\frac{3\eta}{[\xi-\eta]^{1/2}}u$, ...].

i) $p(x-1)\,e^{3x-2}=1$, $p(v)=e^{-3v-1} \to \boxed{u=e^{3xy-3x-2y^3+3y^2-1}}=e^{(y-1)(3x-2y^2+y+1)}$ (única; $T\equiv 1$).

ii) $u(0,y)=p(-y^2)\,e^{-2y^3}=0 \to p(v)\equiv 0$, si $v\leq 0$, pero indeterminada si $v>0 \to u\equiv 0$, si $x\leq y^2$, indeterminada si $x>y^2 \to$ solución única excepto en un entorno del origen ($T=-2y$).

6 a) $\boxed{u_{yy}+4u_{xy}+5u_{xx}+u_y+2u_x=x}$ $B^2-4AC=-4$ elíptica $\begin{cases}\xi=x-2y\\ \eta=y\end{cases}$, $\begin{cases}u_y=-2u_\xi+u_\eta\\ u_x=u_\xi\end{cases}$, $\begin{cases}u_{yy}=4u_{\xi\xi}-4u_{\xi\eta}+u_{\eta\eta}\\ u_{xy}=-2u_{\xi\xi}+u_{\xi\eta}\\ u_{xx}=u_{\xi\xi}\end{cases}$

$\to \boxed{u_{\xi\xi}+u_{\eta\eta}+u_\eta=\xi+2\eta}$ (no resoluble).

b) $\boxed{u_{yy}+6u_{xy}+9u_{xx}+9u=9}$ $B^2-4AC=0$ parabólica $\begin{cases}\xi=x-3y\\ \eta=y\end{cases} \to \begin{cases}u_{xx}=u_{\xi\xi}\\ u_{xy}=-3u_{\xi\xi}+u_{\xi\eta}\\ u_{yy}=9u_{\xi\xi}-6u_{\xi\eta}+u_{\eta\eta}\end{cases} \to \boxed{u_{\eta\eta}+9u=9}$

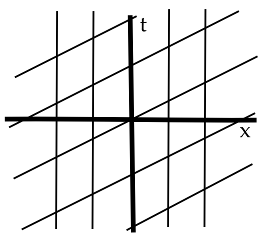
$\xrightarrow{\mu^2+9=0}$ $u=p(\xi)\cos 3\eta+q(\xi)\,\mathrm{sen}\,3\eta+1$, $\boxed{u(x,y)=p(x-3y)\cos 3y+q(x-3y)\,\mathrm{sen}\,3y+1}$ [problema de **R**, solución en **R**].

c) $\boxed{u_{xx}+4u_{xy}-5u_{yy}+6u_x+3u_y=9u}$ Hiperbólica $\begin{cases}\xi=x-\frac{y}{5}\\ \eta=x+y\end{cases}$ ó $\begin{cases}\xi=5x-y\\ \eta=x+y\end{cases} \to \boxed{4u_{\xi\eta}+3u_\xi+u_\eta=u}$ no resoluble

d) $\boxed{3u_{tt}-2u_{xt}-u_{xx}+8u_t-8u_x=0}$ $B^2-4AC=16$ hiperbólica $\begin{cases}\xi=x+t\\ \eta=x-\frac{t}{3}\end{cases}$, $\begin{cases}u_t=u_\xi-\frac{1}{3}u_\eta\\ u_x=u_\xi+u_\eta\end{cases}$, $\begin{cases}u_{tt}=u_{\xi\xi}-\frac{2}{3}u_{\xi\eta}+\frac{1}{9}u_{\eta\eta}\\ u_{xt}=u_{\xi\xi}+\frac{2}{3}u_{\xi\eta}-\frac{1}{3}u_{\eta\eta}\\ u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\end{cases}$, $\begin{matrix}u_{\xi\eta}+2u_\eta=0\\ v_\xi=-2v\end{matrix}$

$v=p^*(\eta)\,e^{-2\xi}=u_\eta$, $u=p(\eta)\,e^{-2\xi}+q(\xi)$, $\boxed{u=p\left(x-\frac{t}{3}\right)e^{-2x-2t}+q(x+t)}$.

7 (E) $\boxed{u_{tt}+2u_{xt}=2}$ Hiperbólica $\begin{cases}\xi=x-2t\\ \eta=x\end{cases} \to \begin{cases}u_{tt}=4u_{\xi\xi}\\ u_{tx}=-2u_{\xi\xi}-2u_{\xi\eta}\end{cases} \to$



$u_{\xi\eta}=-\frac{1}{2} \to u=p(\xi)+q(\eta)-\frac{\xi\eta}{2}=\boxed{u=p(x-2t)+q(x)-\frac{x^2}{2}+xt}$

i) $y=0$ no tangente a las características → solución única con:

$\begin{cases}0=u(x,0)=p(x)+q(x)-\frac{1}{2}x^2 & q(x)=\frac{1}{4}x^2-C\\ 0=u_t(x,0)=-2p'(x)+x \to p(x)=\frac{1}{4}x^2+C \nearrow\end{cases} \to \boxed{u=t^2}$

ii) $u(0,t)=0$, $u_x(0,t)=t$ son datos sobre característica: $\left.\begin{matrix}p(-2t)+q(0)=0\\ p'(-2t)+q'(0)+t=t\end{matrix}\right\}$ Cada q con $q'(0)=0$, y $p(t)\equiv -q(0)$ da una solución distinta (infinitas).

[(E) se puede resolver también: $u_t=v$, $v_t+2v_x=2$, ... O bien: $[u_t+2u_x]_t=2$, $u_t+2u_x=2t+q(x)$, ...]

8 a) $\boxed{u_{tt}+4u_{tx}+4u_{xx}+u_t+2u_x=0}$ $\underset{\text{parabólica}}{B^2-4AC=0}$ $\begin{cases}\xi=x-2t\\ \eta=t\end{cases} \to \begin{cases}u_x=u_\xi\\ u_t=-2u_\xi+u_\eta\end{cases}, \begin{cases}u_{xx}=u_{\xi\xi}\\ u_{xt}=-2u_{\xi\xi}+u_{\xi\eta}\\ u_{tt}=4u_{\xi\xi}-4u_{\xi\eta}+u_{\eta\eta}\end{cases} \to$

$\boxed{u_{\eta\eta}+u_\eta=0} \xrightarrow{\lambda(\lambda+1)=0} u=p(\xi)+q(\xi)\,e^{-\eta}$, $\boxed{u(x,t)=p(x-2t)+q(x-2t)\,e^{-t}}$

$\to u_t=-2p'(x-2t)-\big[2q'(x-2t)+q(x-2t)\big]e^{-t}$.

$$\begin{cases}u(x,0)=p(x)+q(x)=1-x\,,\ p'(x)+q'(x)=-1\\ u_t(x,0)=-2p'(x)-2q'(x)-q(x)=1\,,\ q(x)=1\ \to\ p(x)=-x\end{cases} \nearrow \boxed{u(x,t)=2t-x+e^{-t}}.$$

$u_t=2-e^t$, $u_x=-1$, $u_{tt}=e^t$, $u_{tx}=u_{xx}=0$; $2-e^t+2-e^t-1=0$; $u(x,0)=-x+1$, $u_t(x,0)=2-1$.

b) $\boxed{u_{yy}+4u_{xy}+4u_{xx}-4u=5e^{x+y}}$ $B^2-4AC=0$ parabólica, $\begin{cases}\xi=x-2y\\ \eta=y\end{cases} \to \begin{cases}u_y=-2u_\xi+u_\eta\\ u_x=u_\xi\end{cases}, \begin{cases}u_{yy}=4u_{\xi\xi}-4u_{\xi\eta}+u_{\eta\eta}\\ u_{xy}=-2u_{\xi\xi}+u_{\xi\eta}\\ u_{xx}=u_{\xi\xi}\end{cases}$

$\to \boxed{u_{\eta\eta}-4u=5e^{\xi+3\eta}}$ forma canónica EDO lineal de 2° orden en η con ξ constante al resolverla.

Homogénea: $\mu^2-4=0$, $\mu=\pm2$. Para la particular se prueba $u_p=Ae^{3\eta}\to 9A-4A=5e^{\xi}$, $A=e^{\xi}\to$

$u=p(\xi)\,e^{2\eta}+q(\xi)\,e^{-2\eta}+e^{\xi+3\eta}=\boxed{p(x-2y)\,e^{2y}+q(x-2y)\,e^{-2y}+e^{x+y}}$ solución general

$$\begin{cases}p(x)+q(x)+e^x=0,\ q=-p-e^x \searrow \qquad\qquad q(x)=0\\ -2(p'+q')+2(p-q)+e^x=e^x\quad 2e^x+4p+2e^x=0,\ p(x)=-e^x\uparrow\end{cases} \to \boxed{u(x,y)=e^{x+y}-e^x}$$ (fácil de comprobar).

c) $\boxed{u_{yy}-2u_{xy}+u_{xx}+u=x+y}$ $\underset{\text{parabólica}}{B^2-4AC=0}$ $\begin{cases}\xi=x+y\\ \eta=y\end{cases} \to \begin{cases}u_y=u_\xi+u_\eta\\ u_x=u_\xi\end{cases}, \begin{cases}u_{yy}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\\ u_{xy}=u_{\xi\xi}+u_{\xi\eta}\\ u_{xx}=u_{\xi\xi}\end{cases} \to \boxed{u_{\eta\eta}+u=\xi}$.

$\mu^2+1=0$, $\mu=\pm i$. $u_p=\xi$ a ojo. $u=p(\xi)\cos\eta+q(\xi)\,\mathrm{sen}\,\eta+\xi=\boxed{p(x+y)\cos y+q(x+y)\,\mathrm{sen}\,y+x+y}$.

$$\begin{cases}p(x)+x=x,\ p(x)=0\searrow\\ \qquad q(x)\cos0+1=0,\ q(x)=-1\end{cases} \to \boxed{u(x,y)=x+y-\mathrm{sen}\,y}.$$ $\mathrm{sen}\,y-0+0+x+y-\mathrm{sen}\,y=x+y$.
$u(x,0)=x+0-0$, $u_y(x,0)=1-\cos0=0$.

9 a) $\boxed{\begin{matrix}u_{tt}-4u_{xx}=e^{-t},\ x,t\in\mathbf{R}\\ u(x,0)=x^2,\ u_t(x,0)=-1\end{matrix}}$ $u=\frac12\big[(x+2t)^2+(x-2t)^2\big]+\frac14\int_{x-2t}^{x+2t}ds+\frac14\int_0^t\!\int_{x-2[t-\tau]}^{x+2[t-\tau]}e^{-\tau}\,ds\,d\tau$
$=x^2+4t^2+e^{-t}-1$.

Una solución particular que sólo depende de t es: $v_{tt}=e^{-t}\to v=e^{-t}$. Con $w=u-e^{-t}$ se tiene:

$$\begin{cases}w_{tt}-w_{xx}=0\\ w(x,0)=x^2-1,\ w_t(x,0)=0\end{cases} \to w=\tfrac12\big[(x+2t)^2-1+(x-2t)^2-1\big]=x^2+4t^2-1$$, como antes.

b) $\boxed{\begin{matrix}u_{tt}-4u_{xx}=16,\ x,t\in\mathbf{R}\\ u(0,t)=t,\ u_x(0,t)=0\end{matrix}}$ Lo más sencillo es cambiar papeles de x y t aplicar D'Alembert: $\xrightarrow{x\leftrightarrow t} \begin{cases}u_{tt}-\frac14u_{xx}=-4,\ x,t\in\mathbf{R}\\ u(x,0)=x,\ u_t(x,0)=0\end{cases} \to$

$$u=\tfrac12\big[(x+\tfrac{t}{2})+(x-\tfrac{t}{2})\big]-4\int_0^t\!\int_{x-\frac12(t-\tau)}^{x+\frac12(t-\tau)}ds\,d\tau=x-4\int_0^t(t-\tau)d\tau=x-2t^2 \xrightarrow{x\leftrightarrow t} \boxed{u=t-2x^2}.$$

Podríamos ahorrarnos esta integral doble con una solución v que sólo dependiese de una variable:

$$v''(t)=-4,\ v=-2t^2 \underset{w=u-v}{\longrightarrow} \begin{cases}w_{tt}-\frac14w_{xx}=0\\ w(x,0)=x,\ w_t(x,0)=0\end{cases},\ w=\tfrac12\big[(x+\tfrac{t}{2})+(x-\tfrac{t}{2})\big]=x\to u=x-2t^2$$

$$v''(x)=16,\ v=8x^2 \underset{w=u-v}{\longrightarrow} \begin{cases}w_{tt}-\frac14w_{xx}=0\\ w(x,0)=x-8x^2,\ w_t(x,0)=0\end{cases},\ w=x-4\big[(x+\tfrac{t}{2})^2+(x-\tfrac{t}{2})^2\big]=x-8x^2-2t^2\ldots$$

Sin atajos: $\begin{cases}\xi=x+2t\\ \eta=x-2t\end{cases} \xrightarrow{\text{forma canónica}} u_{\xi\eta}=-1\to$ $u=p(\xi)+q(\eta)-\xi\eta=p(x+2t)+q(x-2t)+4t^2-x^2$

$$\begin{cases}u(0,t)=p(2t)+q(-2t)+4t^2=t\to 2p'(2t)-2q'(-2t)=1-8t\\ u_x(0,t)=p'(2t)+q'(-2t)=0\ \to q'(-2t)=-p'(2t)\end{cases} \to p'(2t)=\tfrac14-2t,\ p'(v)=\tfrac14-v,\ p(v)=\tfrac{v}{4}-\tfrac{v^2}{2}+K$$

$\to q(-v)=\frac{v}{2}-v^2-p(v)=\frac{v}{4}-\frac{v^2}{2}-K$, $q(v)=-\frac{v}{4}-\frac{v^2}{2}-K\to u=\frac{x+2t}{4}-\frac{x-2t}{4}-\frac{(x+2t)^2}{2}-\frac{(x-2t)^2}{2}+4t^2-x^2\uparrow$

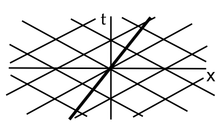
c) $\boxed{\begin{matrix}u_{tt}-4u_{xx}=2\\ u(x,x)=x^2,\ u_t(x,x)=x\end{matrix}}$ Directamente: en apuntes características y cambio: $\begin{cases}\xi=x+2t\\ \eta=x-2t\end{cases}\to -16u_{\xi\eta}=2$

$\to u_\xi=p^*(\xi)-\frac{\eta}{8}\ \to\ u=p(\xi)+q(\eta)-\frac{\xi\eta}{8}=p(x+2t)+q(x-2t)+\frac{4t^2-x^2}{8}$

$$\begin{cases}u(x,x)=p(3x)+q(-x)+\frac{3x^2}{8}=x^2\ \to\ 3p'(3x)-q'(-x)=\frac{5x}{4}\\ u_y(x,x)=2p'(3x)-2q'(-x)+x=x\ \to\ q'(-x)=p'(3x)\end{cases} \to p'(3x)=\tfrac{5x}{8},\ p'(v)=\tfrac{5v}{24},\ p(v)=\tfrac{5v^2}{48}\to$$

$q(-x)=x^2-\frac{3x^2}{8}-\frac{15x^2}{16}=-\frac{5x^2}{16}$, $q(v)=-\frac{5v^2}{16}$. $u=\frac{5(x+2t)^2-15(x-2t)^2+24t^2-6x^2}{48}=\boxed{\frac13\big[5xt-t^2-x^2\big]}$.

Haciendo $w=u-t^2\to \begin{cases}w_{tt}-4w_{xx}=0\\ w(x,x)=0,\ w_t(x,x)=-x\end{cases}$, $w=p(x+2t)+q(x-2t)$ [apuntes].



$$\begin{cases}w(x,x)=p(3x)+q(-x)=0\\ w_y(x,x)=2p'(3x)-2q'(-x)=-x\end{cases}\to p'(3x)=\tfrac{x}{4},\ p(v)=\tfrac{v^2}{24},\ q(v)=-\tfrac{3v^2}{8},\ \ldots$$

10 (E) $\boxed{Au_{yy}+Bu_{xy}+Cu_{xx}+Du_y+Eu_x+Hu=F(x,y)}$ si no es parabólica, $B^2-4AC\neq 0$. $u=e^{py}e^{qx}w\to$

$$u_y=[pw+w_y]\,e^{py+qx}\qquad u_{yy}=[p^2w+2pw_y+w_{yy}]\,e^{py+qx}$$
$$u_x=[qw+w_x]\,e^{py+qx}\qquad u_{xy}=[pqw+pw_x+qw_y+w_{xy}]\,e^{py+qx}\to$$
$$u_{xx}=[q^2w+2qw_x+w_{xx}]\,e^{py+qx}$$
$$Aw_{yy}+Bw_{xy}+Cw_{xx}+(2pA+qB+D)w_y+(2qC+pB+E)w_x+(p^2A+pqB+q^2C+pD+qE+H)w=e^{-py-qx}F(x,y)$$

Si $\left.\begin{matrix}2pA+qB+D=0\\2qC+pB+E=0\end{matrix}\right\}\to\boxed{p=\frac{2CD-BE}{B^2-4AC}}$, es: (E*) $\boxed{Aw_{yy}+Bw_{xy}+Cw_{xx}+\left[\frac{AE^2+CD^2-BDE}{B^2-4AC}+H\right]w=e^{-py-qx}F(x,y)}$.

Si el corchete se anula y la ecuación es hiperbólica, se puede poner $u_{\xi\eta}=F^*$ y es resoluble.

$\boxed{u_{xy}+2u_y+3u_x+6u=1}\to B^2-4AC=1,\ p=-3,\ q=-2,\ [\ \ ]=\left[\frac{-6}{1}+6\right]=0;$

$u=e^{-3y-2x}w\to w_{xy}=e^{3y+2x},\ w=\frac{1}{6}e^{3y+2x}+p(x)+q(y)\to\boxed{u=\frac{1}{6}+e^{-3y}e^{-2x}[p(x)+q(y)]}$.

Si (E) es parabólica se puede poner en la forma canónica: $u_{\eta\eta}+D^*u_\eta+E^*u_\xi+H^*u=F^*(\xi,\eta)$.

Si $E^*=0$, la ecuación (lineal de segundo orden con coeficientes constantes en η) es resoluble.

$u=e^{py}e^{qx}w\to w_{\eta\eta}+(2p+D^*)w_\eta+E^*w_\xi+(p^2+pD^*+qE^*+H^*)w=e^{-py-qx}F^*(\xi,\eta)\equiv F^{**}(\xi,\eta)$.

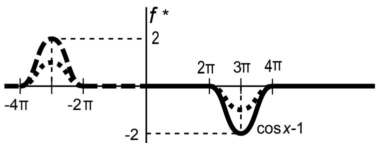
Si $E^*\neq 0$, con $\boxed{p=-\frac{D^*}{2}}$, $\boxed{q=\frac{1}{E^*}\left[\frac{D^{*2}}{4}-H^*\right]}$ se convierte en la del calor: $\boxed{w_\xi+\frac{1}{E^*}w_{\eta\eta}=F^{***}(\xi,\eta)}$.

11 $\boxed{\begin{matrix}u_{tt}-4u_{xx}=0,\ x\geq 0,\ t\in\mathbf{R}\\ u(x,0)=\begin{cases}\cos x-1,\ x\in[2\pi,4\pi]\\0,\ x\in[0,2\pi]\cup[4\pi,\infty)\end{cases}\\ u_t(x,0)=u(0,t)=0\end{matrix}}$

Extendemos a f^* impar definida en todo $\mathbf{R}$.

$u=\frac{1}{2}\left[f^*(x+2t)+f^*(x-2t)\right]\to$

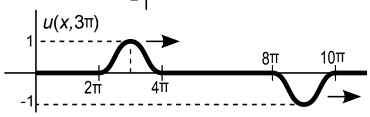
$u(3\pi,3\pi)=\frac{1}{2}\left[f^*(9\pi)+f^*(-3\pi)\right]=\frac{-f(3\pi)}{2}=\boxed{1}$



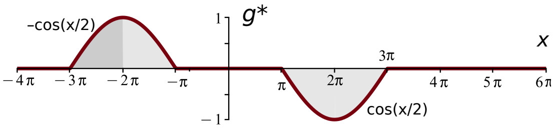
Para dibujar $u(x,3\pi)=\frac{1}{2}\left[f^*(x+6\pi)+f^*(x-6\pi)\right]$ basta $\frac{1}{2}f^*$:

Si $t\geq 3\pi$ es $u(2\pi,t)=\frac{1}{2}\left[f^*(2\pi+2t)+f^*(2\pi-2t)\right]=\boxed{0}$, pues $\begin{matrix}2\pi+2t\geq 8\pi,\\2\pi-2t\leq-4\pi.\end{matrix}$

[A partir de ese t es superado 2π por la onda rebotada viajando hacia la derecha].



12 $\boxed{\begin{matrix}u_{tt}-u_{xx}=0,\ x\geq 0,\ t\in\mathbf{R}\\ u_t(x,0)=\begin{cases}\cos(x/2),\ x\in[\pi,3\pi]\\0,\ x\in[0,\pi]\cup[3\pi,\infty)\end{cases}\\ u(x,0)=u(0,t)=0\end{matrix}}$ $\quad g^*(x)=\begin{cases}-\cos\frac{x}{2},\ x\in[-3\pi,\pi]\\ \cos\frac{x}{2},\ x\in[\pi,3\pi]\\ 0,\ \text{resto de }\mathbf{R}\end{cases}$



$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t}g^*\to u(2\pi,3\pi)=\frac{1}{2}\int_{-\pi}^{5\pi}g^*=\frac{1}{2}\int_{\pi}^{3\pi}\cos\frac{x}{2}\,dx=\operatorname{sen}\frac{x}{2}\Big]_{\pi}^{3\pi}=\boxed{-2}$.

$u(2\pi,4\pi)=\frac{1}{2}\int_{-2\pi}^{6\pi}g^*=(g^*\text{ impar})=\frac{1}{2}\int_{2\pi}^{3\pi}\cos\frac{x}{2}\,dx=\operatorname{sen}\frac{x}{2}\Big]_{2\pi}^{3\pi}=\boxed{-1}$.

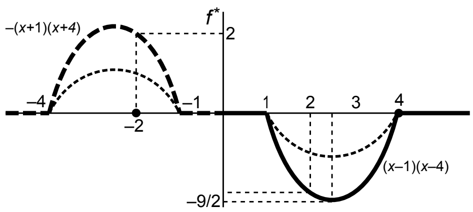
Por ser, si $t\geq 5\pi$, $2\pi-t\leq-3\pi$ y $2\pi+t\geq 7\pi>3\pi$ y ser g^* impar, será $u(2\pi,t)=\frac{1}{2}\int_{2\pi-t}^{2\pi+t}g^*=\boxed{0}$.

13 $\boxed{\begin{matrix}u_{tt}-u_{xx}=0,\ x\geq 0,\ t\in\mathbf{R}\\ u(x,0)=\begin{cases}(x-1)(x-4),\ x\in[1,4]\\0,\ x\in[0,1]\cup[4,\infty)\end{cases}\\ u_t(x,0)=u(0,t)=0\end{matrix}}$

Hay que extender impar a todo $\mathbf{R}$:

a] $u(1,3)=\frac{1}{2}\left[f^*(4)+f^*(-2)\right]$

$=\frac{1}{2}\left[f(4)-f(2)\right]=\frac{1}{2}[0+2]=\boxed{1}$.

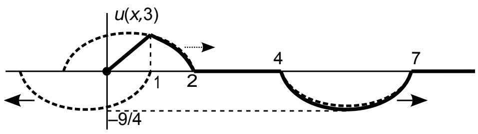


b] $u(x,3)=\frac{1}{2}\left[f^*(x+3)+f^*(x-3)\right]$. Hay que trasladar la gráfica de $\frac{1}{2}f^*$ a izquierda y derecha 3 unidades y sumar:

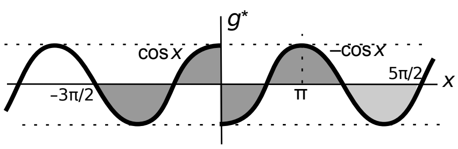
En ese instante están la onda que va hacia la derecha y la suma de la que va hacia la izquieda con la extensión que viene. En concreto:

c] Si $x\in[0,1]$, $f^*(x+3)$ la da la incial y $f^*(x-3)$ la extensión:

$u(x,3)=\frac{1}{2}\left[f(x+3)-f(3-x)\right]=\frac{1}{2}\left[(x+2)(x-1)-(2-x)(-x-1)\right]=\boxed{x}$



14 $\boxed{\begin{matrix}u_{tt}-u_{xx}=0,\ x\geq 0,\ t\in\mathbf{R}\\ u(x,0)=u_t(x,0)=0,\ u(0,t)=\operatorname{sen}t\end{matrix}}\ \xrightarrow{w=u-v}\ \begin{cases}w_{tt}-w_{xx}=0,\ x\geq 0\\ w_t(x,0)=-\cos x\\ w(x,0)=w(0,t)=0\end{cases}$



Extendemos de modo impar $g(x)=-\cos x$ a $g^*(x)$ definida $\forall x$.

La solución viene dada entonces por $u(x,t)=v+\frac{1}{2}\int_{x-t}^{x+t}g^*(s)\,ds$.

$u\left(\frac{\pi}{2},2\pi\right)\underset{\operatorname{sen}2\pi=0}{=}0+\frac{1}{2}\int_{-3\pi/2}^{5\pi/2}g^*\underset{\text{impar}}{=}\frac{1}{2}\int_{3\pi/2}^{5\pi/2}g=\frac{1}{2}\left[\operatorname{sen}\frac{3\pi}{2}-\operatorname{sen}\frac{5\pi}{2}\right]=\frac{1}{2}[-1-1]=\boxed{-1}$.

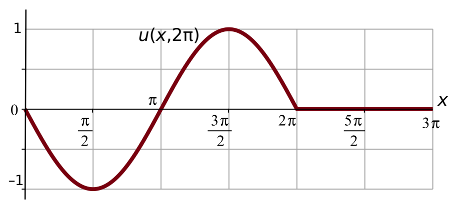
i) Para $x\geq 2\pi$ es $x-2\pi$ positivo, luego: $u(x,2\pi)=0-\frac{1}{2}\int_{x-2\pi}^{x+2\pi}\cos s\,ds=\frac{1}{2}\left[\operatorname{sen}(x-2\pi)-\operatorname{sen}(x+2\pi)\right]=\boxed{0}$.

[Todavía no llegó la onda, que viaja a velocidad 1].

ii) Ahora es $x-2\pi\leq 0$ y $x+2\pi\leq 0$. Usando la imparidad:

$u(x,2\pi)=-\frac{1}{2}\int_{2\pi-x}^{x+2\pi}\cos s\,ds=\frac{1}{2}\left[\operatorname{sen}(2\pi-x)-\operatorname{sen}(x+2\pi)\right]=\boxed{-\operatorname{sen}x}$.

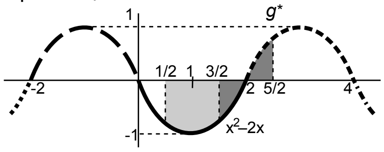
Juntando los resultados se tiene el dibujo de $u(x,2\pi)$ de la derecha: el movimiento del extremo crea la onda que por ahora llega a 2π.



15 $\left\{\begin{array}{l} u_{tt}-u_{xx}=0\,,\ x\in[0,2]\,,t\in\mathbf{R} \\ u(x,0)=0\,,\ u_t(x,0)=(x-1)^2,\ u(0,t)=u(2,t)=t \end{array}\right.$

Para aplicar D'Alembert lo primero es hacer las condiciones de contorno homogéneas. Una v adecuada que las satisface (la que nos dice los apuntes) es $v=t$.

$$\xrightarrow{w=u-v} \left\{\begin{array}{l} w_{tt}-w_{xx}=0\,,\ x\in[0,2],\, t\in\mathbf{R} \\ w(x,0)=0\,,\, w_t(x,0)=x^2-2x \\ w(0,t)=w(2,t)=0 \end{array}\right. \to \left\{\begin{array}{l} w_{tt}-w_{xx}=0\,,\ x,t\in\mathbf{R} \\ w(x,0)=0\,,\, w_t(x,0)=g^* \end{array}\right.$$



con g^* extensión impar y 4-periódica a todo $\mathbf{R}$ de x^2-2x.

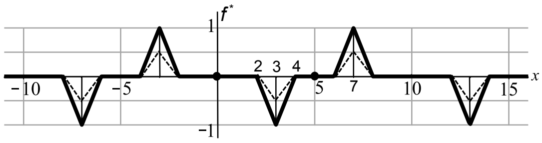
Entonces $u(x,t)=t+\frac{1}{2}\int_{x-t}^{x+t} g^*(s)\,ds \to u(\frac{3}{2},1)=1+\frac{1}{2}\int_{1/2}^{5/2} g^* \underset{g^*\text{ impar}}{=} 1+\frac{1}{2}\int_{1/2}^{3/2}(s^2-2s)\,ds=\boxed{\frac{13}{24}}$.

[No hemos necesitado utilizar la expresión de g^* que en $[2,4]$ sería $-(x-2)(x-4)$].

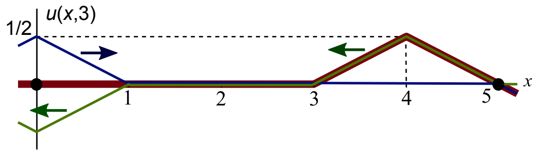
16 $\left\{\begin{array}{l} u_{tt}-u_{xx}=0\,,\ x\in[0,5]\,,t\in\mathbf{R} \\ u(x,0)=\left\{\begin{array}{l} 2-x,\ x\in[2,3]\ ,\ x-4,\ x\in[3,4] \\ 0\,,\ \text{resto de } [0,5] \end{array}\right. \\ u_t(x,0)=u(0,t)=u(5,t)=0 \end{array}\right.$

f^* extensión impar y 10-periódica.

$u=\frac{1}{2}\left[f^*(x+t)+f^*(x-t)\right]$.



$u(4,3)=\frac{1}{2}\left[f^*(7)+f^*(1)\right]=\frac{1}{2}\left[-f(3)+f(1)\right]=\frac{1}{2}[+1+0]=\boxed{\frac{1}{2}}$.



Para dibujar $u(x,3)$ se lleva $\frac{1}{2}f^*$ a derecha e izquierda 3 unidades y se suman las gráficas. En ese instante se cancelan en $[0,1]$. La onda que iba a la derecha ya ha rebotado y se ha invertido, y empieza a ir a la izquierda.

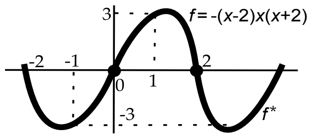
$u(2,t)=\frac{1}{2}\left[f^*(2+t)+f^*(2-t)\right]$. Para $t\in[0,3]$ es $f^*(2-t)=0$.

El valor de $f^*(2+t)=f(2+t)$ depende de t. En concreto: $u(2,t)=\frac{1}{2}\left\{\begin{array}{l} 2-(2+t),\ t\in[0,1] \\ (2+t)-4,\ t\in[1,2] \\ 0\,,\ t\in[2,3] \end{array}\right. = \boxed{\left\{\begin{array}{l} -t/2\,,\ t\in[0,1] \\ t/2-1,\ t\in[1,2] \\ 0\,,\ t\in[2,3] \end{array}\right.}$.

17 $\left\{\begin{array}{l} u_{tt}-4u_{xx}=0\,,\ x\in[0,2],t\in\mathbf{R} \\ u(x,0)=4x-x^3,\ u_t(x,0)=0 \\ u(0,t)=u(2,t)=0 \end{array}\right.$

$\left\{\begin{array}{l} u_{tt}-u_{xx}=0\,,x\in\mathbf{R} \\ u(x,0)=f^*(x),\ u_t(x,0)=0 \end{array}\right.$

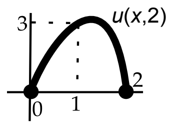
$u=\frac{1}{2}\left[f^*(x+2t)+f^*(x-2t)\right]$.



$u(\frac{3}{2},\frac{3}{4})=\frac{1}{2}\left[f^*(3)+f^*(0)\right] \underset{4\text{-per.}}{=} \frac{1}{2}f^*(-1) \underset{\text{impar}}{=} -\frac{1}{2}f^*(1)=-\frac{3}{2}$.

$u(x,2)=\frac{1}{2}\left[f^*(x+4)+f^*(x-4)\right] \underset{4\text{-per.}}{=} \frac{1}{2}\left[f^*(x)+f^*(x)\right]=f(x)=u(x,0)$.

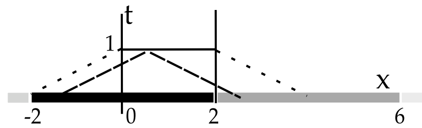
[Sabíamos que era $\frac{2L}{c}$-periódica. Trasladando y sumando sale lo mismo].



Para hallar $u(x,1)$ utilizamos la expresión de f^* en más intervalos:

$f^*(x)=-(x-2)x(x+2)=f(x)$ si $x\in[-2,2]$

$f^*(x)=-(x-6)(x-4)(x-2)$ si $x\in[2,6]$ $\to$

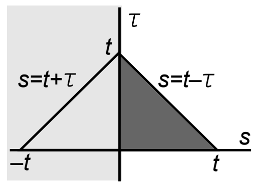


$u(x,1)=\frac{1}{2}\left[f^*(x+2)+f(x-2)\right]=\frac{1}{2}\left[-(x-4)(x-2)x-(x-4)(x-2)x\right]=-(x-4)(x-2)x$.

18 $\left\{\begin{array}{l} u_{tt}-u_{xx}=6x\,,\ x\geq 0,\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=u_x(0,t)=0 \end{array}\right.$

Hay que extender F par respecto a x ($-6x$ si $x\geq 0$).

$$u(0,t)=\frac{1}{2}\int_0^t\int_{-(t-\tau)}^{(t-\tau)} F^*\,ds\,d\tau=\int_0^t\int_0^{t-\tau} 6s\,ds\,d\tau=t^3\,.$$



Otra posibilidad: $v=-x^3$ es solución y cumple el dato de contorno.

$$w=u+x^3 \to \left\{\begin{array}{l} w_{tt}-w_{xx}=0\,,\ x\geq 0 \\ w(x,0)=x^3 \\ w_t(x,0)=w_x(0,t)=0 \end{array}\right. , \left\{\begin{array}{l} w_{tt}-w_{xx}=0\,,\ x\in\mathbf{R} \\ w(x,0)=\left\{\begin{array}{l} x^3,\, x\geq 0 \\ -x^3,\, x\leq 0 \end{array}\right. \\ w_t(x,0)=0 \end{array}\right. \to w(0,t)=\frac{1}{2}\left[t^3+(-(-t)^3)\right]=t^3=u(0,t)\,.$$

19 $\left\{\begin{array}{l} u_{tt}-\left(u_{rr}+\frac{2}{r}u_r\right)=0\,,r\geq 0,\,t\in\mathbf{R} \\ u(r,0)=r\,,\ u_t(r,0)=-2 \end{array}\right.$

$$\xrightarrow{v=ru} \left\{\begin{array}{l} v_{tt}-v_{rr}=0\,,\ r\geq 0 \\ v(r,0)=r^2,\, v_t(r,0)=-2r \text{ [impar]} \\ v(0,t)=0 \end{array}\right. \to \left\{\begin{array}{l} v_{tt}-v_{rr}=0\,,r\in\mathbf{R} \\ v(r,0)=f^*(r)=\left\{\begin{array}{l} r^2\,,\, r\geq 0 \\ -r^2\,,\, r\leq 0 \end{array}\right. \\ v_t(r,0)=-2r \end{array}\right.$$

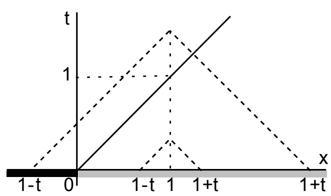
$$v(r,t)=\frac{1}{2}\left[f^*(r+t)+f^*(r-t)\right]-\int_{r-t}^{r+t} s\,ds \to u(r,t)=\frac{1}{2r}\left[f^*(r+t)+f^*(r-t)\right]-2t\,.$$

i] En particular, $u(1,2)=\frac{1}{2}\left[f^*(3)+f^*(-1)\right]-4 \overset{\text{impar}}{=} \frac{1}{2}\left[f(3)-f(1)\right]-4=\boxed{0}$.

ii] Para $u(1,t)$ hay dos casos:

Si $t\leq 1$, $u(1,t)=\frac{1}{2}\left[(1+t)^2+(1-t)^2\right]-2t=\boxed{(1-t)^2}$.

Si $t\geq 1$, $u(1,t)=\frac{1}{2}\left[(1+t)^2-(1-t)^2\right]-2t=\boxed{0}$.



20 a] $\boxed{\begin{cases} 2u_t+u_x=tu \\ u(x,0)=e^{-x^2}\end{cases}}$ i) $\frac{dt}{dx}=2\to\begin{cases}\xi=2x-t\\ \eta=t\end{cases}\to 2u_\eta=\eta u\,,\ u=p(\xi)\,e^{\eta^2/4}=p(2x-t)\,e^{t^2/4}\to$

$u(x,0)=p(2x)=e^{-x^2}\to p(v)=e^{-v^2/4}\to u=e^{-(2x-t)^2/4}\,e^{t^2/4}\to \boxed{u=e^{xt-x^2}}$ [No hay problemas de unicidad: $T=2\ \forall x$].

Haciendo $\eta=x$: $u_\eta=(2\eta-\xi)u\,,\ u=p(\xi)\,e^{\eta^2-\xi\eta}=p(2x-t)\,e^{xt-x^2}\to u(x,0)=p(2x)\,e^{-x^2}=e^{-x^2}\to p(v)\equiv 1$.

ii) $\mathcal{F}(f')=-ik\hat f\,,\ \mathcal{F}(e^{-ax^2})=\frac{e^{-k^2/4a}}{\sqrt{2a}}\to\begin{cases}\hat u_t=\frac{ik}{2}\hat u+\frac{t}{2}\hat u\\ \hat u(k,0)=\frac{1}{\sqrt2}e^{-k^2/4}\end{cases}\to \hat u=p(k)e^{ikt/2}e^{t^2/4}\overset{\text{d. i.}}{\to}\hat u=\frac{1}{\sqrt2}\,e^{t^2/4}e^{-k^2/4}e^{ikt/2}$

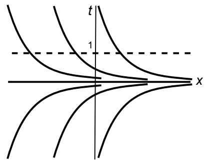
$\to u=e^{t^2/4}\,\mathcal{F}^{-1}\big[\frac{e^{-k^2/4}}{\sqrt2}\,e^{ikt/2}\big]=e^{t^2/4}e^{-(x-\frac{t}{2})^2}=e^{xt-x^2}$, pues $\mathcal{F}^{-1}\big[\frac{e^{-k^2/4}}{\sqrt2}\big]=e^{-x^2}$ y $\mathcal{F}^{-1}\big[\hat f(k)\,e^{iak}\big]=f(x-a)$.

b] $\boxed{\begin{cases} tu_t-u_x=u\\ u(x,1)=f(x)\end{cases}}$ i) $\frac{dt}{dx}=-t$ (lineal), $t=Ce^{-x}\to te^x=C$ características. $\begin{cases}\xi=te^x\\ \eta=t\end{cases}\to\eta u_\eta=u\to$

$u=p(\xi)\,\eta=p\big(te^x\big)\,t$. $u(x,1)=p\big(e^x\big)=f(x)$, $p(v)=f(\ln v)\to\boxed{u(x,t)=tf\big(x+\ln t\big)}$

$\Big[$Con $\begin{cases}\xi=te^x\\ \eta=x\end{cases}$ queda $-u_\eta=u\to u=p^*(\xi)\,e^{-\eta}=p^*\big(te^x\big)\,e^{-x}\overset{d.i.}{\to}p^*(v)\overset{\uparrow}{=}vf(\ln v)\Big]$.

$\big[$Única: $t=1$ no tangente a las características, o $T=1\cdot1-0\cdot(-1)=1\neq0\ \forall x\big]$.



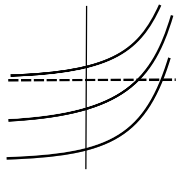
ii) $\begin{cases}t\hat u_t+ik\hat u=\hat u\\ \hat u(k,1)=\hat f(k)\end{cases}\to\hat u_t=\frac{1-ik}{t}\hat u\to\hat u(k,t)=p(k)\,e^{\ln t-ik\ln t}\overset{c.i.}{\longrightarrow}p(k)=\hat f(k)\to\hat u(k,t)=t\hat f(k)\,e^{-ik\ln t}$.

Y como $\mathcal{F}^{-1}\big[\hat f(k)\,e^{ika}\big]=f(x-a)$, la solución es $\boxed{u(x,t)=tf\big(x+\ln t\big)}$, como antes.

c] $\boxed{\begin{cases}u_t+e^tu_x+2tu=0\\ u(x,0)=f(x)\end{cases}}$ i) $\frac{dt}{dx}=\frac{1}{e^t}$, $x=\int e^tdt+C\to x-e^t=C$. $\begin{cases}\xi=x-e^t\\ \eta=t\text{ (mejor)}\end{cases}\to$

$u_\eta=-2tu\to u=p(\xi)\,e^{-\eta^2}=p\big(x-e^t\big)\,e^{-t^2}$. [Con $\eta=x$ queda fea: $u_\eta=\frac{2\log(\eta-\xi)\,u}{\xi-\eta}$].

$u(x,0)=p(x-1)=f(x)$, $p(v)=f(v+1)\to\boxed{u(x,t)=f\big(x-e^t+1\big)\,e^{-t^2}}$.



$\big[$Solución única, pues $t=0$ no es tangente a las características, o porque: $T=1\cdot1-0\cdot1=1\neq0\ \forall x\big]$.

ii) $\begin{cases}\hat u_t-ike^t\hat u+2t\hat u=0\\ \hat u(k,0)=\hat f(k)\end{cases}\to\hat u(k,t)=p(k)\,e^{ike^t-t^2}\overset{c.i.}{\longrightarrow}p(k)\,e^{ik}=\hat f(k)\to\hat u(k,t)=\hat f(k)\,e^{-t^2}e^{ik(e^t-1)}$.

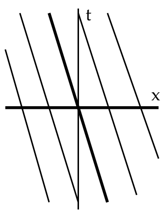
Usando $\mathcal{F}^{-1}\big[\hat f(k)\,e^{ika}\big]=f(x-a)$, llegamos a lo de antes $\boxed{u(x,t)=e^{-t^2}f\big(x-e^t+1\big)}$.

d] $\boxed{\begin{cases}3u_t-u_x=g(x)\\ u(x,0)=0\end{cases}}$ i) $\frac{dt}{dx}=\frac{3}{-1}\to t+3x=C$; Mejor $\begin{cases}\xi=t+3x\\ \eta=x\end{cases}\to u_\eta=-g(\eta)$.

$u=p(\xi)-\int_0^\eta g(s)\,ds=p(t+3x)-\int_0^x g(s)\,ds$. $u(x,0)=p(3x)-\int_0^x g=0$

$\to p(v)=\int_0^{v/3}g\to u=\int_0^{x+\frac{t}{3}}g-\int_0^x g\to u=\int_x^{x+\frac{t}{3}}g(s)\,ds$.

Solución única pues $t=0$ no es tangente a las características $\big[T=1\cdot3-0\cdot(-1)=3\neq0\big]$.



ii] $\begin{cases}\hat u_t+ik\hat u=\hat g(k)\\ \hat u(k,0)=0\end{cases}\to\hat u=p(k)\,e^{-ikt/3}+\frac{\hat g(k)}{ik}\overset{d.i.}{\to}p(k)=-\frac{\hat g(k)}{ik}\to\hat u(k,t)=\hat g(k)\Big[\frac{1-e^{-ikt/3}}{ik}\Big]\to$

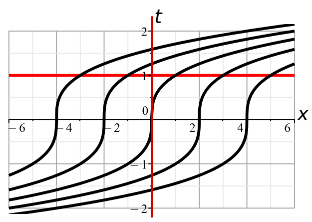
$u(x,t)=\sqrt{2\pi}\,g(x)*h(x)=\int_{-t/3}^0 g(x-u)\,du\underset{x-u=s}{=}-\int_{x+\frac{t}{3}}^x g(s)\,ds$ como antes.

$h(x)=1$ en $[-t/3,0]$, 0 fuera

21 $\boxed{u_t+3t^2u_x=\big(3t^2+1\big)u}$, a] $\frac{dt}{dx}=\frac{1}{3t^2}$. $x-t^3=C$. $\begin{cases}\xi=x-t^3\\ \eta=t\end{cases}$, $u_\eta=\big(3\eta^2+1\big)u$,

$u=p(\xi)\,e^{\eta^3+\eta}$, $u(x,t)=p\big(x-t^3\big)\,e^{t^3+t}$.

$u(x,1)=p(x-1)e^2=f(x)$, $p(v)=f(v+1)e^{-2}$, $\boxed{u=f\big(x-t^3+1\big)\,e^{t^3+t-2}}$.



$\begin{cases}\hat u_t=\big(3t^2ik+3t^2+1\big)\hat u\\ \hat u(k,1)=\hat f(k)\end{cases}\to\hat u(k,t)=p(k)\,e^{t^3ik+t^3+t}\overset{c.i.}{\longrightarrow}\hat u=e^{t^3+t-2}\hat f(k)\,e^{ik(t^3-1)}$.

Como $\mathcal{F}^{-1}\big[\hat f(k)\,e^{ika}\big]=f(x-a)$ es $u(x,t)=e^{t^3+t-2}f\big(x-t^3+1\big)$, como arriba.

Si $f(x)=x$ pasa a ser: $\boxed{u=e^{x+t-1}}$. Comprobamos: $e^{\cdots}+3t^2e^{\cdots}=(3t^2+1)e^{\cdots}$, y el dato: $u(x,1)=e^x$.

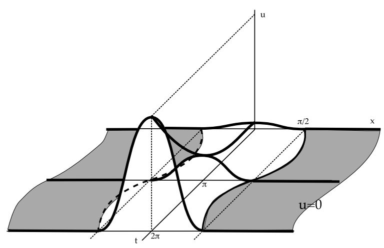
b] Tiene a] claramente **solución única** (dibujo o $T(x)=1\cdot1-0\cdot3\equiv1$), pero la recta $(g,h)=(0,t)$ es tangente a las características en el origen. Lo confirma T: $T(t)=g'A(g,h)-h'B(g,h)=0\cdot1-1\cdot(3t^2)=0$, si $t=0$, $\boxed{(0,0)}$. Imponemos el dato: $u(0,t)=p\big(-t^3\big)=0$ determina $p(v)=0$ para todos los valores v, positivos y negativos. Hay **unicidad a pesar de la tangencia**. La única solución con ese dato es la $u\equiv0$.

22 $\boxed{u_t+(\cos t)u_x=u}$, $u(x,0)=f(x)$. $\begin{cases}\xi=x-\operatorname{sen}t\\ \eta=t\end{cases}$, $u_\eta=u$, $\boxed{u=f(x-\operatorname{sen}t)\,e^t}$.

$\begin{cases}\hat u_t-ik\cos t\,\hat u=\hat u\\ \hat u(k,0)=\hat f(k)\end{cases}\to\hat u=p(k)\,e^t\,e^{ik\operatorname{sen}t}\overset{c.i.}{=}\hat f(k)\,e^t\,e^{ik\operatorname{sen}t}\nearrow$

Si $f(x)=\begin{cases}\cos^2x,\ x\in[-\frac{\pi}{2},\frac{\pi}{2}]\\ 0\text{ en el resto}\end{cases}$, $u\neq0$ si $\operatorname{sen}t-\frac{\pi}{2}\leq x\leq\operatorname{sen}t+\frac{\pi}{2}$, $u(x,n\pi)=e^{n\pi}f(x)$.

La solución avanza siguiendo las características y crece su altura exponencialmente con el tiempo.



23 $\boxed{\begin{cases} u_t - u_x = 2x e^{-x^2} \\ u(x,2)=0 \end{cases}}$ i) $\frac{dt}{dx} = -1$. $\int dt = -\int dx + C$. Características: $\boxed{x+t=C}$. [La recta $t=2$ no lo es y hay solución única].

Mucho más corto: $\begin{cases} \xi = x+t \\ \eta = x \end{cases} \to \begin{cases} u_t = u_\xi \\ u_x = u_\xi + u_\eta \end{cases} \to u_\eta = -2x e^{-x^2} = -2\eta e^{-\eta^2}$. $u = p(\xi) + e^{-\eta^2} = p(x+t) + e^{-x^2}$.

Imponiendo el dato inicial: $u(x,2) = p(x+2) + e^{-x^2} = 0 \to p(v) = -e^{-(v-2)^2}$, $\boxed{u(x,t) = e^{-x^2} - e^{-(x+t-2)^2}}$.

ii) Si $f(x) = e^{-x^2}$, el término de la derecha es $-f'$ y su transformada es $ik\hat{f}$, conocida:

$$\begin{cases} \hat{u}_t + ik\hat{u} = \frac{ik}{\sqrt{2}} e^{-k^2/4} \\ \hat{u}(k,2)=0 \end{cases} \to \hat{u} = p(k)e^{-ikt} + \frac{1}{\sqrt{2}} e^{-k^2/4} \xrightarrow{c.i.} p(k) = -\frac{1}{\sqrt{2}} e^{-k^2/4+2ik},\ \hat{u} = \frac{1}{\sqrt{2}} e^{-k^2/4} - \frac{1}{\sqrt{2}} e^{-k^2/4} e^{ik(2-t)}.$$

Como $\mathcal{F}^{-1}[\hat{f}(k)e^{ika}] = f(x-a)$, es $u(x,t) = e^{-x^2} - e^{-(x+t-2)^2}$. [Resolver con $\mathcal{F}$ es transformar e imponer también el dato inicial].

24 a) $\boxed{\begin{cases} 21u_{tt} + 2u_{tx} - 3u_{xx} = 0 \\ u(x,0)=f(x),\ u_t(x,0)=3f'(x) \end{cases}}$ i] $B^2-4AC=256$ hiperbólica. $\begin{cases} \xi = x + \frac{1}{3}t \\ \eta = x - \frac{3}{7}t \end{cases} \to u_{\xi\eta}=0$ (sólo hay derivadas de segundo orden).

Luego $u = p(\xi)+q(\eta) = p(x+\frac{1}{3}t) + q(x-\frac{3}{7}t)$, $u_t = \frac{1}{3}p'(x+\frac{1}{3}t) - \frac{3}{7}q'(x-\frac{3}{7}t) \to \begin{matrix} p(x)+q(x)=f(x) \\ \frac{1}{3}p'(x) - \frac{3}{7}q'(x) = 3f'(x) \end{matrix} \to$

$p = \frac{9f}{2}$, $q = -\frac{7f}{2}$. $\boxed{u(x,t) = \frac{9}{2}f(x+\frac{1}{3}t) - \frac{7}{2}f(x-\frac{3}{7}t)}$

ii] $\begin{cases} 21\hat{u}_{tt} - 2ik\hat{u}_t + 3k^2\hat{u} = 0 \\ \hat{u}(k,0) = \hat{f}(k),\ \hat{u}_t(k,0) = -3ik\hat{f}(k) \end{cases}$, $\mu^2 - 2ik\mu + 3k^2$, $\mu = \frac{ik \pm \sqrt{-64k^2}}{21} = \frac{3ik}{7}, -\frac{ik}{3}$, $\hat{u} = p(k)e^{3ikt/7} + q(k)e^{-ikt/3}$.

Con los datos: $\begin{matrix} p(k)+q(k)=\hat{f}(k), & q(k)=\hat{f}(k)-p(k) \searrow & q(k)=\frac{9}{2}\hat{f}(k) \\ \frac{3ik}{7}p(k) - \frac{ik}{3}q(k) = -3ik\hat{f}(k) & (\frac{3}{7}+\frac{1}{3})p(k) = (\frac{1}{3}-3)\hat{f}(k),\ p(k) = -\frac{7}{2}\hat{f}(k) \nearrow & \end{matrix}$.

$\hat{u} = \frac{9}{2}\hat{f}(k)e^{-ikt/3} - \frac{7}{2}\hat{f}(k)e^{3ikt/7} \xrightarrow{\mathcal{F}^{-1}[e^{ika}\hat{f}(k)]=f(x-a)} \boxed{u(x,t) = \frac{9}{2}f(x+\frac{1}{3}t) - \frac{7}{2}f(x-\frac{3}{7}t)}$

b] $\boxed{\begin{cases} u_{tt} - 6u_{tx} + 9u_{xx} = 0 \\ u(x,0)=f(x),\ u_t(x,0)=0 \end{cases}}$ i] $B^2-4AC=0$ parabólica $\begin{cases} \xi = x+3t \\ \eta = t \end{cases}$ $\begin{cases} u_{xx} = u_{\xi\xi} \\ u_{xt} = 3u_{\xi\xi} + u_{\xi\eta} \\ u_{tt} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta} \end{cases} \to u_{\eta\eta} = 0 \to u = p(\xi) + \eta q(\xi)$

$u = p(x+3t) + t\,q(x+3t)$, $\begin{cases} u(x,0) = p(x) = f(x) \\ u_t(x,0) = 3p'(x) + q(x) = 0,\ q(x) = -3f'(x) \end{cases} \to \boxed{u(x,t) = f(x+3t) - 3tf'(x+3t)}$.

ii] $\begin{cases} \hat{u}_{tt} + 6ik\hat{u}_t - 9k^2\hat{u} = 0 \\ \hat{u}(k,0) = \hat{f}(k),\ \hat{u}_t(k,0) = 0 \end{cases}$ $\begin{matrix} \mu^2 + 6ik\mu - 9k^2 = 0 \\ \to \mu = -3ik \text{ doble} \end{matrix} \to \hat{u}(k,t) = [p(k) + t\,q(k)]e^{-3ikt} \xrightarrow{c.i.} \begin{matrix} p(k) = \hat{f}(k) \\ q(k) = 3ik\hat{f}(k) \end{matrix} \to$

$\hat{u}(k,t) = \hat{f}(k)e^{-3ikt} + 3t\,ik\hat{f}(k)e^{-3ikt}$. Como $\begin{matrix} \mathcal{F}^{-1}[ik\hat{f}(k)] = -f'(x) \\ \mathcal{F}^{-1}[\hat{f}(k)e^{ika}] = f(x-a) \end{matrix}$ es $\boxed{u(x,t) = f(x+3t) - 3tf'(x+3t)}$.

c] $\boxed{\begin{cases} u_{tt} + 2u_{xt} + u_{xx} - u_t - u_x = 0 \\ u(x,0)=f(x),\ u_t(x,0)=0 \end{cases}}$ i] $B^2-4AC=0$ parabólica $\begin{cases} \xi = x-t \\ \eta = t \end{cases} \to \begin{cases} u_t = -u_\xi + u_\eta \\ u_x = u_\xi \end{cases}$, $\begin{cases} u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \\ u_{xt} = -u_{\xi\xi} + u_{\xi\eta} \\ u_{xx} = u_{\xi\xi} \end{cases} \to u_{\eta\eta} - u_\eta = 0$.

EDO lineal de 2° orden en η con $\mu^2 - \mu = 0$, $\mu = 0, 1$. $u = p(\xi) + q(\xi)e^\eta$, $u(x,t) = p(x-t) + q(x-t)e^t$.

$\begin{cases} p(x)+q(x) = f(x),\ p'(x)+q'(x) = f'(x) \searrow & p(x) = f(x) - f'(x) \\ -p'(x) - q'(x) + q(x) = 0, & q(x) = f'(x) \uparrow \end{cases} \to \boxed{u(x,t) = f(x-t) + f'(x-t)[e^t - 1]}$.

$\begin{cases} \hat{u}_{tt} - (1+2ik)\hat{u}_t + (ik-k^2)\hat{u} = 0,\ \mu = \frac{1}{2}[1 + 2ik \pm \sqrt{1}] = 1+ik, ik. & u(\hat{k},t) = p(k)e^{t+ikt} + q(k)e^{ikt} \xrightarrow{ci} \\ \hat{u}(k,0) = \hat{f}(k),\ \hat{u}_t(k,0) = 0 & \hat{u} = \hat{f}(k)e^{ikt} - ik\hat{f}(k)e^{ikt}[e^t - 1],\ \boxed{u(x,t) = f(x-t) + f'(x-t)[e^t - 1]} \end{cases}$.

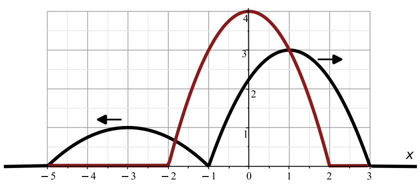
25 $\boxed{\begin{cases} u_{tt} + 2u_{xt} + u_{xx} - u_t - u_x = 0 \\ u(x,0)=f(x),\ u_t(x,0)=0 \end{cases}}$ a] $\begin{cases} \hat{u}_{tt} + 2ik\hat{u}_t + 3k^2\hat{u} = 0 \\ \hat{u}(k,0) = \hat{f}(k),\ \hat{u}_t(k,0) = 0 \end{cases}$, $\mu = ik, -3ik$, $\hat{u} = p(k)e^{ikt} + q(k)e^{-3ikt}$.

$\begin{matrix} p(k)+q(k) = \hat{f}(k), & 4q(k) = \hat{f}(k),\ q(k) = \frac{1}{2}\hat{f}(k) \\ ik\,p(k) - 3ik\,q(k) = 0,\ p(k) = 3q(k)\uparrow & p(k) = \frac{3}{4}\hat{f}(k) \end{matrix}$. $\hat{u} = \frac{1}{4}[3\hat{f}(k)e^{ikt} + \hat{f}(k)e^{-3ikt}]$, $\boxed{u = \frac{1}{4}f(x+3t) + \frac{3}{4}f(x-t)}$.

$B^2-4AC=16$ hiperbólica. $\begin{cases} \xi = x+3t \\ \eta = x-t \end{cases} \to u_{\xi\eta} = 0$, $u = p(\xi) + q(\eta)$, $\boxed{u(x,t) = p(x+3t) + q(x-t)}$ solución general.

Imponiendo datos: $\begin{matrix} p(x)+q(x) = f(x) \\ 3p'(x) - q'(x) = 0 \end{matrix}$. $p = \frac{f}{4}$, $q = \frac{3f}{4}$.

b] Un cuarto de la onda inicial viaja hacia la izquierda a velocidad 3 y tres cuartos van a la derecha velocidad 1. Para $t=1$ la primera está en $[-5,-1]$ y la otra en $[-1,3]$. En el resto es $u(x,1)=0$.



26 $\boxed{\begin{cases} u_{tt} - c^2u_{xx} = 0,\ x,t \in \mathbf{R} \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}}$ $\begin{cases} \hat{u}_{tt} + c^2k^2\hat{u} = 0 \\ \hat{u}(k,0) = \hat{f}(k),\ \hat{u}_t(k,0) = \hat{g}(k) \end{cases} \to \hat{u} = p(k)e^{ickt} + q(k)e^{-ickt} \to \begin{matrix} p(k)+q(k) = \hat{f}(k) \\ ick[p(k)-q(k)] = \hat{g}(k) \end{matrix}$

$p(k) = \frac{1}{2}[\hat{f}(k) + \frac{\hat{g}(k)}{ick}]$, $q(k) = \frac{1}{2}[\hat{f}(k) - \frac{\hat{g}(k)}{ick}] \to \hat{u} = \frac{1}{2}\hat{f}(k)[e^{ickt} + e^{-ickt}] + \frac{1}{2}\hat{g}(k)[\frac{e^{ickt} - e^{-ickt}}{ick}]$

$\to u = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}g(x) * \sqrt{2\pi}h(x)$, con $h(x) = \begin{cases} 1 \text{ si } x \in [-ct,ct] \\ 0 \text{ si } x \notin [-ct,ct] \end{cases}$

$$\| $$

$$\frac{1}{2c}\int_{-\infty}^{\infty} g(x-s)h(s)\,ds = \frac{1}{2c}\int_{-ct}^{ct} g(x-s)\,ds \underset{u=x-s}{=} -\frac{1}{2c}\int_{x+ct}^{x-ct} g(u)\,du$$

27 $\begin{cases} u_{tt}-4u_{xx}-2u_t+4u_x=0\,,\ x,t\in\mathbf{R} \\ u(x,0)=f(x)\,,\ u_t(x,0)=f(x) \end{cases}$ $\begin{cases} \hat u_{tt}+4k^2\hat u-2\hat u_t-4\mathrm{i}k\hat u=0 \\ \hat u(k,0)=\hat f(k),\ \hat u_t(k,0)=\hat f(k) \end{cases}$ $\mu^2-2\mu+4k^2-4\mathrm{i}k=0\to$ $\mu=1\pm\sqrt{1+4\mathrm{i}k-4k^2}=1\pm(1+2\mathrm{i}k)$ $\to$

$\hat u=p(k)\,e^{2(1+\mathrm{i}k)t}+q(k)\,e^{-2\mathrm{i}kt}\xrightarrow{c.i.}$ $p(k)+q(k)=\hat f(k)\,,\ q(k)=\hat f(k)-p(k)\searrow$ $2(1+\mathrm{i}k)p(k)-2\mathrm{i}kq(k)=\hat f(k)\quad 2(1+2\mathrm{i}k)p(k)=(1+2\mathrm{i}k)\hat f(k)\nearrow$ $p(k)=\frac{\hat f(k)}{2}=q(k)$,

$\hat u(k,t)=\frac{1}{2}e^{2t}\hat f(k)\,e^{\mathrm{i}k2t}+\frac{1}{2}\hat f(k)\,e^{-\mathrm{i}k2t}\to$ $\boxed{u(x,t)=\frac{1}{2}\big[f(x-2t)\,e^{2t}+f(x+2t)\big]}$. Para $f(x)=1$, $\boxed{u=\frac{1}{2}\big[e^{2t}+1\big]}$.

[Fácilmente comprobable: $u(x,0)=\frac{2}{2}$, $u_t(x,0)=e^0$, $u_{tt}-4u_{xx}-2u_t+4u_x=2e^{2t}-0-2e^{2t}+0=0$].

Las características son las de la ecuación de ondas (lo dicen las derivadas segundas):

$\begin{cases}\xi=x+2t\\ \eta=x-2t\end{cases}$, $\begin{cases}u_x=u_\xi+u_\eta\\ u_t=2u_\xi-2u_\eta\end{cases}$, $\begin{cases}u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\\ u_{tt}=4u_{\xi\xi}-8u_{\xi\eta}+4u_{\eta\eta}\end{cases}$ $\to -16u_{\xi\eta}+8u_\eta=0$, $\boxed{u_{\xi\eta}-\frac{1}{2}u_\eta=0}$ forma canónica.

$u_\eta=v$, $v_\xi=\frac{1}{2}v\to v=p^*(\eta)\,e^{\xi/2}$, $u=p(\eta)\,e^{\xi/2}+q(\xi)$, $\boxed{u(x,t)=p(x-2t)\,e^{x/2+t}+q(x+2t)}$ solución general.

28 $\begin{cases}u_t-u_{xx}=(x^2-1)\,e^{-x^2/2}\,,\ x\in\mathbf{R},\ t>0\\ u(x,0)=0\,,\ u\text{ acotada}\end{cases}$ a] Como $\mathcal{F}[f'']=-k^2\hat f$ y $\mathcal{F}[e^{-ax^2}]=\frac{1}{\sqrt{2a}}\,e^{-k^2/4a}\xrightarrow{a=1/2}e^{-k^2/2}$:

$\begin{cases}\hat u_t+k^2\hat u=-k^2e^{-k^2/2}\\ \hat u(k,0)=0\end{cases}\xrightarrow[x_p\text{ a ojo}]{}\hat u(k,t)=p(k)\,e^{-k^2t}-e^{-k^2/2}\xrightarrow[d.i.]{}$

$\hat u(k,t)=e^{-k^2/2}\,e^{-k^2t}-e^{-k^2/2}$, $u(x,t)=\mathcal{F}^{-1}\big[e^{-k^2(t+\frac{1}{2})}\big]-e^{-x^2/2}=$ $\boxed{\frac{1}{\sqrt{1+2t}}\,e^{-x^2/(4t+2)}-e^{-x^2/2}}\xrightarrow[t\to\infty]{}-e^{-x^2/2}$ (•)

b] $v=-e^{-\frac{x^2}{2}}$ satisface la ecuación, $\xrightarrow{w=u-v}\begin{cases}w_t-w_{xx}=0\\ u(x,0)=e^{-x^2/2}\end{cases}\xrightarrow{\text{formulario}}w=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-s^2/2}\,e^{-(x-s)^2/4t}ds$.

Para evaluar la integral completamos cuadrados buscando $\int_{-\infty}^{\infty}e^{-p^2}dp=\sqrt{\pi}$:

$$-\frac{(2t+1)s^2-2xs+x^2}{4t}=-\frac{\big[\sqrt{2t+1}\,s-\frac{x}{\sqrt{2t+1}}\big]^2}{(2\sqrt{t})^2}-\frac{x^2}{4t}+\frac{x^2}{4t(2t+1)}=-\Big[\frac{\sqrt{2t+1}\,s}{2\sqrt{t}}-\frac{x}{2\sqrt{t}\sqrt{2t+1}}\Big]^2-\frac{x^2}{4t+2}$$

Llamando p al último corchete $\Big[dp=\frac{\sqrt{2t+1}}{2\sqrt{t}}ds\Big]$: $w=\frac{1}{2\sqrt{\pi t}}\frac{2\sqrt{t}}{\sqrt{2t+1}}\,e^{-x^2/(4t+2)}\int_{-\infty}^{\infty}e^{-p^2}dp=\frac{1}{\sqrt{2t+1}}e^{-x^2/(4t+2)}$ $u=v+w\nwarrow$

(•) [Estamos todo el rato sacando calor en $[-1,1]$ y dándolo (menos cantidad según nos alejamos) fuera de ese intervalo. Las temperaturas acaban siendo negativas y menores cerca del origen].

29 $\begin{cases}u_t-u_{xx}=e^{-x^2/4}\,,\ x\in\mathbf{R},\ t>0\\ u(x,0)=0\,,\ u\text{ acotada}\end{cases}$ Su solución: $u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-k^2}-e^{-k^2(t+1)}}{k^2}e^{-ikx}dk$ [en $k=0$ decente].

Como $\mathcal{F}[e^{-ax^2}]=\frac{1}{\sqrt{2a}}\,e^{-k^2/4a}\xrightarrow{a=1/4}\begin{cases}\hat u_t+k^2\hat u=\sqrt{2}\,e^{-k^2}\\ \hat u(k,0)=0\end{cases}\xrightarrow{\hat u_p\text{ a ojo}}\hat u(k,t)=p(k)\,e^{-k^2t}+\frac{\sqrt{2}}{k^2}e^{-k^2}\xrightarrow{d.i.}$

$p(k)=-\frac{\sqrt{2}}{k^2}e^{-k^2}$, $\hat u(k,t)=\frac{\sqrt{2}}{k^2}\big[e^{-k^2}-e^{-k^2(t+1)}\big]$. Y de $u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat u(k,t)e^{-ikx}dk$, sale lo de arriba.

$u(0,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-k^2}-e^{-k^2(t+1)}}{k^2}dk=-\Big[\frac{e^{-k^2}-e^{-k^2(t+1)}}{\sqrt{\pi}\,k}\Big]_{-\infty}^{\infty}-\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-k^2}dk+\frac{2(t+1)}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-k^2(t+1)}dk=\boxed{2\sqrt{t+1}-2}$.

[Es normal que tienda a ∞. Estamos constantemente metiendo calor en toda la varilla].

30 a] $\begin{cases}u_t-\frac{1}{4}u_{xx}+u_x=0\\ u(x,0)=e^{-x^2}\end{cases}$ $\begin{cases}\hat u_t=\big(ik-\frac{k^2}{4}\big)\hat u\\ \hat u(k,0)=\frac{1}{\sqrt{2}}e^{-k^2/4}\end{cases}\to\hat u=\frac{1}{\sqrt{2}}\,e^{ikt}\,e^{-\frac{k^2(t+1)}{4}}\to$ $\boxed{u=\frac{1}{\sqrt{t+1}}\,e^{-\frac{(x-t)^2}{t+1}}}$.

b] $\begin{cases}u_t-2tu_{xx}-u_x=0\\ u(x,1)=e^{-x^2/4}\end{cases}$ $\begin{cases}\hat u_t+2tk^2\hat u+\mathrm{i}k\hat u=0\\ \hat u(k,1)=\sqrt{2}\,e^{-k^2}\end{cases}\to\hat u=p(k)\,e^{-t^2k^2-\mathrm{i}kt}\xrightarrow[c.i.]{}p(k)\,e^{-k^2-\mathrm{i}k}=\sqrt{2}\,e^{-k^2}$, $p(k)=\sqrt{2}\,e^{\mathrm{i}k}$,

$\hat u(k,t)=\sqrt{2}\,e^{-t^2k^2}e^{\mathrm{i}k(1-t)}$. $\mathcal{F}^{-1}\big(\sqrt{2}\,e^{-t^2k^2}\big)=\frac{1}{t}\,e^{-x^2/4t^2}$ y $\mathcal{F}^{-1}\big[\hat f(k)\,e^{\mathrm{i}ka}\big]=f(x-a)\to$ $\boxed{u(x,t)=\frac{1}{t}\,e^{-(x+t-1)^2/4t^2}}$.

c] $\begin{cases}u_{tt}-4u_{xx}=0\,,\ x\in\mathbf{R},\ t\in\mathbf{R}\\ u(x,0)=2e^{-x^2/2},\ u_t(x,0)=0\end{cases}$ $\begin{cases}\hat u_{tt}+4k^2\hat u=0\\ \hat u(k,0)=2e^{-k^2/2},\ \hat u_t(k,0)=0\end{cases}\xrightarrow{\mu=\pm2k\mathrm{i}}\hat u(k,t)=p(k)\,e^{2k\mathrm{i}t}+q(k)\,e^{-2k\mathrm{i}t}\xrightarrow{c.i.}$

$\begin{matrix}p(k)+q(k)=2e^{-k^2/2}\\ 2k\mathrm{i}\big[p(k)-q(k)\big]=0\to q(k)=p(k)\end{matrix}$ $\to p(k)=e^{-k^2/2}=q(k)$, $\hat u(k,t)=e^{-k^2/2}\,e^{\mathrm{i}k(2t)}+e^{-k^2/2}\,e^{\mathrm{i}k(-2t)}$.

Como $\mathcal{F}^{-1}\big[e^{-k^2/2}\big]=e^{-x^2/2}$, $\mathcal{F}^{-1}\big[\hat f(k)\,e^{\mathrm{i}ka}\big]=f(x-a)$, es $\boxed{u(x,t)=e^{-(x-2t)^2/2}+e^{-(x+2t)^2/2}}$ [la que daría D'Alembert]

d] $\begin{cases}u_t-2tu_{xx}=0\\ u(x,0)=\delta(x)\end{cases}$ $\begin{cases}\hat u_t+2tk^2\hat u=0\\ \hat u(k,0)=1/\sqrt{2\pi}\end{cases}\to\hat u=p(k)\,e^{-t^2k^2}\to\hat u=\frac{1}{\sqrt{2\pi}}\,e^{-t^2k^2}\to$ $\boxed{u=\frac{1}{2\sqrt{\pi}\,t}e^{-x^2/4t^2}}$

Soluciones de problemas 2 de MII(C) (2023-24)

1 a) $\boxed{(1+x^2)y''-2y=0}$ $y=\sum_{k=0}^{\infty}c_kx^k$, $\sum_{k=2}^{\infty}\left[k(k-1)c_kx^{k-2}+k(k-1)c_kx^k\right]-\sum_{k=0}^{\infty}2c_kx^k=0$, $c_k=-\frac{k-4}{k}c_{k-2}$,

$c_2=c_0$, $c_4=c_6=\cdots=0$, $c_3=\frac{c_1}{3}$, $c_5=-\frac{c_1}{3\cdot 5}$, $c_7=\frac{c_1}{5\cdot 7}$, ... $\boxed{y=c_0\left[1+x^2\right]+c_1\left[x+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{(2n-1)(2n+1)}\right]}$ si $|x|<1$.

$y_1=1+x^2 \to y_2=(1+x^2)\int\frac{dx}{(1+x^2)^2} \to y=c_0[1+x^2]+c_1\left[x+(1+x^2)\arctan x\right]$.

b) $\boxed{(1-x)(1-2x)y''+2xy'-2y=0}$ $x=0$ regular, $y=\sum_{k=0}^{\infty}c_kx^k \to c_k=\frac{3(k-2)}{k}c_{k-1}-\frac{2(k-3)}{k}c_{k-2}$, $x^0\to c_2=c_0$,

$x^1\to c_3=c_2$, $\forall c_1$. Si $c_{k-1}=c_{k-2}$, $c_k=\frac{k}{k}c_{k-2}=c_{k-2} \to$ $\boxed{y=c_0\left[1+x^2+\cdots\right]+c_1x=c_0\left[\frac{1}{1-x}-x\right]+c_1x}$.

$\Big[$O bien, $y_1=x$, $e^{-\int a}=e^{-\int(\frac{2}{1-x}-\frac{2}{1-2x})}$, $y_2=x\int\frac{1-2x}{x^2(1-x)^2}=\int\left(\frac{1}{x^2}-\frac{1}{(1-x)^2}\right)=-\frac{1}{1-x}$, $y=c_1x+\frac{c_2}{1-x}$ ↑ $\Big]$

La serie converge en $(-1,1)$ $\big[$el teorema aseguraba que lo hacía al menos en $(-1/2,1/2)\big]$.

c) $\boxed{\cos x\, y''+(2-\operatorname{sen} x)y'=0}$ Resoluble, $v=-\frac{C\cos x}{(1+\operatorname{sen}x)^2}$ $\left[\int\frac{-2}{\cos x}=2\log\frac{1+\operatorname{sen}x}{\cos x}\right]$, $y=K+\frac{C}{1+\operatorname{sen}x}\to$

$y=K+C\left[1-\operatorname{sen}x+\operatorname{sen}^2x-\operatorname{sen}^3x+\operatorname{sen}^4x-\cdots\right]=$ $\boxed{K+C\left[1-x+x^2+\left(\frac{1}{6}-1\right)x^3+\left(1-\frac{1}{3}\right)x^4+\cdots\right]}$ $|x|<\frac{\pi}{2}$.

O bien, $\left[1-\frac{x^2}{2}-\frac{x^4}{24}+\cdots\right]\left[2c_2+6c_3x+12c_4x^2+\cdots\right]+\left[2-x+\frac{x^3}{6}-\cdots\right]\left[c_1+2c_2x+3c_3x^2+\cdots\right]=0$,

x^0: $2c_2+2c_1=0$, $c_2=-c_1$; x^1: $6c_3+4c_2-c_1=0$, $c_3=-\frac{5}{6}c_1$; x^2: $12c_4+6c_3-3c_2=0$, $c_3=\frac{2}{3}c_1$; . . .

O bien, $y''(0)=-2y'(0)$; $\cos x\, y'''+(2-2\operatorname{sen}x)y''-\cos t\, y'=0$, $y'''(0)=5y'(0)$; . . .

2 $\boxed{y''+2xy'+2y=0}$ $x=0$ es regular. Probamos pues $y=\sum_{k=0}^{\infty}c_kx^k$, sabiendo que será $c_0=1$, $c_1=0$.

$\sum_{k=2}^{\infty}k(k-1)c_kx^{k-2}+\sum_{k=1}^{\infty}2kc_kx^k+\sum_{k=0}^{\infty}2c_kx^k=0 \to$ x^0: $2c_2+2c_0=0\to c_2=-1$;

x^1: $6c_3+4c_1=0\to c_3=-\frac{2}{3}c_1=0$; x^2: $12c_4+6c_2=0\to c_4=-\frac{1}{2}c_2=\frac{1}{2}$ $\to$ $\boxed{y=1-x^2+\frac{1}{2}x^4+\cdots}$.

$\big[$O bien: $y''(0)+y(0)=0$, $y''(0)=-2$; $y'''+2xy''+4y'=0$, $y'''(0)=-4y'(0)=0$; $y^{iv}+2xy'''+6y''=0$, $y^{iv}(0)=-6y''(0)=12\big]$.

x^k: $(k+2)(k+1)c_{k+2}+2(k+1)c_k=0$, $c_{k+2}=-\frac{2}{k+2}c_k$ ó $c_k=-\frac{2}{k}c_{k-2}$ $\to$ $c_6=-\frac{1}{3}c_4=-\frac{1}{6}$, $c_8=-\frac{1}{4}c_4=\frac{1}{4!}$, . . . ;

$c_{2k}=-\frac{1}{k}c_{2k-2}=\frac{1}{k(k-1)}c_{2k-4}=\cdots$ $\to$ $\boxed{y=1-x^2+\frac{1}{2!}x^4-\frac{1}{3!}x^6+\cdots+\frac{(-1)^k}{k!}x^{2k}+\cdots=e^{-x^2}}$.

3 $\boxed{y''+(2-2x)y'+(1-2x)y=0}$ $x=0$ regular $\to$ probamos $y=\sum_{k=0}^{\infty}c_kx^k$, sabiendo que $c_0=0$ y $c_1=1$.

$$\sum_{k=2}^{\infty}k(k-1)c_kx^{k-2}+\sum_{k=1}^{\infty}\left[2kc_kx^{k-1}-2kc_kx^k\right]+\sum_{k=0}^{\infty}\left[c_kx^k-2c_kx^{k+1}\right]=0\to$$

x^0 : $2c_2+2c_1+c_0=2c_2+2=0\to c_2=-1$; x^1 : $6c_3+4c_2-c_1-2c_0=6c_3-5=0\to c_3=\frac{5}{6}$;

x^2 : $12c_4+6c_3-3c_2-2c_1=12c_4+6=0\to c_4=-\frac{1}{2}$. Así: $\boxed{y=x-x^2+\frac{5}{6}x^3-\frac{1}{2}x^4+\cdots}$.

O bien: $y''(0)+2y'(0)+y(0)=0$, $y''(0)=-2$ ↗. Y derivando la ecuación:

$y'''+(2-2x)y''-(1+2x)y'-2y=0\to y'''(0)+2y''(0)-y'(0)-2y(0)=0$, $y'''(0)=5$ ↑

$y^{iv}+(2-2x)y'''-(3+2x)y''-4y'=0\to y^{iv}(0)+2y'''(0)-3y''(0)-4y'(0)=0$, $y^{iv}(0)=-12$ ↑

$y_1=e^{-x}$, $e^{-\int a}=e^{x^2-2x}\to y=ce^{-x}+ke^{-x}\int_0^x e^{s^2}ds \overset{\text{d.i.}}{\to}$ $\boxed{y=e^{-x}\int_0^x e^{s^2}ds}$ $=e^{-x}\int_0^x\left[1+s^2+\frac{1}{2}s^4+\cdots\right]ds$

$=\left[1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots\right]\left[x+\frac{1}{3}x^3+\frac{1}{10}x^5+\cdots\right]=x-x^2+\left[\frac{1}{3}+\frac{1}{2}\right]x^3-\left[\frac{1}{3}+\frac{1}{6}\right]x^4+\cdots$ ↑

4 $\boxed{2\sqrt{x}\,y''-y'=0}$ $x=0$ no es singular regular $\left(a^*(x)=-\frac{\sqrt{x}}{2}\right.$ no es analiítica en $\left.x=0\right)$.

$x-1=s$↘ $2[1+s]^{1/2}y''-y'=0$, $2\left[1+\frac{s}{2}-\frac{s^2}{8}+\cdots\right]\left[2c_2+6c_3s+\cdots\right]-\left[c_1+2c_2s+3c_3s^2+\cdots\right]=0$,

s^0: $4c_2-c_1=0$, $c_2=\frac{1}{4}$; s^1: $12c_3+2c_2-2c_2=0$, $c_3=0$; . . . $\boxed{y=1+(x-1)+\frac{1}{4}(x-1)^2+\cdots}$.

O bien, $y''(1)=\frac{y'(1)}{2}=\frac{1}{2}$; $2\sqrt{x}\,y'''+\left(\frac{1}{\sqrt{x}}-1\right)y''=0$, $y'''(1)=0$; . . . $y=y(1)+y'(1)(x-1)+\cdots$ ↑

$\Big[$Solución calculable sin series: $v'=\frac{v}{2\sqrt{x}}$, $v=Ce^{\sqrt{x}}$, $y=K+C\left(\sqrt{x}-1\right)e^{\sqrt{x}} \overset{\text{d.i.}}{\to} y=1+2\left(\sqrt{x}-1\right)e^{\sqrt{x}}\Big]$.

5 a) $\boxed{xy''+2y'=x}$ $\lambda(\lambda-1)+2\lambda=0 \to y=c_1+c_2x^{-1}+y_p$ solución de la no homogénea.

$\begin{vmatrix}1 & x^{-1}\\ 0 & -x^{-2}\end{vmatrix}=-x^{-2}$, $y_p=\frac{1}{x}\int\frac{1\cdot 1}{-x^{-2}}-1\int\frac{1/x\cdot 1}{-x^{-2}}=\frac{x^2}{6}$. O mejor, $y_p=Ax^2\ \left(Ae^{2s}\right)\to 2A+4A=1$.

O también: $xv'+2v=x \to v=\frac{x}{3}+\frac{C}{x^2} \to y=\frac{x^2}{6}-\frac{C}{x}+K$, como antes (con otro nombre de las constantes).

b) $\boxed{x^2y''-3xy'+3y=9\ln x}$ $\lambda^2-4\lambda+3=0 \to y=c_1x+c_2x^3+y_p$. Con variación de constantes:

$|\mathbf{W}|(x)=2x^3$; $y_p=x^3\int\frac{9\ln x\,dx}{2x^4}-x\int\frac{9\ln x\,dx}{2x^2}\underset{\text{partes}}{=}3\ln x+4 \to y=c_1x+c_2x^3+3\ln x+4$.

O bien, $y_p=As+B=A\ln x+B$, $y_p'=\frac{A}{x}$, $y_p''=-\frac{A}{x^2} \to -A-3A+3A\ln x+3B=9\ln x \to y_p=3\ln x+4$.

c) $\boxed{x^2y''+4xy'+2y=e^x}$ $y=\frac{c_1}{x}+\frac{c_2}{x^2}+y_p$, $|W|=-x^{-4}$, $y_p=-x^{-2}\int xe^x+x^{-1}\int e^x=\frac{e^x}{x^2}$.

6 a) $\boxed{y''+xy'+y=0}$ $x=0$ regular. $y=\sum_{k=0}^{\infty}c_kx^k \to \sum_2 k(k-1)c_kx^{k-2}+\sum_1 kc_kx^k+\sum_0 c_kx^k=0$, con $c_0=0$ y $c_1=1$.
[para que se anule en $x=0$]

x^0 : $2c_2+c_0=0$, $c_2=\frac{1}{2}c_0=0$; x^1 : $6c_3+2c_1=0$, $c_3=-\frac{1}{3}c_1=-\frac{1}{3}$; x^{k-2} : $k(k-1)c_k+(k-1)c_{k-2}=0$,

$\boxed{c_k=-\frac{1}{k}c_{k-2}} \to c_4=c_6=\cdots=0$, $c_5=-\frac{1}{5}c_3=\frac{1}{15} \to \boxed{y=x-\frac{1}{3}x^3+\frac{1}{15}x^5-\cdots}$.

[Más largo (y sin recurrencia). Derivando la ecuación: $y'''+xy''+2y'=0$, $y^{iv}+xy'''+3y''=0$, $y^{v}+x^{iv}+4y'''=0 \to$ $y(0)=0$, $y'(0)=1 \to y'''(0)=-2y'(0)=-2$, $y^{iv}(0)=-3y''(0)=0$, $y^{v}(0)=-4y'''(0)=8 \to y=x-\frac{2}{6}x^3+\frac{8}{120}x^5-\cdots$].

Todas las soluciones, como en todo punto regular, están **acotadas en** $x=0$.

b) $\boxed{3xy''+y'+xy=0}$ $x=0$ singular regular de $x^2y''+x\frac{1}{3}y'+\frac{x^2}{3}y=0$ con $\lambda(\lambda-1)+\frac{1}{3}\lambda=0 \to \lambda_1=\frac{2}{3}$, $\lambda_2=0$.

Se anula en $x=0$ la solución (no analítica, ni siquiera derivable) $y_1=\sum_{k=0}^{\infty}c_kx^{k+2/3}$, $c_0\neq 0$ [convergerá en todo **R**, pues lo hacen a^* y b^*].

La otra solución y_2 (analítica) también está acotada en 0 .

$\to \sum_0\left[3(k+\frac{2}{3})(k-\frac{1}{3})c_kx^{k-1/3}+(k+\frac{2}{3})c_kx^{k-1/3}+c_kx^{k+5/3}\right]=\sum_0\left[k(3k+2)c_kx^{k-1/3}+c_kx^{k+5/3}\right]=0$

$\to$ $x^{-1/3}$: $0\cdot c_0=0$, c_0 indeterminado; $x^{2/3}$: $5c_1=0$, $c_1=0$; $x^{5/3}$: $16c_2+c_0=0$, $c_2=-\frac{1}{16}c_0$;

$x^{k-1/3}$: $c_k=-\frac{1}{k(3k+2)}c_{k-2} \to c_3=c_5=\cdots=0$, $c_4=-\frac{1}{56}c_2=\frac{1}{896}c_0 \to \boxed{y_1=x^{2/3}\left[1-\frac{1}{16}x^2+\frac{1}{896}x^4-\cdots\right]}$.

c) $\boxed{xy''-2y'+4e^xy=0}$ $r(r-1)-2r=0 \to r_1=3$, $r_2=0 \to y_1=\sum_{k=0}^{\infty}c_kx^{k+3}$ se anula en $x=0$.

$x\left(6c_0x+12c_1x^2+20c_2x^3+30c_2x^4+\cdots\right)-2\left(3c_0x^2+4c_1x^3+5c_2x^4+6c_3x^5+\cdots\right)$
$+\left(4+4x+2x^2+\cdots\right)\left(c_0x^3+c_1x^4+c_2x^5+\cdots\right)=0 \to$

x^2 : $0c_0=0 \to c_0$ cualquiera; x^3 : $12c_1-8c_1+4c_0=0 \to c_1=-c_0$; x^4 : $20c_2-10c_2+4c_1+4c_0=0 \to c_2=0$;

x^5 : $30c_3-12c_3+4c_2+4c_1+2c_0=0 \to c_3=\frac{1}{9}c_0$. $\boxed{y_1=x^3-x^4+\frac{1}{9}x^6+\cdots}$ (sin regla de recurrencia decente).

Tanto y_1 como $y_2=\sum_{k=0}^{\infty}b_kx^k+d\,x^3(1-x+\cdots)\ln x$ acotadas en $x=0$ $\left(x^3\ln x\underset{x\to 0}{\longrightarrow}0\right) \Rightarrow$ **todas acotadas**.

7 $\boxed{2x^2y''+x(3-2x)y'-(1+2x)y=0}$ Acotada $y_1=\sum_{k=0}^{\infty}c_kx^{k+1/2}$, no lo está $y_2=\sum_{k=0}^{\infty}b_kx^{k-1}$. Ninguna analítica.

$\sum_{k=0}^{\infty}\left[2(k^2-\frac{1}{4})c_kx^{k+1/2}+3(k+\frac{1}{2})c_kx^{k+1/2}-c_kx^{k+1/2}-(2k+1)c_kx^{k+3/2}-2c_kx^{k+3/2}\right]=0 \to$

$\sum_{k=0}^{\infty}\left[k(2k+3)c_kx^{k+1/2}-(2k+3)c_kx^{k+3/2}\right]=0$, $x^{1/2}$: c_0 cualquiera, $x^{3/2}$: $5c_1-3c_0=0$, $c_1=\frac{3}{5}c_0$,

$x^{5/2}$: $c_2=\frac{5}{14}c_1=\frac{3}{14}c_0$, $x^{7/2}$: $c_3=\frac{7}{27}c_2=\frac{1}{18}c_0$, ... Luego $\boxed{y_1=x^{1/2}\left[1+\frac{3}{5}x+\frac{3}{14}x^2+\frac{1}{18}x^3+\cdots\right]}$.

Si $y_2=\frac{1}{x}$, $e^{-\int a}=\frac{e^x}{x^{3/2}}$, $y_1=\frac{1}{x}\int x^{1/2}e^xdx=\frac{1}{x}\int\left[x^{1/2}+x^{3/2}+\frac{x^{5/2}}{2}+\frac{x^{7/2}}{6}+\cdots\right]dx=\boxed{x^{1/2}\left[\frac{2}{3}+\frac{2}{5}x+\frac{1}{7}x^2+\frac{1}{27}x^3+\cdots\right]}$.

[Calculando la no acotada: $\sum_{k=0}^{\infty}\left[2(k-1)(k-2)b_kx^{k-1}+3(k-1)b_kx^{k-1}-b_kx^{k-1}-2(k-1)b_kx^k-2b_kx^k\right]=0$
$\to x^{-1}$: $0b_0=0$, x^0: $-b_1=0$, x^{k-1}: $k(2k-3)b_k=2(k-1)b_{k-1}$, $b_2=b_3=\cdots=0$, $y_2=\frac{1}{x}$].

8 $\boxed{3xy''+(2-6x)y'+2y=0}$ $x=0$ es singular regular con $\lambda(\lambda-1)+\frac{2}{3}\lambda=0 \to \lambda=\frac{1}{3}$, 0 . Es no analítica

$y_1=x^{1/3}\sum_{k=0}^{\infty}c_kx^k \to \sum_{k=0}^{\infty}\left[3(k+\frac{1}{3})(k-\frac{2}{3})c_kx^{k-2/3}+2(k+\frac{1}{3})c_kx^{k-2/3}-6(k+\frac{1}{3})c_kx^{k+1/3}+2c_kx^{k+1/3}\right] \to$

$x^{-2/3}$: $0c_0=0$; $x^{1/3}$: $4c_1=0$; $x^{k-2/3}$: $c_k=\frac{6(k-1)}{k(3k+1)}c_{k-1} \to c_2=c_3=\cdots=0 \to \boxed{y_1=x^{1/3}}$.

$y_2=x^{1/3}\int\frac{e^{\int(2-\frac{2}{3x})}}{x^{2/3}}dx=x^{1/3}\int\frac{1+2x+2x^2+\frac{4}{3}x^3+\cdots}{x^{4/3}}dx=-3(1-x-\frac{2}{5}x^2-\frac{1}{6}x^4+\cdots)=-3\sum\frac{2^nx^n}{n!(1-3n)}$. O bien:

$y_2=\sum_{k=0}^{\infty}b_kx^k \to \sum_{k=0}^{\infty}\left[(3k-1)kb_kx^{k-1}-2(3k-1)b_kx^k\right]=0 \to x^0 : b_1=-b_0$; $x^1 : b_2=\frac{2}{5}b_1=-\frac{2}{5}b_0$;

$x^{k-1} : b_k=\frac{2(3k-4)}{k(3k-1)}b_{k-1} \to b_3=\frac{5}{12}b_2=-\frac{1}{6}b_0$; ... $\to \boxed{y_2=1-x-\frac{2}{5}x^2-\frac{1}{6}x^3+\cdots}$.

9 $\boxed{3xy''-y'-3x^2y=0}$ $x=0$ es singular regular con $r_1=\frac{4}{3}$ y $r_2=0$. Se anula en $x=0$: $y_1=\sum_{k=0}^{\infty}c_kx^{k+4/3}$

$\rightarrow \sum_{k=0}^{\infty}\left[3(k+\frac{4}{3})(k+\frac{1}{3})c_kx^{k+1/3}-(k+\frac{4}{3})c_kx^{k+1/3}-3c_kx^{k+10/3}\right]=\sum_{k=0}^{\infty}\left[k(3k+4)c_kx^{k+1/3}-3c_kx^{k+10/3}\right]=0\rightarrow$

$x^{1/3}: 0c_0=0$; $x^{4/3}: 7c_1=0$, $c_1=0$; $x^{7/3}: 20c_2=0$, $c_2=0$; $x^{k+1/3}: k(3k+4)c_k-3c_{k-3}=0$, $c_k=\frac{3}{k(3k+4)}c_{k-3}$

$\rightarrow c_3=\frac{1}{13}c_0$. $\left[c_{3k}=\frac{c_{3k-3}}{k(9k+4)}\right.$ y $\left.c_{3k+1}=c_{3k+2}=0\right]$. $\boxed{y_1=x^{4/3}\left[1+\frac{1}{13}x^3+\cdots\right]=x^{4/3}+\frac{1}{13}x^{13/3}+\cdots}$.

$y_2=\sum_{k=0}^{\infty}b_kx^k$ es analítica, pero y_1 no lo es. Todas son derivables. Claramente lo es y_2.

Y también y_1, producto de derivables ($x^{4/3}$ lo es, aunque no tiene derivada segunda).

10 $\boxed{4xy''+2y'+y=0}$ $x=0$ singular regular, $r=\frac{1}{2},0$; $y_2=\sum_{n=0}^{\infty}c_kx^k$ analítica; $c_k=-\frac{1}{2k(2k-1)}c_{k-1}$,

$y_2=\sum_{n=0}^{\infty}\frac{(-1)^k}{(2k)!}x^k=\cos\sqrt{x}$, $x\geq 0$; $y_1=\cos\sqrt{x}\int\frac{x^{-1/2}}{\cos^2\sqrt{x}}=2\,\mathrm{sen}\sqrt{x}\rightarrow$ $\boxed{y=c_1\cos\sqrt{x}+c_2\,\mathrm{sen}\sqrt{x}}$.

$s=x^{1/2}\rightarrow\frac{dy}{dx}=\frac{dy}{ds}\frac{1}{2s}$, $\frac{d^2y}{dx^2}=\frac{d^2y}{ds^2}\frac{1}{4s^2}-\frac{dy}{ds}\frac{1}{4s^3}$, $\frac{d^2y}{ds^2}+y=0$, $y=c_1\cos s+c_2\,\mathrm{sen}\,s$ ↑

11 $\boxed{x(2+x^2)y''+2y'-2xy=0}$ $x^2y''+x\frac{2}{2+x^2}y'-2x^2y=0$, $x=0$ s. reg., $r=0$ doble. Analítica $y_1=\sum_{n=0}^{\infty}c_kx^k\rightarrow$

$\left[y_2=x\sum b_kx^k+y_1\ln x\right.$ no es analítica$\left.\right]$.

$\sum_2\left[2k(k-1)c_kx^{k-1}+k(k-1)c_kx^{k+1}\right]+\sum_1 2kc_kx^{k-1}+\sum_0 -2c_kx^{k+1}=\sum_0\left[2k^2c_kx^{k-1}+(k-2)(k+1)c_kx^{k+1}\right]=0$.

x^0: $c_1=0$. x^1: $c_2=\frac{1}{4}c_0$. x^{k-1}: $c_k=-\frac{(k-4)(k-1)}{2k^2}c_{k-2}\rightarrow c_4=0=c_3=c_5=c_6=\cdots\rightarrow$ $\boxed{y_1=1+\frac{1}{4}x^2}$.

12 $\boxed{x^2y''+x(7+2x)y'+9y=0}$ $r=-3$ doble. Ninguna analítica: $y_1=\sum_{k=0}^{\infty}c_kx^{k-3}$, $y_2=\sum_{k=0}^{\infty}b_kx^{k-2}+y_1\ln x$.

$\sum_{k=0}^{\infty}\left[(k-3)(k-4)c_kx^{k-3}+7(k-3)c_kx^{k-3}+2(k-3)c_kx^{k-2}+9c_kx^{k-3}\right]=\sum_{k=0}^{\infty}\left[k^2c_kx^{k-3}-2(3-k)c_kx^{k-2}\right]=0$.

$\rightarrow x^{-3}$: $0c_0=0\ \forall c_0$, x^{-2}: $c_1-6c_0=0$, $c_1=6c_0$, x^{k-3}: $k^2c_k-2(4-k)c_{k-1}=0$, $c_k=\frac{2(4-k)}{k^2}c_{k-1}$.

$c_2=\frac{4}{4}c_1=6c_0$, $c_3=\frac{2}{9}c_2=\frac{4}{3}c_0$, $c_4=0=c_5=\cdots$, $\boxed{y_1=x^{-3}+6x^{-2}+6x^{-1}+\frac{4}{3}}$.

13 $\boxed{x^2(1+x^2)y''-6y=0}$ $x=0$ singular regular de $x^2y''-\frac{6}{1+x^2}y=0$ con $\lambda(\lambda-1)-6=0\rightarrow\lambda_1=3,\lambda_2=-2$.

Está acotada en $x=0$ la $y_1=\sum_{k=0}^{\infty}c_kx^{k+3}$, $c_0\neq 0$ [la serie convergerá al menos en $(-1,1)$, donde lo hace $b^*(x)$].

$\rightarrow\sum_0\left[(k+3)(k+2)c_kx^{k+3}+(k+3)(k+2)c_kx^{k+5}-6c_kx^{k+3}\right]=0\rightarrow x^3$: $6c_0-6c_0=0$, c_0 indeterminado;

x^4: $6c_1=0$, $c_1=0$; x^5: $20c_2+6c_0-6c_2=0$, $c_2=-\frac{3}{7}c_0$; x^{k+3}: $(k+5)kc_k+(k+1)kc_{k-2}=0$, $c_k=-\frac{k+1}{k+5}c_{k-2}$

$\rightarrow c_3=c_5=\cdots=0$, $c_4=-\frac{5}{9}c_2=\frac{5}{21}c_0\rightarrow$ $\boxed{y_1=x^3-\frac{3}{7}x^5+\frac{5}{21}x^7-\cdots}$. El coeficiente de x^{2012} es $\boxed{0}$.

14 $\boxed{x(1+x)y''+(2+3x)y'+y=0}$ $x=0$ singular regular, $r=0,-1\rightarrow y_1=\sum_{k=0}^{\infty}c_kx^k$ acotada en $x=0$.

$\sum_{k=2}^{\infty}[k(k-1)c_kx^{k-1}+k(k-1)c_kx^k]+\sum_{k=1}^{\infty}[2kc_kx^{k-1}+3kc_kx^k]+\sum_{k=0}^{\infty}c_kx^k=0\rightarrow c_{k+1}=-\frac{k+1}{k+2}c_k\rightarrow$

$\boxed{y_1=1-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3+\cdots+(-1)^k\frac{1}{k+1}x^k+\cdots}=\frac{\log(1+x)}{x}$, función no analítica en $x=-1$.

Si no se identifica la serie: $s=x+1\rightarrow s(s-1)y''+(3s-1)y'+y=0\rightarrow r=0$ doble $\rightarrow$

$y_1=\sum_{k=0}^{\infty}c_ks^k$ analítica, pero $y_2=s\sum_{k=0}^{\infty}b_ks^k+y_1\log s$ no analítica en $s=0$ ($x=-1$).

Utilizando que $y_1=\frac{1}{x}$: $e^{-\int\frac{2+3x}{x(1+x)}}=e^{-\int[\frac{2}{x}+\frac{1}{1+x}]}=\frac{1}{x^2(1+x)}$, $y_2=\frac{1}{x}\int\frac{dx}{1+x}=\frac{\log(1+x)}{x}$,

solución acotada en $x=0$ con el desarrollo de arriba y claramente no analítica en $x=-1$.

15 $\boxed{xy''-(1+x)y'+y=0}$ $x=0$ es singular regular con $r(r-1)-r+0=0$, $r=2,0$. Se anula en $x=0$:

$y_1=\sum_{k=0}^{\infty}c_kx^{k+2}\rightarrow\sum_{k=0}^{\infty}\left[(k+2)(k+1)c_kx^{k+1}-(k+2)c_kx^{k+1}-(k+2)c_kx^{k+2}+c_kx^{k+2}\right]=0\rightarrow$

x^1: $2c_0-2c_0=0$, $\forall c_0$. x^2: $3c_1-c_0=0$, $c_1=\frac{1}{3}c_0$. x^{k+1}: $(k+2)kc_k-kc_{k-1}=0\rightarrow c_k=\frac{1}{k+2}c_{k-1}$

$\rightarrow c_2=\frac{1}{4}c_1=\frac{1}{12}c_0$, $c_3=\frac{1}{5}c_2=\frac{1}{60}c_0,\ldots\rightarrow$ $\boxed{y_1=x^2+\frac{1}{3}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5+\cdots}$ $\left[=2(e^x-1-x)\right]$.

La otra solución $y_2=\sum_{k=0}^{\infty}b_kx^k+dy_1\ln x$ también está acotada en $x=0$ (sea o no $d=0$), porque $x^2\ln x\xrightarrow{x\to 0}0$.

Es analítica y_1. Si $d=0$ lo será $y_2\rightarrow\sum_{k=0}^{\infty}\left[k(k-2)b_kx^{k-1}-(k-1)b_kx^k\right]+d\left[2y_1'-\frac{2}{x}y_1-y_1\right]=0\rightarrow$ x^0: $b_1=b_0$. x^1: $d[2]=0$.

[Con un poco de vista o alguna integral se puede dar la solución general sin series: $y=c_1e^x+c_2(1+x)$].

16 $\boxed{(1-x^2)y''-2xy'+y=0}$ Como $x=0$ es regular (a y b son analíticas para $|x|<1$), probamos:

$$\sum_{k=0}^{\infty}c_kx^k \to \sum_{k=2}^{\infty}[k(k-1)c_kx^{k-2}-k(k-1)c_kx^k]+\sum_{k=1}^{\infty}-2kc_kx^k+\sum_{k=0}^{\infty}c_kx^k=0 \to$$

$x^0:\ 2c_2+c_0=0 \to c_2=-\frac{c_0}{2}$; $x^1:\ 6c_3-2c_1+c_1=0 \to c_3=\frac{c_1}{6}$; $x^k:\ (k+2)(k+1)c_{k+2}-(k^2+k-1)c_k=0$

$\to c_{k+2}=\frac{k^2+k-1}{(k+2)(k+1)}c_k \to c_4=0\,,\ c_5=\frac{11}{20}c_3=\frac{11}{120}c_1 \to \boxed{y=x+\frac{1}{6}x^3+\frac{11}{120}x^5+\cdots}\ \big[\,c_0=y(0)\,,\ c_1=y'(0)\,\big].$

De otra forma: $y''(0)+y(0)=0 \to y''(0)=0$. Y derivando: $(1-x^2)y'''-4xy''-y'=0 \to y'''(0)=y'(0)=1$, $(1-x^2)y^{IV}-6xy'''-5y''=0 \to y^{IV}(0)=5y''(0)=0$, $(1-x^2)y^{V}-8xy^{IV}-11y'''=0 \to y^{V}(0)=11y'''(0)=11$.

Haciendo $x+1=s$ obtenemos $(2s-s^2)y''+2(1-s)y'+y=0$, $s^2y''+s\frac{2(1-s)}{2-s}y'+\frac{s}{2-s}y=0 \to$

$r^2=0 \to$ ni $y_1=c_0+c_1s+\cdots$ ni $y_2=b_0s+\cdots+y_1\ln|s|$ tienden a 0 si $s\to 0$ $(x\to -1)$.

17 $\boxed{xy''+2y'-xy=0}$ $x=0$ singular regular con $r=0,-1 \to y_1=\sum_{k=0}^{\infty}c_kx^k$. $y_2=\sum_{k=0}^{\infty}b_kx^{k-1}+dy_1\ln x$.

Probamos $\sum_{k=0}^{\infty}b_kx^{k-1} \to \sum_{k=0}^{\infty}\big[(k-1)(k-2)b_kx^{k-2}+2(k-1)b_kx^{k-2}-b_kx^k\big]=\sum_{k=0}^{\infty}\big[k(k-1)b_kx^{k-2}-b_kx^k\big]=0 \to$

$x^{-2}:\ 0b_0=0\,,\ \forall b_0$, $x^{-1}:\ 0b_1=0\,,\ \forall b_1$, $x^0:\ 2b_2-b_0=0\,,\ b_2=\frac{1}{2}b_0$, $x^1:\ 6b_3-b_1=0\,,\ b_3=\frac{1}{6}b_1$.

$x^{k-2}:\ k(k-1)b_k-b_{k-2}=0\,,\ b_k=\frac{b_{k-2}}{k(k-1)} \to b_4=\frac{b_2}{4\cdot 3}=\frac{b_0}{4!}\,,\ b_5=\frac{b_3}{5\cdot 4}=\frac{b_1}{5!}\,,\ \ldots\,,\ b_{2k}=\frac{b_0}{(2k)!}\,,\ b_{2k+1}=\frac{b_1}{(2k+1)!}$.

La solución es: $\boxed{y=b_0\frac{1}{x}\big[1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots\big]+b_1\frac{1}{x}\big[x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots\big]=b_0\frac{\operatorname{ch}x}{x}+b_1\frac{\operatorname{sh}x}{x}}$.

$y=\frac{v}{x} \to y'=\frac{v'}{x}-\frac{v}{x^2}\,,\ y''=\frac{v''}{x}-\frac{2v'}{x^2}+\frac{2v}{x^3} \to v''-\frac{2v'}{x}+\frac{2v}{x^2}+\frac{2v'}{x}-\frac{2v}{x^2}-v=v''-v=0\uparrow$

18 $\boxed{x(x+1)y''+(x-1)y'=0}$ $x=0$ singular regular con $a^*(x)=\frac{x-1}{x+1}$, $b^*(x)=0$, $r=2,0$. Se anula en $x=0$:

$$y_1=\sum_{k=0}^{\infty}c_kx^{k+2} \to \sum_{k=0}^{\infty}\big[(k+2)(k+1)c_kx^{k+2}+(k+2)(k+1)c_kx^{k+1}+(k+2)c_kx^{k+2}-(k+2)c_kx^{k+1}\big]=0 \to$$

$x^1:2c_0-2c_0=0\,,\ \forall c_0$. $x^2:4c_0+3c_1=0\,,\ c_1=-\frac{4}{3}c_0$.

$x^{k+1}:\ (k+1)^2c_{k-1}+k(k+2)c_k=0 \to c_k=-\frac{(k+1)^2}{k(k+2)}c_{k-1} \to c_2=-\frac{9}{8}c_1=\frac{3}{2}c_0\,,\ \ldots \to$

$\boxed{y_1=x^2-\frac{4}{3}x^3+\frac{3}{2}x^4+\cdots}$ $\big[=2\ln(1+x)-\frac{2x}{1+x}$ (no acotada) ya que $v'=\big[\frac{1}{x}-\frac{2}{x+1}\big]v,\ v=\frac{Cx}{(x+1)^2}\,,\ldots\big]$.

$x=\frac{1}{s} \to \frac{1}{s}\big(1+\frac{1}{s}\big)\big[s^4\ddot{y}+2s^3\dot{y}\big]-s^2\big(\frac{1}{s}-1\big)\dot{y}=0$, $s(1+s)\ddot{y}+(1+3s)\dot{y}=0$, $r=0$ doble. La obvia $y_1=1$ es acotada, pero $y_2=s\sum_{k=0}^{\infty}c_ks^k+\ln s$ **no está acotada** para $s\to 0^+$ $(x\to\infty)$. [Las series de $x=0$ no informan sobre el infinto].

19 $\boxed{x(1-x)y''-(1+x)y'+y=0}$ $x=0$ singular regular con $r=2,0$. Se anula en $x=0$:

$$y_1=\sum_{k=0}^{\infty}c_kx^{k+2} \to \sum_{k=0}^{\infty}\big[(k+2)(k+1)c_kx^{k+1}-(k+2)(k+1)c_kx^{k+2}-(k+2)c_kx^{k+1}-(k+2)c_kx^{k+2}+c_kx^{k+2}\big]=0$$

$\to x^1:2c_0-2c_0=0\,,\ \forall c_0$. $x^2:6c_1-2c_0-3c_1-2c_0+c_0=0\,,\ c_1=c_0$.

$x^{k+1}:\ k(k+2)c_k-k(k+2)c_{k-1}=0 \to c_k=c_{k-1}\,,\ c_2=c_1=c_0\,,\ \ldots \to \boxed{y_1=x^2+x^3+x^4-\cdots=\frac{x^2}{1-x}}$ no acotada.

La solución $y_2=1+x$ casi salta a la vista y sale (largo) con Frobenius. $y=\frac{c_1x^2}{1-x}+c_2(1+x)\ \xrightarrow{c_1=c_2=1}\ \frac{1}{1-x}\ \xrightarrow{x\to\infty}\ 0$.

También se puede hallar la solución general desde la primera: $e^{-\int a}=\frac{x}{(1-x)^2}$, $y_2=y_1\int\frac{1}{x^3}=-\frac{1}{2(1-x)}$.

O bien, $x=\frac{1}{s} \to \frac{1}{s}\big(1-\frac{1}{s}\big)\big[s^4\ddot{y}+2s^3\dot{y}\big]+s^2\big(1+\frac{1}{s}\big)\dot{y}+y=s^2(s-1)\ddot{y}+s(3s-1)\dot{y}+y=0$, $r=\pm 1$, $y_1=s\sum \underset{s\to 0}{\longrightarrow} 0$.

20 $\boxed{x(x-1)y''+y'-py=0}$ $x=0$ singular regular, $r=2,0$; $y_1=\sum_{k=0}^{\infty}c_kx^{k+2}$; $y_2=\sum_{k=0}^{\infty}b_kx^k+dy_1\ln x$.

$x^1:-2c_0+2c_0=0\,,\ \forall c_0$. $x^2:(2-p)c_0-3c_1=0\,,\ c_1=\frac{2-p}{3}c_0$. $x^{k+1} \to c_k=\frac{(k+1)k-p}{k(k+2)}c_{k-1}$

$\to y_1$ será polinomio si $\boxed{p=n(n+1)}$, $n=1,2,\ldots$ $P_1=x^2$, $P_2=x^2-\frac{4}{3}x^3$, ...

Si $\boxed{p=0}$, y_1 no lo es, pero lo será la clara $y_2=1$ (son cero d y demás b_k del teorema de Frobenius).

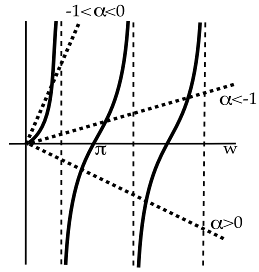
Para $p=2$, $y_2=x^2\int\frac{e^{-\int[1/(x^2-x)]}}{x^4}=x^2\int\frac{1}{x^3(x-1)}=x^2\ln\big|\frac{x-1}{x}\big|+x+\frac{1}{2}\underset{x\to\infty}{\longrightarrow}0$ $\Big[\frac{\ln(1-s)+s+\frac{s^2}{2}}{s^2}\underset{s\to 0^+}{\longrightarrow}0\Big]$.

O bien, $x=\frac{1}{s} \to s^2(1-s)\ddot{y}+s(2-3s)\dot{y}-2y=0$, $r=1,-2$, hay soluciones $y_1=s\sum c_ks^k\underset{s\to 0}{\longrightarrow}0$.

Soluciones de problemas 3 de MII(C) (2023-24)

1 $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y'(0)-\alpha y(0)=y(1)=0 \end{array}}$ $\lambda>0$: Si $\alpha=0$, $\lambda_n=\frac{(2n-1)^2\pi^2}{2^2}$, $y_n=\big\{\cos\frac{(2n-1)\pi x}{2}\big\}$.

Si $\alpha\neq 0$: $\left.\begin{array}{l} wc_2-\alpha c_1=0 \\ c_1\cos w+c_2\,\mathrm{sen}\,w=0 \end{array}\right\}$ $\tan w_n=-\frac{w_n}{\alpha}$, $y_n=\{\mathrm{sen}\,w_n(x-1)\}$.



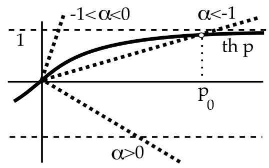
$\lambda=0$: $\left.\begin{array}{l} c_2-\alpha c_1=0 \\ c_1+c_2=0 \end{array}\right\}\to$ Autovalor si $\alpha=-1$, con autofunción $y_0\equiv\{1-x\}$.

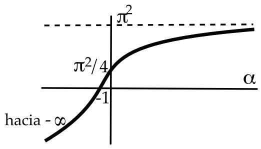
$\lambda<0$: $y=c_1\,e^{px}+c_2\,e^{-px}$, $\left.\begin{array}{l} (p-\alpha)c_1-(p+\alpha)c_2=0 \\ c_1\,e^{p}+c_2\,e^{-p}=0 \end{array}\right\}\to$

$p[e^p+e^{-p}]+\alpha[e^p-e^{-p}]=0\to \mathrm{th}\,p=-\frac{p}{\alpha}$

Si $\alpha<-1$ hay un $\lambda=-p_0^2$ $\big[y_0\equiv\{\alpha\,\mathrm{sh}\,p_0x+p_0\,\mathrm{ch}\,p_0x\}\big]$.

El menor autovalor es negativo si $\alpha<-1$, 0 si $\alpha=-1$ y positivo si $\alpha>-1$ (para $\alpha=0$ es $\lambda=\frac{\pi^2}{4}$).





2 a) $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y(0)-2y'(0)=y(1)-2y'(1)=0 \end{array}}$ $\lambda<0$, $y=c_1e^{px}+c_2e^{-px}\to \begin{array}{l} c_1+c_2-2p[c_1-c_2]=0 \\ c_1e^p+c_2e^{-p}-2p[c_1e^p-c_2e^{-p}]=0 \end{array}$.

$\begin{vmatrix} 1-2p & 1+2p \\ [1-2p]e^p & [1+2p]e^{-p} \end{vmatrix}=[1-2p][1+2p][e^{-p}-e^p]=0$ si $p=\frac{1}{2}$ (y es $c_2=0$). $\boxed{\lambda_0=-\frac{1}{4}}$ e $\boxed{y_0=\{e^{x/2}\}}$.

$\lambda=0$, $y=c_1+c_2x\to \begin{array}{l} c_1-2c_2=0 \\ c_1-c_2=0 \end{array}\Rightarrow c_1=c_2=0$. No autovalor.

$\lambda>0$, $y=c_1\cos wx+c_2\,\mathrm{sen}\,wx\to \begin{array}{l} c_1-2wc_2=0,\ c_1=2wc_2 \\ c_1\cos w+c_2\,\mathrm{sen}\,w-2w[-c_1\,\mathrm{sen}\,w+c_2\cos w]=0 \end{array}\to c_1(1+4w^2)\,\mathrm{sen}\,w=0$.

Por tanto, $w_n=n\pi$, $\boxed{\lambda_n=n^2\pi^2}$, $\boxed{y_n=\{2n\pi\cos n\pi x+\mathrm{sen}\,n\pi x\}}$, $n=1,2,\dots$

$\langle y_0,y_0\rangle=\int_0^1 e^x dx=\boxed{e-1}$. $\langle y_n,y_n\rangle=\int_0^1\big[2n^2\pi^2(1+\cos 2n\pi x)+2n\pi\,\mathrm{sen}\,2n\pi x+\frac{1-\cos 2n\pi x}{2}\big]dx=\boxed{\frac{1}{2}+2n^2\pi^2}$.

b) $\boxed{\begin{array}{l} y''+2y'+\lambda y=0 \\ y(0)=y(\pi)=0 \end{array}}$ Si $\lambda>1$: $\mu^2+2\mu+\lambda=0$, $\mu=-1\pm i\,w$, con $w=\sqrt{\lambda-1}\to y=(c_1\cos wx+c_2\,\mathrm{sen}\,wx)\,e^{-x}$.

$y(0)=c_1=0\to y(\pi)=e^{-\pi}c_2\,\mathrm{sen}\,w\pi=0\to w_n=n$, $\boxed{\lambda_n=1+n^2}$, $\boxed{y_n=\{e^{-x}\,\mathrm{sen}\,nx\}}$.

La ecuación en forma autoadjunta queda $\big[e^{2x}y'\big]'+e^{2x}\lambda y=0$, con lo que el peso es $r(x)=e^{2x}$.

Por tanto, $\langle y_n,y_n\rangle=\int_0^\pi r\,y_n^2\,dx=\int_0^\pi e^{2x}\,e^{-2x}\,\mathrm{sen}^2 nx\,dx=\int_0^\pi\big(\frac{1}{2}-\frac{\cos 2nx}{2}\big)dx=\frac{\pi}{2}-\big[\frac{\mathrm{sen}\,2nx}{4n}\big]_0^\pi=\boxed{\frac{\pi}{2}}$.

c) $\boxed{\begin{array}{l} x^2y''+xy'+[\lambda x^2-\frac{1}{4}]y=0 \\ y(1)=y(4)=0 \quad \text{[casi Bessel]} \end{array}}$ $u=\sqrt{x}\,y\to \left.\begin{array}{l} u''+\lambda u=0 \\ u(1)=u(4)=0 \end{array}\right\}\xrightarrow{x=s+1}\left.\begin{array}{l} u''+\lambda u=0 \\ u(0)=u(3)=0 \end{array}\right\}\to u_n=\{\mathrm{sen}\frac{n\pi s}{3}\}=\{\mathrm{sen}\frac{n\pi(x-1)}{3}\}$.

$s=\sqrt{\lambda}\,x=wx\to s^2y''+sy'+[s^2-\frac{1}{4}]y=0$, $y=c_1\frac{\cos wx}{\sqrt{x}}+c_2\frac{\mathrm{sen}\,wx}{\sqrt{x}}\underset{\text{c.c.}}{\to}\lambda_n=\frac{n^2\pi^2}{9}$, $y_n=\big\{\frac{1}{\sqrt{x}}\,\mathrm{sen}\frac{n\pi(x-1)}{3}\big\}$, $n=1,2,\dots$

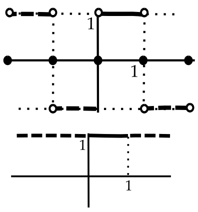
$[xy']'-\frac{y}{4x}+\lambda xy=0$. $\langle y_n,y_n\rangle=\int_1^4 x\frac{1}{x}\,\mathrm{sen}^2\frac{n\pi(x-1)}{3}\,dx=\frac{1}{2}\int_1^4\big(1-\cos\frac{2n\pi(x-1)}{3}\big)dx=\frac{3}{2}$.

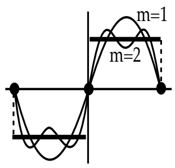
3 a) $\boxed{f(x)=1}$ Su serie en senos es: $\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\mathrm{sen}(2m-1)\pi x}{2m-1}$.

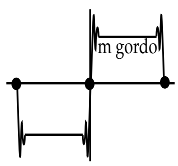
Tiende hacia la extensión 2-periódica de $f(x)=\begin{cases}-1, & -1<x<0 \\ 1, & 0<x<1\end{cases}$,

y la suma es 0 si $x\in\mathbf{Z}$. Cerca de ellos convergerá mal.

La serie en cosenos es la propia constante $1=1+0+0+\cdots$ (es uno de los elementos de la base de Fourier).



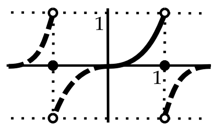


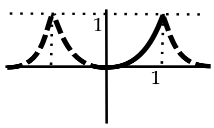


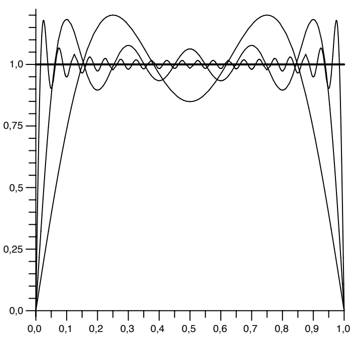
b) $\boxed{f(x)=x^2}$ $=\sum_{n=1}^{\infty}\big[\frac{2(-1)^{n+1}}{\pi n}+\frac{4[(-1)^n-1]}{\pi^3n^3}\big]\mathrm{sen}\,n\pi x$.

[En la serie en senos aparecerán picos cerca de 1].

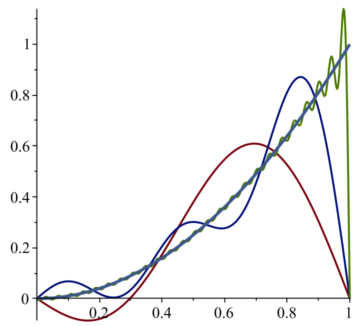
$x^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$ converge uniformemente en $[0,1]$.



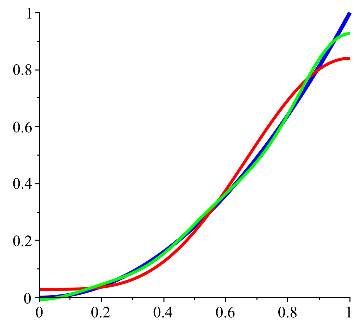




a) sen, $m=2,5,20$



b) sen, $n=2,5,50$



b) cos, $n=2,5$

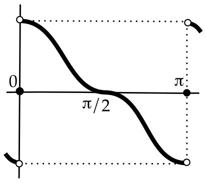
4 i) Para desarrollar en cosenos basta escribir $\boxed{\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x}$, $x\in[0,\pi]$.

(La 'serie' claramente 'converge uniformemente' en todo $[0,\pi]$ hacia la f dada).

ii) $b_n = \frac{2}{\pi}\int_0^\pi \cos^3 x \,\mathrm{sen}\, nx\, dx = \frac{3}{4\pi}\int_0^\pi [\mathrm{sen}(n+1)x + \mathrm{sen}(n-1)x]\, dx + \frac{1}{4\pi}\int_0^\pi [\mathrm{sen}(n+3)x + \mathrm{sen}(n-3)x]\, dx$

$= -\frac{3}{4\pi}\Big[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\Big]_0^\pi - \frac{1}{4\pi}\Big[\frac{\cos(n+3)x}{n+3} + \frac{\cos(n-3)x}{n-3}\Big]_0^\pi$

$= \frac{3}{4\pi}\Big[\frac{1+(-1)^n}{n+1} + \frac{1+(-1)^n}{n-1}\Big] + \frac{1}{4\pi}\Big[\frac{1+(-1)^n}{n+3} + \frac{1+(-1)^n}{n-3}\Big] = \frac{2n(n^2-7)[1+(-1)^n]}{\pi(n^2-1)(n^2-9)}$



$$\boxed{\cos^3 x = \sum_{m=1}^{\infty} \frac{8m(4m^2-7)}{\pi(4m^2-1)(4m^2-9)}\,\mathrm{sen}\, 2mx} \ , \ x\in(0,\pi) \ .$$

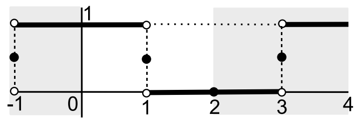
Converge hacia f en los puntos de continuidad de su extensión impar y 2π-periódica, es decir, hacia $\cos^3 x$ en $(0,\pi)$ y a 0 (evidentemente) si $x=0,\pi$. Hay convergencia uniforme en todo $[a,b]\subset(0,\pi)$.

5 $\boxed{f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & 1<x\le 2\end{cases}}$ Si $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, es $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$. En este caso:

$a_0 = \frac{2}{2}\int_0^1 dx = 1$, $a_n = \int_0^1 \cos\frac{n\pi x}{2}\, dx = \frac{2}{n\pi}\,\mathrm{sen}\,\frac{n\pi}{2} = \begin{cases} 0\ , & n \text{ par}\\ \frac{2(-1)^m}{(2m+1)\pi}\ , & n=2m+1\end{cases} \to \ \frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos\frac{(2m+1)\pi x}{2}$.

i) En $x=1$ es f discontinua y la serie tenderá hacia $\frac{1}{2}[f(1^-)+f(1^+)] = \frac{1}{2}$, como se comprueba fácil:

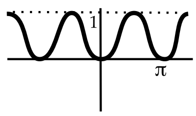
$\frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos\frac{(2m+1)\pi}{2} = \frac{1}{2}$ [los cosenos se anulan].



ii) Como tiende en todo **R** hacia la extensión par y 4-periódica de f, en $x=2$ ha de tender hacia $f(2)=0$. Sustituyendo:

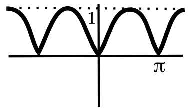
$\frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos(2m+1)\pi = \frac{1}{2} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} = 0$, ya que la última serie $1-\frac{1}{3}+\frac{1}{5}+\cdots = \arctan 1 = \frac{\pi}{4}$.

6 a) $\boxed{f(x)=\mathrm{sen}^2 x} = \frac{1}{2} - \frac{1}{2}\cos 2x$, ya desarrollada $\big[\, a_0 = \frac{1}{2}\, ,\ a_2 = -\frac{1}{2}$ resto a_n y b_n son $0\,\big]$.



b) $\boxed{f(x)=|\mathrm{sen}\, x|}$ par $\to$ $b_n = 0$. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\mathrm{sen}\, x|\, dx = \frac{2}{\pi}\int_0^\pi \mathrm{sen}\, x\, dx = \frac{4}{\pi}$.

$a_1 = \frac{2}{\pi}\int_0^\pi \mathrm{sen}\, x\cos x\, dx = 0$.

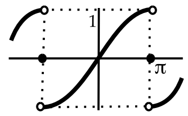


$a_n = \frac{2}{\pi}\int_0^\pi \mathrm{sen}\, x\cos nx\, dx = \frac{1}{\pi}\int_0^\pi [\mathrm{sen}(1+n)x + \mathrm{sen}(1-n)x]\, dx = -\frac{1}{\pi}\Big[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\Big]_0^\pi$

$= \frac{1}{\pi}\Big[\frac{1+\cos n\pi}{1+n} + \frac{1+\cos n\pi}{1-n}\Big] = \frac{2}{\pi}\frac{1+(-1)^n}{1-n^2} \ \to\ |\mathrm{sen}\, x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{1-4m^2}$.

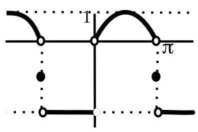
c) $\boxed{f(x)=\mathrm{sen}\,\frac{x}{2}}$ impar. $a_n = 0$. $b_n = \frac{2}{\pi}\int_0^\pi \mathrm{sen}\,\frac{x}{2}\,\mathrm{sen}\, nx\, dx$

$= \frac{1}{\pi}\int_0^\pi \Big[\cos\frac{(1-2n)x}{2} - \cos\frac{(1+2n)x}{2}\Big] dx = \frac{8}{\pi}\frac{(-1)^n n}{1-4n^2}$.



d) $\boxed{f(x) = \begin{cases} -\pi, & \text{si } -\pi\le x<0\\ \mathrm{sen}\, x, & \text{si } 0\le x<\pi\end{cases}}$ $a_n = \frac{1}{\pi}\int_0^\pi \mathrm{sen}\, x\cos nx\, dx - \int_{-\pi}^0 \cos nx\, dx$, $n=0,1,\dots$

$b_n = \frac{1}{\pi}\int_0^\pi \mathrm{sen}\, x\,\mathrm{sen}\, nx\, dx - \int_{-\pi}^0 \mathrm{sen}\, nx\, dx$, $n=1,2,\dots$



$\frac{1}{2}a_0 = \frac{1}{\pi} - \frac{\pi}{2}$; $a_1 = 0$; $a_n = \frac{1}{\pi}\frac{1+(-1)^n}{1-n^2}$, $n=2,3,\dots$; $b_1 = \frac{5}{2}$; $b_n = \frac{1-(-1)^n}{n}$, $n=2,3,\dots$

7 a) $\boxed{\begin{array}{l} y''+\lambda y=0\\ y(0)=y'(1)=0\end{array}}$ $\lambda\ge 0$ (teor 1). $\lambda=0$: $y=c_1+c_2x$, $\left.\begin{array}{l} y(0)=c_1=0\\ y'(1)=c_2=0\end{array}\right\} \to y\equiv 0$. $\lambda=0$ no autovalor.

$\lambda>0$: $y=c_1\cos wx + c_2\,\mathrm{sen}\, wx$, $\left.\begin{array}{l} y(0)=c_1=0\\ y'(1)=wc_2\cos w=0\end{array}\right\} \to \lambda_n = \frac{(2n-1)^2\pi^2}{2^2}$, $y_n = \big\{\mathrm{sen}\,\frac{(2n-1)\pi x}{2}\big\}$, $n=1,2,\dots$.

$x = \sum_{n=1}^{\infty} c_n\,\mathrm{sen}\,\frac{(2n-1)\pi x}{2}$. Conocido $\langle y_n, y_n\rangle = \frac{1}{2}$. Es $c_n = 2\int_0^1 x\,\mathrm{sen}\,\frac{(2n-1)\pi x}{2}\, dx = \frac{8(-1)^{n+1}}{\pi^2(2n-1)^2}$.

b) $\boxed{\begin{array}{l} y''+\lambda y=0\\ y(-1)=y(1)=0\end{array}}$ $\overset{s=x+1}{\to}$ $\left.\begin{array}{l} y''+\lambda y=0\\ y(0)=y(2)=0\end{array}\right\} \to \lambda_n = \frac{n^2\pi^2}{2^2}$, $y_n \equiv \big\{\mathrm{sen}\,\frac{n\pi s}{2}\big\} = \big\{\mathrm{sen}\big(\frac{n\pi x}{2} + \frac{n\pi}{2}\big)\big\}$, $n=1,2,\dots$.

Directamente ($\lambda>0$): $\left.\begin{array}{l} c_1\cos w - c_2\,\mathrm{sen}\, w=0\\ c_1\cos w + c_2\,\mathrm{sen}\, w=0\end{array}\right\} \to \big|\ \ \big| = \mathrm{sen}\, 2w = 0$, $\lambda_n = \frac{n^2\pi^2}{2^2} \to$ $\begin{array}{l} n \text{ par, } c_1=0 \to \text{sen}\\ n \text{ impar, } c_2=0\to \text{cos}\end{array}$.

$x = \sum_{n=1}^{\infty} c_n\,\mathrm{sen}\,\frac{n\pi(x+1)}{2}$. $r=1$. $\langle y_n, y_n\rangle = \int_{-1}^1 \mathrm{sen}^2\frac{n\pi(x+1)}{2} = 1$, $c_n = \int_{-1}^1 x\,\mathrm{sen}\,\frac{n\pi(x+1)}{2}\, dx = -\frac{2[1+(-1)^n]}{n\pi}$.

c) $\boxed{\begin{array}{l} x^2y''+xy'+\lambda y=0\\ y'(1)=y'(\mathrm{e})=0\end{array}}$ $\big(xy'\big)' + \lambda\frac{1}{x}y = 0$. $\lambda\ge 0$. $\lambda=0$: $y=c_1+c_2\ln x$, $\begin{array}{l} y'(1)=c_2=0\\ y'(2)=c_2/\mathrm{e}=0\end{array} \to y_0=\{1\}$.

$\lambda>0$: $y=c_1\cos(w\ln x)+c_2\,\mathrm{sen}(w\ln x)$ $\begin{array}{l} y'(1)=wc_2=0 \downarrow\\ y'(2)=-\frac{w}{\mathrm{e}}c_1\,\mathrm{sen}\, w=0 \nearrow\end{array}$ $\lambda_n = n^2\pi^2$, $y_n = \big\{\cos(n\pi\ln x)\big\}$, $n=1,2,\dots$

$\Big[$O haciendo $x=\mathrm{e}^s \to \frac{d^2y}{ds^2}+\lambda y=0$, $y(s=0)=y'(s=1)$, problema conocido con esos λ_n e $y_n(s)=\{\cos n\pi s\}\uparrow\Big]$.

$x = \sum_{n=0}^{\infty} c_n\cos(n\pi\ln x)$. $\begin{array}{l} \langle x,1\rangle = \int_1^{\mathrm{e}}\frac{x}{x}dx = \mathrm{e}-1\ ,\ \langle 1,1\rangle = \int_1^{\mathrm{e}}\frac{1}{x}dx = 1 \ \to\ c_0 = \frac{\langle x,1\rangle}{\langle 1,1\rangle} = \mathrm{e}-1 \ .\\ \langle y_n, y_n\rangle = \int_1^{\mathrm{e}}\frac{1}{x}\cos^2(n\pi\ln x)\, dx = \frac{1}{2} \ \to\ c_n = 2\int_1^{\mathrm{e}}\cos(n\pi\ln x)\, dx = 2\,\frac{(-1)^n\mathrm{e}-1}{1+n^2\pi^2}\ ,\ n\ge 1\ .\end{array}$

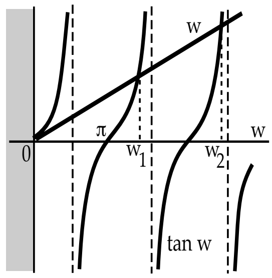
8 a) $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y(0)=y(1)-y'(1)=0 \end{array}}$ $\lambda=0:\ y=c_1+c_2x \to \left.\begin{array}{l} c_1=0 \\ c_1+c_2-c_2=0 \end{array}\right\} \to c_1=0 \to \lambda_0=0$ autovalor con $\boxed{y_0=\{x\}}$.

$\lambda>0:\ y=c_1\cos wx+c_2 \,\mathrm{sen}\, wx$, $y'=-wc_1\,\mathrm{sen}\, wx+wc_2\cos wx$, $w=\sqrt{\lambda}\ \to$

$\left.\begin{array}{l} c_1=0 \searrow \\ c_2(\mathrm{sen}\, w-w\cos w)=0 \end{array}\right\}$ c_2 queda indeterminado si $\mathrm{sen}\, w=w\cos w$.

Hay infintos w_n con $w_n=\tan w_n \to \lambda_n=w_n^2$, $y_n=\{\mathrm{sen}\, w_nx\}$.

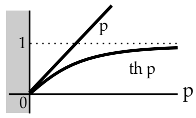
[Con algún método numérico se irían calculando: $w_1\approx 4.493$, $w_2\approx 7.725, \dots$].



Como $\beta\beta'<0$ podría haber $\lambda<0$. El dato nos dice que no habrá:

$y=c_1\mathrm{e}^{px}+c_2\mathrm{e}^{-px} \to \left.\begin{array}{l} c_1=-c_2 \searrow \\ c_2(p[\mathrm{e}^p+\mathrm{e}^{-p}]-[\mathrm{e}^p-\mathrm{e}^{-p}])=0 \end{array}\right\} \to y\equiv 0$

$\big[$no existe $p>0$ con $p=\mathrm{th}\,p$, pues $(\mathrm{th}\,p)'(0)=1\big]$.



Los coeficientes del desarrollo $1=\sum\limits_{n=0}^{\infty} c_ny_n(x)=c_0\,x+\sum\limits_{n=1}^{\infty}\mathrm{sen}\, w_nx$ vienen dados por $c_n=\frac{\langle 1,y_n\rangle}{\langle y_n,y_n\rangle}$.

En particular, $\langle 1,x\rangle=\int_0^1 x\,dx=\frac{1}{2}$ y $\langle x,x\rangle=\int_0^1 x^2\,dx=\frac{1}{3}$. Por tanto, $c_0=\frac{1/2}{1/3}=\boxed{\frac{3}{2}}$, $1=\frac{3}{2}x+\cdots$

El resto de coeficientes: $\langle \mathrm{sen}\, w_nx,\mathrm{sen}\, w_nx\rangle=\int_0^1\frac{1-\cos 2w_nx}{2}\,dx=\frac{1}{2}-\frac{\mathrm{sen}\, 2w_n}{4w_n}=\frac{1}{2}-\frac{\mathrm{sen}\, w_n\cos w_n}{2w_n}=\frac{1-\cos^2 w_n}{2}$.

$\langle 1,\mathrm{sen}\, w_nx\rangle=\int_0^1\mathrm{sen}\, w_nx\,dx=\frac{1}{w_n}[1-\cos w_n]$, $c_n=\frac{2}{w_n(1+\cos w_n)}$.

b) $\boxed{\begin{array}{l} y''+2y'+\lambda y=0 \\ y(0)+y'(0)=y(1/2)=0 \end{array}}$ En forma autoadjunta: $\big(y'\mathrm{e}^{2x}\big)'+\lambda\mathrm{e}^{2x}y=0$ [problema de S-L regular].
$\mu^2+2\mu+\lambda=0 \to \mu=-1\pm\sqrt{1-\lambda}$. En principio, puede haber λ negativos.

$\lambda<1$, $\sqrt{1-\lambda}=p \to y=c_1\mathrm{e}^{(p-1)x}+c_2\mathrm{e}^{-(p+1)x} \to \begin{array}{l} y(0)+y'(0)=p(c_1-c_2)=0 \\ y(\frac{1}{2})=(c_1\mathrm{e}^{p/2}+c_2\mathrm{e}^{-p/2})\mathrm{e}^{-1/2}=0 \end{array} \to c_1=c_2=0$.

$\lambda=1 \to y=(c_1+c_2x)\mathrm{e}^{-x} \to \begin{array}{l} y(0)+y'(0)=c_2=0 \\ y(\frac{1}{2})=(c_1+\frac{c_2}{2})\mathrm{e}^{-1/2}=0 \end{array} \to c_1=c_2=0$.

$\lambda>1$, $\sqrt{\lambda-1}=w \to y=(c_1\cos wx+c_2\,\mathrm{sen}\, wx)\,\mathrm{e}^{-x} \to y(0)+y'(0)=c_2w=0 \to c_2=0 \to$

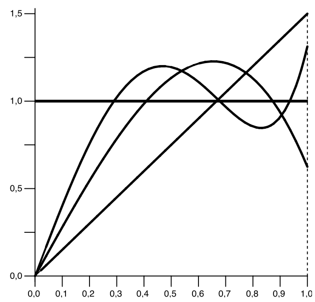
$y\big(\frac{1}{2}\big)=c_1\cos\frac{w}{2}\mathrm{e}^{-1/2}=0 \to w_n=(2n-1)\pi$, $\lambda_n=1+(2n-1)^2\pi^2$, $y_n=\{\mathrm{e}^{-x}\cos(2n-1)\pi x\}$, $n=1,2,\dots$

Por tanto: $1=\sum\limits_{n=1}^{\infty}\frac{\langle 1,y_n\rangle}{\langle y_n,y_n\rangle}y_n$, con $\langle 1,y_n\rangle=\int_0^{1/2}\mathrm{e}^x\cos(2n-1)\pi x\,dx$, $\langle y_n,y_n\rangle=\int_0^{1/2}\cos^2(2n-1)\pi x\,dx$

Como $\int_0^{1/2}\cos^2 bx=\frac{1}{2}\int_0^{1/2}(1+\cos 2bx)=\frac{1}{4}+\frac{\mathrm{sen}\, b}{4b} \to \langle y_n,y_n\rangle=\frac{1}{4}$ e $\int\mathrm{e}^x\cos bx\,dx=\frac{(\cos bx+b\,\mathrm{sen}\, bx)\mathrm{e}^x}{1+b^2}$,

concuimos que: $1=4\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)\pi\mathrm{e}^{1/2}-1}{1+(2n-1)^2\pi^2}\,\mathrm{e}^{-x}\cos(2n-1)\pi x$.

9 $\boxed{\begin{array}{l} \big([1-x^2]y'\big)'+\lambda y=0 \\ y(0)=0,\ y \text{ acotada en } 1 \end{array}}$ Los P_{2n-1} son las únicas soluciones de Legendre que pasan por el origen y están acotadas en $x=1$.



$\lambda_n=2n(2n-1)$, $y_n=\{P_{2n-1}\}$, $n\in\mathbf{N}$. $\int_{-1}^1 P_n^2=\frac{2}{2n+1} \to \int_0^1 P_{2n-1}^2=\frac{1}{4n-1}$.

$1=\sum\limits_{n=1}^{\infty}c_nP_{2n-1}(x)$
$r(x)=1$

$c_1=3\int_0^1 x\,dx=\frac{3}{2}$.

$c_2=7\int_0^1\big[\frac{5}{2}x^3-\frac{3}{2}x\big]dx=-\frac{7}{8}$.

$c_3=11\int_0^1\big[\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\big]dx=\frac{11}{16}$.

10 $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y'(\frac{\pi}{2})=y(\pi)=0 \end{array}}$ Ya autoadjunta $\big(y'\big)'+\lambda y=0$. $q\equiv 0=\alpha\alpha'=\beta\beta' \Rightarrow \lambda=-1$ no autovalor. O directamente:
$y=c_1\,\mathrm{e}^x+c_2\,\mathrm{e}^{-x} \overset{cc}{\to} \begin{array}{l} c_1\mathrm{e}^{\pi/2}-c_2\mathrm{e}^{-\pi/2}=0,\ c_2=\mathrm{e}^{\pi}c_1\downarrow \\ c_1\mathrm{e}^{\pi}+c_2\mathrm{e}^{-\pi}=0 \qquad c_1\big(\mathrm{e}^{\pi}+1\big)=0,\ c_1=0\uparrow \end{array}$ $c_2=0$. No es autovalor.

Si $\lambda=1$, $y=c_1\cos x+c_2\,\mathrm{sen}\, x \overset{cc}{\to} \begin{array}{l} -c_1=0 \\ -c_1=0 \end{array}$, $c_1=0$, $\forall c_2$. La autofunción asociada es: $\boxed{y_1=\{\mathrm{sen}\, x\}}$.

Podríamos dar todas las y_n: $s=x-\frac{\pi}{2}$, $y''+\lambda y=0$, $y'(0)=y\big(\frac{\pi}{2}\big)=0$, $\lambda_n=(2n-1)^2$, $y_n=\{\cos(2n-1)s\}=\{\mathrm{sen}(2n-1)x\}$.

$\big(y'\big)'+y=x-a$. Infinitas si: $\int_{\pi/2}^{\pi}(x-a)\,\mathrm{sen}\, x\,dx=(a-x)\cos x\big]_{\pi/2}^{\pi}+\int_{\pi/2}^{\pi}\cos x\,dx=\pi-a-1=0 \to \boxed{a=\pi-1}$.

$\big[$Directamente. $y_p=x-a$ a simple vista, $y=c_1\cos x+c_2\,\mathrm{sen}\, x+x-a$. Imponiendo aquí los datos:
$\begin{array}{l} -c_1+1=0 \\ -c_1+\pi-a=0 \end{array}$. Cuando $\pi-a=1$ será $y=C\,\mathrm{sen}\, x+\cos x+x+1-\pi$, y para otros a es imposible$\big]$.

11 $\boxed{\begin{array}{l} x^2y''-2xy'+\lambda y=0 \\ y'(1)=y(2)=0 \end{array}}$ $\mu^2-3\mu+\lambda=0$, $\mu=\frac{3\pm\sqrt{9-4\lambda}}{2}$. $\lambda=-4$, $\mu=4,-1$. $y=c_1x^4+c_2x^{-1}$. $\left.\begin{array}{l} 4c_1-c_2=0 \\ 16c_1+c_2/2=0 \end{array}\right\}$
No autovalor. [0 $\alpha\alpha'=\cdots$].

$\lambda=2 \to \mu=2,1$. $y=c_1x^2+c_2x$. $\left.\begin{array}{l} 2c_1+c_2=0 \\ 2(2c_1+c_2)=0 \end{array}\right\} \to c_2=-2c_1$. **Es autovalor** con $\boxed{y_n=\{x^2-2x\}}$.

$\big[\frac{y'}{x^2}\big]'+\frac{\lambda}{x^4}y=0$. $r(x)=\frac{1}{x^4}$, $\langle y_n,y_n\rangle=\int_1^2\frac{1}{x^4}x^2(x-2)^2dx=\int_1^2\big(1-\frac{4}{x}+\frac{4}{x^2}\big)dx=\boxed{3-4\ln 2}$.

Para $\lambda=-4$ claramente solución **única**. Para $\lambda=2$ habrá infinitas o ninguna según se anule o no la integral:

$\int_1^2\underbrace{\frac{4}{x^4}}_{f(x)}(x^2-2x)\,dx=\int_1^2\big(\frac{4}{x^2}-\frac{8}{x^3}\big)dx=-1\neq 0$. **Sin solución**. $\big[$O desde la solución general $y=c_1x^2+c_2x+2\big]$.

12 $\boxed{\begin{array}{l} xy''+2y'+\lambda xy=0 \\ y(\pi)+\pi y'(\pi)=y(2\pi)+2\pi y'(2\pi)=0 \end{array}}$ Si $\lambda=0$, con $y'=v$, $v'=-\frac{2}{x}v$, $v=Ce^{-\int 2/x}=\frac{C}{x^2}=y' \to y=\frac{C}{x}+K$.
O Euler con $r(r-1)+2r=0$. O $u=xy$.

Imponiendo los datos de contorno: $\begin{array}{l} y(\pi)+\pi y'(\pi)=K=0 \\ y(2\pi)+2\pi y'(2\pi)=K=0 \end{array}$. $\boxed{y_0=\{\frac{1}{x}\}}$ autofunción del autovalor $\lambda=0$.

Si $\lambda=4$, $y=c_1\frac{\cos 2x}{x}+c_2\frac{\operatorname{sen} 2x}{x}$, $y=c_1\big[\frac{-2\operatorname{sen}2x}{x}-\frac{\cos 2x}{x^2}\big]+c_2\big[\frac{2\cos 2x}{x}-\frac{\operatorname{sen}2x}{x^2}\big]$ para cumplir los datos:

$\begin{array}{l} \frac{c_1}{\pi}-\pi\frac{c_1}{\pi^2}+\pi\frac{2c_2}{\pi}=0 \\ \frac{c_1}{2\pi}-2\pi\frac{c_1}{4\pi^2}+2\pi\frac{2c_2}{2\pi}=0 \end{array}$, luego $c_2=0$ y la autofunción es: $\boxed{y_4=\{\frac{\cos 2x}{x}\}}$.

Escribimos la ecuación en forma autoadjunta: $e^{\int \frac{2}{x}dx}=x^2$, $x^2y''+2xy'+4x^2y=2x$, $(x^2y')'+4x^2y=2x$.

Como $\lambda=4$ es autovalor, hay solución sólo si es cero: $\int_\pi^{2\pi}2x\frac{\cos 2x}{x}\,dx=\int_\pi^{2\pi}2\cos 2x\,dx=\operatorname{sen}2x\big]_\pi^{2\pi}=0$.

Luego **existen infinitas soluciones del problema**. [Se puede comprobar que son $y=\frac{1}{2x}+C\frac{\cos 2x}{x}$].

[Haciendo $u=xy$ en el problema se obtiene el sencillo $u''+\lambda u=0$, $u'(\pi)=u'(2\pi)=0$, ..., $u_4=\{\cos 2x\}$,
y la ecuación de **c]** pasa a ser $[u']'+4u=2$ de clara solución general $u=c_1\cos 2x+c_2\operatorname{sen}2x+\frac{1}{2}$].

13 $\boxed{\begin{array}{l} y''-2y'+\lambda y=0 \\ y'(0)=y'(1)=0 \end{array}}$ Autoadjunta: $\big[e^{-2x}y'\big]'+\lambda e^{-2x}y=0$. Peso $r(x)=e^{-2x}$. $\mu^2-2\mu+\lambda=0$, $\mu=1\pm\sqrt{1-\lambda}$.

$\lambda=0$, $\mu=0,2$: $y=c_1+c_2e^{2x}$, $y'=2c_2e^{2x} \to \left.\begin{array}{l} 2c_2=0 \\ 2c_2e^2=0 \end{array}\right\} \to \lambda=0$ autovalor, $\boxed{y_0=\{1\}}$.

$\lambda=-3$ no puede ser autovalor pues $\alpha\alpha'=\beta\beta'=0\equiv q(x)$. O directamente:

$\mu=0,2$, $y=c_1e^{3x}+c_2e^{-x}$, $y'=3c_1e^{3x}-c_2e^{-x} \to \left.\begin{array}{l} 3c_1-c_2=0 \to \\ 3c_1e^3-c_2e^{-1}=0 \end{array}\right\} \to c_2\big[e^3-e^{-1}\big] \to c_2=c_1=0$.

Si $\lambda=2$, $\mu=1\pm i$: $\begin{array}{l} y=[c_1\cos x+c_2\operatorname{sen}x]\,e^x \\ y'=\big[(c_1+c_2)\cos x+(c_2-c_1)\operatorname{sen}x\big]e^{-x} \end{array} \to \begin{array}{l} c_1+c_2=0 \to \\ 2c_2 e\operatorname{sen}1=0 \end{array} \to c_2=c_1=0$. No autovalor.

$e^x=c_0+\sum_{n=1}^{\infty}c_n y_n(x) \to c_0=\frac{\langle e^x,y_0\rangle}{\langle y_0,y_0\rangle}=\frac{\int_0^1 e^{-x}dx}{\int_0^1 e^{-2x}dx}=2\frac{1-e^{-1}}{1-e^{-2}}=2e\frac{e-1}{e^2-1}=\boxed{\frac{2e}{e+1}}$.

Como el problema homogéneo tiene sólo la solución trivial $y\equiv 0$, el no homogéneo tiene **solución única**.

[Directamente: La solución general de la no homogénea es: $y=c_1e^{3x}+c_2e^{-x}-1 \xrightarrow{c.c.} y=-1$].

14 $\boxed{\begin{array}{l} y''+\lambda y=\operatorname{sen}x \\ y(0)=y'(\frac{\pi}{2})=0 \end{array}}$ Autovalores y autofunciones del homogéneo: $\lambda_n=(2n-1)^2$, $y_n=\big\{\operatorname{sen}(2n-1)x\big\}$.
$n=1,2,\ldots$

Para cualquier $\lambda\neq(2n-1)^2$ el homogéneo sólo tiene la solución trivial y el no homogéneo **solución única**.

[Por ejemplo para $\lambda=0$ la solución es $y=c_1+c_2x-\operatorname{sen}x \overset{c.c.}{\to} y=-\operatorname{sen}x$ única solución del problema].

Para $\lambda=(2n-1)^2$ el homogéno tiene infinitas y el no homogéneo tendrá $\begin{array}{c}\text{infinitas}\\ \text{ninguna}\end{array}$ según sea $\begin{array}{c}=0\\ \neq 0\end{array}$ la

integral $I=\int_0^{\pi/2}\operatorname{sen}x\,\operatorname{sen}(2n-1)x\,dx$ [la ecuación ya está en forma autoadjunta: $[y']'+\lambda y=\operatorname{sen}x$].

Por tanto, **no hay solución** sólo si $\lambda=1$ $\big[\int_0^{\pi/2}\operatorname{sen}^2x\,dx\neq 0\big]$, ya que para los otros autovalores $\lambda=9,25,\ldots$

la integral es cero [sin calcularla: $\operatorname{sen}x$ es ortogonal a las otras autofunciones] y hay **infinitas soluciones**.

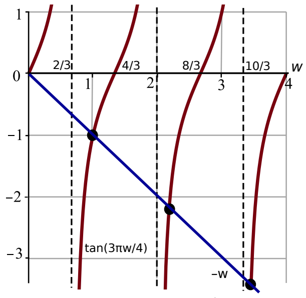
[La integral no es difícil de calcular: $I=\frac{1}{2}\int_0^{\pi/2}\big[\cos 2(n-1)x-\cos 2nx\big]dx \overset{n\neq 1}{=} \frac{\operatorname{sen}(n-1)\pi}{4(n-1)}-\frac{\operatorname{sen}n\pi}{4n}=0$,

y para $n=1$: $I=\frac{1}{2}\int_0^{\pi/2}\big[1-\cos 2x\big]dx=\frac{\pi}{4}\neq 0$].

[Por ejemplo, si $\lambda=9$ la solución de la no homogénea es $y=c_1\cos 3x+c_2\operatorname{sen}3x+\frac{\operatorname{sen}x}{8} \overset{c.c.}{\to} y=c_2\operatorname{sen}3x+\frac{\operatorname{sen}x}{8}$ $\forall c_2$.

En cambio, para $\lambda=1$, $y=c_1\cos x+c_2\operatorname{sen}x-\frac{x}{2}\cos x \overset{c.c.}{\to} \left\{\begin{array}{l} c_1=0 \\ \frac{\pi}{4}-c_1=0 \end{array}\right.$ imposible; no hay solución].

15 $\boxed{\begin{array}{l} y''+\lambda y=0 \\ y(0)=y(\frac{3\pi}{4})+y'(\frac{3\pi}{4})=0 \end{array}}$ $y=c_1\cos x+c_2\operatorname{sen}x$. $\left.\begin{array}{l} c_1=0 \to \\ c_2\big[\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\big]=0 \end{array}\right\}$ $\forall c_2$. Autovalor.



con autofunción $\boxed{y_1=\{\operatorname{sen}x\}}$. La ecuación está en forma autoadjunta y $r\equiv 1$.

$1=c_1\operatorname{sen}x+\sum_{n=2}^{\infty}c_ny_n \to c_1=\frac{\langle 1,y_1\rangle}{\langle y_1,y_1\rangle}=\frac{\int_0^{3\pi/4}\operatorname{sen}x\,dx}{\int_0^{3\pi/4}\operatorname{sen}^2x\,dx}=\frac{2+\sqrt{2}}{\int_0^{3\pi/4}(1-\cos 2x)\,dx}=\boxed{\frac{4(2+\sqrt{2})}{3\pi+2}}$.

$\int_0^{3\pi/4}(6\operatorname{sen}x\cos x-a)\operatorname{sen}x\,dx=\big[2\cos^3x+a\cos x\big]_0^{3\pi/4}=\frac{1}{\sqrt{2}}-a\big(\frac{1}{\sqrt{2}}+1\big)=0$

$\to a=\frac{1}{\sqrt{2}+1}=\boxed{\sqrt{2}-1}$. [Más largo usando $y=c_1\cos x+c_2\operatorname{sen}x-\operatorname{sen}2x-a$].

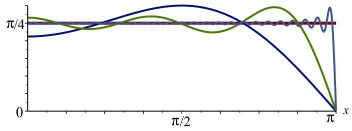
$y=c_1\cos wx+c_2\operatorname{sen}wx$. $y(0)=c_1=0 \to c_2\big[\operatorname{sen}\frac{3\pi w}{4}+w\cos\frac{3\pi w}{4}\big]=0$. w_n son soluciones de $\tan\frac{3\pi w}{4}\overset{\uparrow}{=}-w$.

w_2 está a la derecha de la asíntota en 2 $\Rightarrow \lambda_2=w_2^2>4$. [Y es menor que $\frac{8}{3} \to \lambda_2<\frac{64}{9}$. Numéricamente ≈ 4.76].

Soluciones de problemas 4 de MII(C) (2023-24)

1 $\boxed{\begin{array}{l} X''+\lambda X=0 \\ X'(0)=X(\pi)=0 \end{array}}$ $\rightarrow\ \lambda_n=\frac{(2n-1)^2}{4}$, $X_n=\left\{\cos\frac{(2n-1)x}{2}\right\}$, $n=1,2,\dots$

$$c_n=\frac{1}{2}\int_0^\pi \cos\frac{(2n-1)x}{2}\,dx=\frac{1}{2n-1}\operatorname{sen}\frac{(2n-1)x}{2}\Big]_0^\pi=\frac{(-1)^{n+1}}{2n-1}\,.$$



[Arriba dibujo hecho con maple de 1 y las sumas de 2, 5 y 50 términos].

Por tanto: $\boxed{\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\frac{(2n-1)x}{2}=\cos\frac{x}{2}-\frac{1}{3}\cos\frac{3x}{2}+\frac{1}{5}\cos\frac{5x}{2}-\cdots}$.

i) Como f es continua en $\frac{\pi}{2}$ la suma es $f\big(\frac{\pi}{2}\big)=\frac{\pi}{4}$. ii) Cada sumando es par y lo será la suma de la serie. Por ser la f extendida continua en 0 también sumará $\frac{\pi}{4}$. O es claro que $1-\frac{1}{3}+\frac{1}{5}+\cdots=\arctan 1=\frac{\pi}{4}$.
[Esos cosenos son impares respecto a π y por eso debe converger a 0 ahí, y lo hace por anularse los cosenos].

$\boxed{\begin{cases} u_t-4u_{xx}=0\,,\ x\in(0,\pi),\ t>0 \\ u(x,0)=f(x)\,,\ u_x(0,t)=u(\pi,t)=0 \end{cases}}$ Separando variables en este problema homogéneo (apuntes y formulario) y usando las condiciones de contorno sale:

$\begin{cases} X''+\lambda X=0 \\ X'(0)=X(\pi)=0 \end{cases}$ que da los λ_n y X_n de **a]**, y además $T'=-4\lambda_n T=-(2n-1)^2T\ \rightarrow\ T_n=\left\{e^{-(2n-1)^2t}\right\}$.

Probamos pues $u(x,t)=\sum_{n=1}^{\infty}c_n\,e^{-(2n-1)^2t}\cos\frac{(2n-1)x}{2}$. Y por el dato incial: $u(x,0)=\sum_{n=1}^{\infty}c_n\cos\frac{(2n-1)x}{2}=f(x)$.

Para i) $f(x)=\frac{\pi}{4}$, los c_n son los de arriba, y por tanto es: $\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\,e^{-(2n-1)^2t}\cos\frac{(2n-1)x}{2}}$.

Para ii) $f(x)=\cos\frac{x}{2}$, $c_n=1$ y los demás $c_n=0$, con lo que la solución es ahora: $\boxed{u(x,t)=e^{-t}\cos\frac{x}{2}}$.

2 $\boxed{\begin{cases} u_t-u_{xx}=0\,,\ x\in(0,\pi),\ t>0 \\ u(x,0)=0,\ u_x(0,t)=u_x(\pi,t)=t \end{cases}}$ Casi a ojo se ve que $v=xt$ cumple las condiciones de contorno.

$w=u-xt\rightarrow\begin{cases} w_t-w_{xx}=-x \\ w(x,0)=0,\ w_x(0,t)=w_x(\pi,t)=0 \end{cases}\rightarrow\begin{cases} X''+\lambda X=0 \\ X'(0)=X'(\pi)=0 \end{cases}\rightarrow X_n=\{\cos nx\}\,,\ n=0,1,\dots\rightarrow$

$w=T_0(t)+\sum_{n=1}^{\infty}T_n(t)\cos nx\ \rightarrow\ T_0'+\sum_{n=1}^{\infty}[T_n'+n^2T_n]\cos nx=-x=\frac{b_0}{2}+\sum_{n=1}^{\infty}b_n\cos nx$, con $b_n=-\frac{2}{\pi}\int_0^\pi x\cos nx\,dx$:

$$b_0=-\frac{2}{\pi}\frac{\pi^2}{2}=-\pi\ ,\quad b_n=-\frac{2}{n\pi}x\operatorname{sen}nx\Big]_0^\pi+\frac{2}{n\pi}\int_0^\pi\operatorname{sen}nx\,dx=\frac{2}{n^2\pi}[1-\cos n\pi]=\begin{cases} 4/(n^2\pi)\,,\ n \text{ impar} \\ 0\,,\ n \text{ par} \end{cases}$$

$$\begin{cases} T_0'=-\frac{\pi}{2} \\ T_0(0)=0 \end{cases}\rightarrow T_0(t)=-\frac{\pi}{2}t\ ,\quad \begin{cases} T_n'+n^2T_n=b_n\rightarrow Ce^{-n^2t}+\frac{b_n}{n^2} \\ T_n(0)=0 \end{cases}\rightarrow T_n(t)=\frac{b_n}{n^2}[1-e^{-n^2t}]\ .$$

$\boxed{u(x,t)=t\big(x-\frac{\pi}{2}\big)+\sum_{m=1}^{\infty}\frac{4}{\pi(2m-1)^4}\,[1-e^{-(2m-1)^2t}]\cos(2m-1)x}$ $\rightarrow\infty$, si $x\in(\pi/2,\pi)$
$\rightarrow 0$, si $x=\pi/2$
$\rightarrow-\infty$, si $x\in(0,\pi/2)$

3 $\boxed{\begin{cases} u_t-4u_{xx}=0\,,\ x\in(0,\pi),\ t>0 \\ u(x,0)=0\,,\ u(0,t)=t\,,\ u_x(\pi,t)=0 \end{cases}}$ $\xrightarrow{w=u-t}\begin{cases} w_t-4w_{xx}=-1 \\ w(x,0)=w(0,t)=w_x(\pi,t)=0 \end{cases}\xrightarrow{u=XT}\begin{cases} X''+\lambda X=0 \\ X(0)=X'(\pi)=0 \end{cases}$

$\rightarrow\ X_n=\left\{\operatorname{sen}\frac{(2n-1)x}{2}\right\}$, $n=1,2,\dots$ $\big[$y $T'+4\lambda T=0\,\big]$.

Probamos $w(x,t)=\sum_{n=1}^{\infty}T_n(t)\operatorname{sen}\frac{(2n-1)x}{2}\ \rightarrow\ \sum_{n=1}^{\infty}\big[T_n'+(2n-1)^2T_n\big]\operatorname{sen}\frac{(2n-1)x}{2}=-1=\sum_{n=1}^{\infty}B_n\operatorname{sen}\frac{(2n-1)x}{2}$,

con $B_n=-\frac{2}{\pi}\int_0^\pi\operatorname{sen}\frac{(2n-1)x}{2}\,dx=\frac{-4}{\pi(2n-1)}$. Como $w(x,0)=\sum_{n=1}^{\infty}T_n(0)\operatorname{sen}\frac{(2n-1)x}{2}=0$,

$$\begin{cases} T_n'+(2n-1)^2T_n=B_n \\ T_n(0)=0 \end{cases}\rightarrow T_n(t)=Ce^{-(2n-1)^2t}+\frac{B_n}{(2n-1)^2}\xrightarrow{d.i.}C=\frac{-B_n}{(2n-1)^2}\ .$$

Deshaciendo el cambio tenemos la solución: $\boxed{u(x,t)=t-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\Big[1-e^{-(2n-1)^2t}\Big]\operatorname{sen}\frac{(2n-1)x}{2}}$.

O bien, tanteando un poco se encuentra $v=t+\frac{x^2}{8}-\frac{\pi x}{4}$ $\xrightarrow{w=u-v}\begin{cases} w_t-4w_{xx}=0 \\ w(x,0)=\frac{\pi x}{4}-\frac{x^2}{8}\,,\ w(0,t)=w_x(\pi,t)=0 \end{cases}$

$\xrightarrow{u=XT}\begin{cases} X''+\lambda X=0 \\ X(0)=X'(\pi)=0 \end{cases}\rightarrow X_n=\left\{\operatorname{sen}\frac{(2n-1)x}{2}\right\}$, $n=1,2,\dots$ y $T'+4\lambda T=0\ \rightarrow\ T_n=\left\{e^{-(2n-1)^2t}\right\}$.

$$w(x,t)=\sum_{n=1}^{\infty}c_ne^{-(2n-1)^2t}\operatorname{sen}\frac{(2n-1)x}{2}\ \rightarrow\ w(x,0)=\sum_{n=1}^{\infty}c_n\operatorname{sen}\frac{(2n-1)x}{2}=\frac{2\pi x-x^2}{8}\ ,$$

$$c_n=\frac{2}{\pi}\int_0^\pi\frac{2\pi x-x^2}{8}\operatorname{sen}\frac{(2n-1)x}{2}\,dx=\frac{x^2-2\pi x}{2\pi(2n-1)}\cos\frac{(2n-1)x}{2}\Big]_0^\pi-\frac{1}{\pi(2n-1)}\int_0^\pi(\pi-x)\cos\frac{(2n-1)x}{2}\,dx$$

$$=\frac{2(\pi-x)}{\pi(2n-1)^2}\operatorname{sen}\frac{(2n-1)x}{2}\Big]_0^\pi+\frac{2}{\pi(2n-1)^2}\int_0^\pi\operatorname{sen}\frac{(2n-1)x}{2}\,dx=\frac{4}{\pi(2n-1)^3}\ .$$

Otra expresión de la solución única es: $\boxed{u(x,t)=t+\frac{x^2}{8}-\frac{\pi x}{4}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2t}\operatorname{sen}\frac{(2n-1)x}{2}}$.

4 a) $\boxed{\begin{cases} u_t-u_{xx}=\mathrm{e}^{-2t},\ x\in(0,\pi),\ t>0\\ u(x,0)=u(0,t)=u(\pi,t)=0\end{cases}}$ Separando la homogénea: $\begin{cases}X''+\lambda X=0\\ X(0)=X(\pi)=0\end{cases}\to \lambda_n=n^2,\ X_n=\{\operatorname{sen} nx\},$ $n=1,2\ldots$

$u(x,t)=\sum\limits_{n=1}^{\infty}T_n(t)\operatorname{sen}nx \to \sum\limits_{n=1}^{\infty}\big[T_n'+n^2T_n\big]\operatorname{sen}nx=\mathrm{e}^{2t}=\mathrm{e}^{2t}\sum\limits_{n=1}^{\infty}B_n\operatorname{sen}nx$, con $B_n=\frac{2}{\pi}\int_0^{\pi}\operatorname{sen}nx\,dx=\frac{2[1-(-1)^n]}{n\pi}$

Del dato inicial: $u(x,0)=\sum\limits_{n=1}^{\infty}T_n(0)\operatorname{sen}nx=0\to T_n(0)=0\ \forall n$. Debemos resolver: $\begin{cases}T_n'+n^2T_n=B_n\mathrm{e}^{-2t}\\ T_n(0)=0\end{cases}\to$

$T_n=C\mathrm{e}^{-n^2t}+T_{np}$. $T_{np}=A\mathrm{e}^{-2t}\to[-2+n^2]A=B_n$, $T_n=C\mathrm{e}^{-n^2t}+\frac{B_n}{n^2-2}\mathrm{e}^{-2t}\overset{d.i.}{\to}C=-\frac{B_n}{n^2-2}$. Por tanto:

$$u(x,t)=\sum_{n=1}^{\infty}\frac{B_n}{n^2-2}\big[\mathrm{e}^{-2t}-\mathrm{e}^{-n^2t}\big]\operatorname{sen}nx=\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)[(2m-1)^2-2]}\big[\mathrm{e}^{-2t}-\mathrm{e}^{-(2m-1)^2t}\big]\operatorname{sen}(2m-1)x\,.$$

b) $\boxed{\begin{cases}u_t-2tu_{xx}=\mathrm{e}^{-t^2}\cos x,\ x\in\big(0,\frac{\pi}{2}\big),\ t>0\\ u(x,0)=\cos 3x,\ u_x(0,t)=u\big(\frac{\pi}{2},t\big)=0\end{cases}}$ $u=XT$, $\frac{X''}{X}=\frac{T'}{2tT}=-\lambda$, $\begin{cases}X''+\lambda X=0\\ X'(0)=X(\pi/2)=0\end{cases}$, $X_n=\{\cos(2n-1)x\}$. $n=1,2\ldots$

$u(x,t)=\sum\limits_{n=1}^{\infty}T_n(t)\cos(2n-1)x\overset{\text{EDP}}{\to}\sum\limits_{n=1}^{\infty}\big[T_n'+2(2n-1)^2tT_n\big]\cos(2n-1)x=\mathrm{e}^{-t^2}\cos x$ (ya desarrollada).

D.I.↘ $u(x,0)=\sum\limits_{n=1}^{\infty}T_n(0)\cos(2n-1)x=\cos 3x\to T_2(0)=1$ y demás $T_n(0)=0$.

$\begin{cases}T_1'+2tT_1=\mathrm{e}^{-t^2}\\ T_1(0)=0\end{cases}\to T_1=C\mathrm{e}^{-t^2}+\mathrm{e}^{-t^2}\int\mathrm{e}^{t^2}\mathrm{e}^{-t^2}=C\mathrm{e}^{-t^2}+t\,\mathrm{e}^{-t^2}\overset{d.i.}{\to}C=0$. $\begin{cases}T_2'+18tT_2=0\\ T_2(0)=1\end{cases}\to T_2=\mathrm{e}^{-9t^2}$.

El resto de T_n son nulas, porque 0 es solución y hay solución única. $u(x,t)=t\,\mathrm{e}^{-t^2}\cos x+\mathrm{e}^{-9t^2}\cos 3x$.

c) $\boxed{\begin{cases}u_t-u_{xx}+u=0,\ x\in(0,\pi),\ t>0\\ u(x,0)=0,\ u_x(0,t)=u_x(\pi,t)=\mathrm{e}^{-t}\end{cases}}$ $\overset{w=u-x\mathrm{e}^{-t}}{\longrightarrow}\begin{cases}w_t-w_{xx}+w=0\\ w(x,0)=-x\\ w_x(0,t)=w_x(\pi,t)=0\end{cases}$. $w=XT\to\begin{matrix}X''+\lambda X=0,\ X'(0)=X'(\pi)=0\\ T'=-(\lambda+1)T\end{matrix}$

$\lambda_n=n^2$, $n=0,1,\ldots$, $X_n=\{\cos nx\}$. $T_n=\big\{\mathrm{e}^{-(n^2+1)t}\big\}$. Probamos $w(x,t)=\frac{c_0}{2}\mathrm{e}^{-t}+\sum\limits_{n=1}^{\infty}c_n\,\mathrm{e}^{-(n^2+1)t}\cos nx$.

$w(x,0)=\frac{c_0}{2}+\sum\limits_{n=1}^{\infty}c_n\cos nx=-x\to c_n=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2x\operatorname{sen}nx}{n\pi}\Big]_0^{\pi}+\int_0^{\pi}\frac{2\operatorname{sen}nx\,dx}{n\pi}=2\,\frac{1-(-1)^n}{\pi n^2}$.

Y además $c_0=\frac{2}{\pi}\int_0^{\pi}x\,dx=\pi\to u(x,t)=\big(x-\frac{\pi}{2}\big)\mathrm{e}^{-t}+\frac{4}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,\mathrm{e}^{-(4n^2-4n+2)t}\cos(2n-1)x$.

d) $\boxed{\begin{cases}u_t-\frac{1}{t}u_{xx}=2\cos x,\ x\in\big(0,\frac{\pi}{2}\big),\ t>1\\ u(x,1)=\cos 3x,\ u_x(0,t)=u\big(\frac{\pi}{2},t\big)=0\end{cases}}$ $\overset{u=XT}{\to}\frac{X''}{X}=\frac{tT'}{T}=-\lambda$, $\begin{cases}X''+\lambda X=0\\ X'(0)=X\big(\frac{\pi}{2}\big)=0\end{cases}$, $X_n=\{\cos(2n-1)x\}$, $n=1,2,\ldots$

$u=\sum\limits_{n=1}^{\infty}T_n(t)\cos(2n-1)x\overset{EDP}{\longrightarrow}\sum\limits_{n=1}^{\infty}\big[T_n'+\frac{(2n-1)^2}{t}T_n\big]\cos(2n-1)x=2\cos x$, $u(x,1)=\sum\limits_{n=1}^{\infty}T_n(1)\cos(2n-1)x=\cos 3x$.
(ya desarrollada) (ya desarrollada)

$\begin{cases}T_1'=-\frac{1}{t}T_1+2\\ T_1(1)=0\end{cases}\to T_1=\frac{C}{t}+t\overset{d.i.}{\longrightarrow}T_1=t-\frac{1}{t}$. $\begin{cases}T_2'=-\frac{9}{t}T_2\\ T_2(1)=1\end{cases}\to T_2=Ct^{-9}\overset{d.i.}{\longrightarrow}C=1$. $\boxed{u=\big(t-t^{-1}\big)\cos x+t^{-9}\cos 3x}$.

e) $\boxed{\begin{cases}u_t-u_{xx}+3u=\pi,\ x\in(0,\pi),\ t>0\\ u(x,0)=u(0,t)=u(\pi,t)=0\end{cases}}$ $u=XT\to\frac{X''}{X}=\frac{T'}{T}+3=-\lambda\to\begin{cases}X''+\lambda X=0\\ X(0)=X(\pi)=0\end{cases}\to X_n=\{\operatorname{sen}nx\}$, $n=1,2,\ldots$

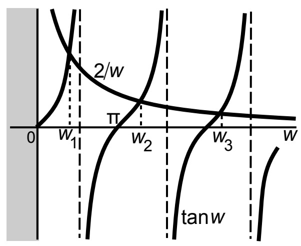
Llevamos la serie $u(x,t)=\sum\limits_{n=1}^{\infty}T_n(t)\operatorname{sen}nx$ a la EDP y al dato inicial para calcular los T_n:

$\sum\limits_{n=1}^{\infty}\big[T_n'+n^2T_n+3T_n\big]\operatorname{sen}nx=\pi=\sum\limits_{n=1}^{\infty}\frac{4}{2n-1}\operatorname{sen}(2n-1)x=4\operatorname{sen}x+\frac{4}{3}\operatorname{sen}3x+\cdots$ $\big[c_n=\frac{2}{\pi}\int_0^{\pi}\pi\operatorname{sen}nx\,dx=2\,\frac{1-(-1)^n}{n}\big]$

Además: $u(x,0)=\sum\limits_{n=1}^{\infty}T_n(0)\operatorname{sen}nx=0\to T_n(0)=0$ para todo n. Debemos resolver los problemas:

$\begin{cases}T_n'+(n^2+3)T_n=c_n\\ T_n(0)=0\end{cases}$. $T_n=\frac{c_n}{n^2+3}\big[1-\mathrm{e}^{-(n^2+3)t}\big]$. Por tanto: $\boxed{u=\sum\limits_{n=1}^{\infty}\frac{1-\mathrm{e}^{-4(n^2-n+1)t}}{(2n-1)(n^2-n+1)}\operatorname{sen}(2n-1)x}$.

f) $\boxed{\begin{cases}u_t-u_{xx}=0,\ x\in[0,1],\ t>0\\ u(x,0)=2-x^2,\ u_x(0,t)=u_x(1,t)+2u(1,t)=0\end{cases}}$ $\begin{cases}X''+\lambda X=0\\ X'(0)=X'(1)+2X(1)=0\end{cases}$ problema no conocido, y además $T'=-\lambda_nT\to T_n=\big\{\mathrm{e}^{-\lambda_nt}\big\}$.



$\alpha\alpha'=0$, $\beta\beta'=2$, $q\equiv0\Rightarrow\lambda\geq0$. $\lambda=0$: $X=c_1+c_2x\overset{c.c.}{\to}\begin{cases}c_2=0\\ 2c_1+3c_2=0\end{cases}\to c_1=c_2=0$.

$\lambda>0$: $X=c_1\cos wx+c_2\operatorname{sen}ww$, $w=\sqrt{\lambda}\overset{c.c.}{\to}\begin{cases}c_2=0\ \searrow\\ c_1[-w\operatorname{sen}w+2\cos w]=0\to\end{cases}$

Los w_n son las infinitas raíces de $\tan w=\frac{2}{w}$, $\lambda_n=w_n^2$ y $X_n=\{\cos w_nx\}$.

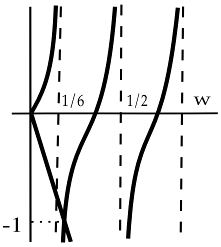
Probamos pues $u(x,t)=\sum\limits_{n=1}^{\infty}c_n\,\mathrm{e}^{-w_n^2t}\cos w_nx$. Y por el dato incial:

$u(x,0)=\sum\limits_{n=1}^{\infty}c_n\cos w_nx=f(x)\to c_n=\frac{\langle y_n,f\rangle}{\langle y_n,y_n\rangle}$. $\langle y_n,y_n\rangle=\int_0^1\cos^2w_nx\,dx=\frac{1}{2}+\frac{\operatorname{sen}2w_n}{4w_n}=\frac{2+\operatorname{sen}^2w_n}{4}$ $\big[\frac{\cos w_n}{w_n}=\frac{\operatorname{sen}w_n}{2}\big]$.

$\langle y_n,f\rangle=\int_0^1(2-x^2)\cos w_nx=\frac{2-x^2}{w_n}\operatorname{sen}w_nx\Big]_0^1+\frac{2}{w_n}\int_0^1x\operatorname{sen}w_nx=\frac{\operatorname{sen}w_n}{w_n}-\frac{2\cos w_n}{w_n^2}+\frac{2}{w_n^2}\int_0^1\cos w_nx=\frac{2\operatorname{sen}w_n}{w_n^3}$.

La solución única del problema es: $\boxed{u(x,t)=\sum\limits_{n=1}^{\infty}\frac{8\operatorname{sen}w_n}{w_n^3(2+\operatorname{sen}^2w_n)}\,\mathrm{e}^{-w_n^2t}\cos w_nx}$.

5 $\begin{cases} u_t-u_{xx}-au=0,\ x\in(0,3\pi),\ t>0 \\ u(x,0)=1\ u(0,t)-4u_x(0,t)=u(3\pi,t)=0 \end{cases}$ $\frac{T'-aT}{T}=\frac{X''}{X}=-\lambda \to$ $T'=(a-\lambda)T$, $\begin{cases} X''+\lambda X=0 \\ X(0)-4X'(0)=X(3\pi)=0 \end{cases}$



$\lambda=0$ no autovalor. $\lambda>0$: $\begin{cases} c_1=4c_2w \\ c_2[4w\cos 3\pi w+\operatorname{sen} 3\pi w]=0 \end{cases}$, $\tan 3\pi w_n=-4w_n$ $\left[w_1=\frac{1}{4}\right]$

$\to\ \lambda_n=w_n^2\ \left[\lambda_1=\frac{1}{16}\right]$, $X_n=\{\operatorname{sen} w_nx+4w_n\cos w_nx\}$ $\left[X_1=\{\operatorname{sen}\frac{x}{4}+\cos\frac{x}{4}\}\right]$.

$u=c_1\,e^{(a-\frac{1}{16})t}(\operatorname{sen}\frac{x}{4}+\cos\frac{x}{4})+\sum_{n=2}^{\infty}c_ne^{(a-\lambda_n)t}X_n(x)$, con $c_1=\frac{\int_0^{3\pi}X_1dx}{\int_0^{3\pi}X_1^2dx}=\frac{4[\sqrt{2}+1]}{3\pi+2}$.

Si $a<\frac{1}{16}$, $u\underset{t\to\infty}{\longrightarrow}0$. Si $a=\frac{1}{16}$, $u\underset{t\to\infty}{\longrightarrow}\frac{4[\sqrt{2}+1]}{3\pi+2}(\operatorname{sen}\frac{x}{4}+\cos\frac{x}{4})$. Si $a>\frac{1}{16}$, $u\underset{t\to\infty}{\longrightarrow}\infty$ $\left[e^{(a-\frac{1}{16})t}\right.$ manda y $\left.X_1>0\right]$.

6 a] $\begin{cases} X''+2X'+\lambda X=0 \\ X(0)=X(1)+X'(1)=0 \end{cases}$ $\mu^2+2\mu+\lambda=0$, $\mu=-1\pm\sqrt{1-\lambda}$. En forma S-L queda $\left[e^{2x}X'\right]'+\lambda e^{2x}X=0$.
Es $\lambda\geq 0$, pero se debe discutir $\lambda<,=,>1$. Llamamos $p=\sqrt{1-\lambda}$, $w=\sqrt{\lambda-1}$.

$\lambda<1$: $X=c_1\,e^{(-1+p)x}+c_2\,e^{(-1-p)x}$, $X'=c_1(p-1)\,e^{(-1+p)x}-c_2(1+p)\,e^{(-1-p)x}$ $\to$

$\left.\begin{matrix} c_1+c_2=0 \\ c_1pe^{-1+p}-c_2pe^{-1-p}=0 \end{matrix}\right\}\to c_1pe^{-1}[e^p+e^{-p}]=0\to c_1=c_2=0$ no autovalor.

$\lambda=1$: $X=[c_1+c_2x]\,e^{-x}$, $X'=[c_2-c_1-c_2x]\,e^{-x}$ $\to c_1=c_2=0\to X\equiv 0$. $\lambda=1$ no autovalor.

$\lambda>1$: $X=[c_1\cos wx+c_2\operatorname{sen} wx]\,e^{-x}\xrightarrow{X(0)=0}c_1=0\to X(1)+X'(1)=c_2\big[w\cos wx\big]e^{-x}\to$

$w_n=\frac{(2n-1)\pi}{2}$, $n=1,2,\dots\to$ $\boxed{\lambda_n=1+w_n^2,\ X_n=\{e^{-x}\operatorname{sen} w_nx\}}$.

b] $\begin{cases} u_t-u_{xx}-2u_x=0,\ x\in(0,1),\ t>0 \\ u(x,0)=e^{-x},\ u(0,t)=u(1,t)+u_x(1,t)=0 \end{cases}$ $u=XT\to\frac{X''+2X'}{X}=\frac{T'}{T}=-\lambda\to$ $\begin{cases} X''+2X'+\lambda X=0 \\ X(0)=X(1)+X'(1)=0 \end{cases}$ ↑ y $T'+\lambda T=0\to T_n=\{e^{-\lambda_nt}\}$

$\to u(x,t)=\sum_{n=1}^{\infty}c_n\,e^{-\lambda_nt}\,e^{-x}\operatorname{sen} w_nx\to u(x,0)=\sum_{n=1}^{\infty}c_n\,e^{-x}\operatorname{sen} w_nx=e^{-x}$. Simplificando e^{-x}:

$1=\sum_{n=1}^{\infty}c_n\operatorname{sen}\frac{(2n-1)\pi x}{2}$, $c_n=2\int_0^1\operatorname{sen}\frac{(2n-1)\pi x}{2}dx=\frac{4}{\pi(2n-1)}$, $\boxed{u(x,t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-x-t-(2n-1)^2\pi^2t/4}}{2n-1}\operatorname{sen}\frac{(2n-1)\pi x}{2}}$.

Peor: $e^{-x}=\sum_{n=1}^{\infty}c_nX_n(x)\to c_n=\frac{\langle e^{-x},X_n\rangle}{\langle X_n,X_n\rangle}=2\int_0^1\underset{\text{peso}}{e^{2x}}e^{-2x}\operatorname{sen}\frac{(2n-1)\pi x}{2}dx$, pues $\langle X_n,X_n\rangle=2\int_0^1\operatorname{sen}^2\frac{(2n-1)\pi x}{2}dx=\frac{1}{2}$.

$u=e^{pt+qx}w\to w_t-w_{xx}-(2q+2)w_x+(p-q^2-2q)w=0\to q=p=-1$ lleva al calor. Así pues:

$w=e^{t+x}u\ \left[w_x=(u+u_x)e^{t+x}\right]\to\begin{cases} w_t-w_{xx}=0 \\ w(x,0)=1,\ w(0,t)=w_x(1,t)=0 \end{cases}\to X_n=\{\operatorname{sen} w_nx\}$, $T_n=\{e^{-w_n^2t}\}$.

$\sum c_nT_nX_n$ lleva al desarrollo de antes y haciendo $u=e^{-t-x}w$ llegamos a la solución de arriba.

7 $\begin{cases} u_t-\left(u_{rr}+\frac{2}{r}u_r\right)=0,\ r\in(\pi,2\pi),\ t>0 \\ u(r,0)=\frac{\operatorname{sen} r}{r},\ u(\pi,t)=u(2\pi,t)=0 \end{cases}$ $u=R(r)T(t)\to\frac{T'}{T}=\frac{rR''+2R}{rR}=-\lambda$, $\begin{cases} rR''+2R'+\lambda rR=0 \\ R(\pi)=R(2\pi)=0 \end{cases}$ y $T'=-\lambda T$.
Problema para R similar a singular conocido.

Con $v=rR$ la EDO pasa a $v''+\lambda v=0$. [En efecto: $R'=\frac{v'}{r}-\frac{v}{r^2}$, $R''=\frac{v''}{r}-\frac{2v'}{r^2}+\frac{2v}{r^3}\to v''-\frac{2v'}{r}+\frac{2v}{r^2}+\frac{2v'}{r}-\frac{2v}{r^2}+\lambda v=v''+\lambda v=0$].

Y el problema: $\begin{cases} v''+\lambda v=0 \\ v(\pi)=v(2\pi)=0 \end{cases}\overset{\bullet}{\to}\lambda_n=n^2,\ v_n=\{\operatorname{sen} nr\}$, $R_n=\left\{\frac{\operatorname{sen} nr}{r}\right\}$, $n=1,2,\dots$, $T_n=\{e^{-n^2t}\}$.

• Llevando el intervalo al origen: $s=r-\pi\to\begin{cases} v''+\lambda v=0 \\ v(0)=v(\pi)=0 \end{cases}\overset{\bullet}{\to}v_n=\{\operatorname{sen} ns\}=\{\operatorname{sen}(nr-n\pi)\}=\{\operatorname{sen} nr\}$.

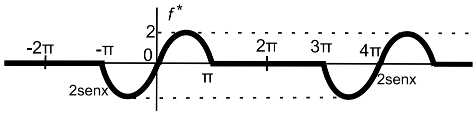
[Bastante más corto que aplicando directamente los datos de contorno a $v=c_1\cos wr+c_2\operatorname{sen} wr$].

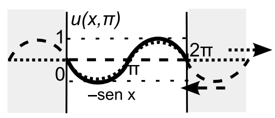
Probamos entonces $u(r,t)=\sum_{n=1}^{\infty}c_n\,e^{-n^2t}\frac{\operatorname{sen} nr}{r}$, a la que sólo le falta cumplir el dato inicial:

$u(r,0)=\sum_{n=1}^{\infty}c_n\frac{\operatorname{sen} nr}{r}=\frac{\operatorname{sen} r}{r}$. El único c_n no nulo es $c_1=1$ y, por tanto, $\boxed{u(r,t)=e^{-t}\frac{\operatorname{sen} r}{r}}$.

8 $\begin{cases} u_{tt}-u_{xx}=0,\ x\in[0,2\pi],\ t\in\mathbf{R} \\ u(x,0)=\begin{cases} 2\operatorname{sen} x,\ x\in[0,\pi] \\ 0,\ x\in[\pi,2\pi] \end{cases},\ u_t(x,0)=0 \\ u(0,t)=u(2\pi,t)=0 \end{cases}$ $u=\frac{1}{2}\big[f^*(x+t)+f^*(x-t)\big]$, con f^* extensión impar y 4π-periódica.
Para dibujar $u(x,\pi)$ basta trasladar $\frac{1}{2}f(x)$ a izquierda y derecha π unidades y sumar en $[0,2\pi]$. [Sólo queda lo que va a la derecha].

Como para $x\in[0,2\pi]$ siempre $x+\pi\in[\pi,3\pi]$ y $x-\pi\in[-\pi,\pi]$, y en todo este intervalo es $f^*(x)=2\operatorname{sen} x$ ($\operatorname{sen} x$ impar), es:





$u(x,\pi)=\frac{1}{2}\big[f^*(x+\pi)+f^*(x-\pi)\big]=\frac{1}{2}\big[0+2\operatorname{sen}(x-\pi)\big]=\boxed{-\operatorname{sen} x}$ $\forall x\in[0,2\pi]$.

$u=XT\to\begin{cases} X''+\lambda X=0 \\ X(0)=X(2\pi)=0 \end{cases}\to\lambda_n=\frac{n^2}{4}$, $X_n=\{\operatorname{sen}\frac{nx}{2}\}$, $n=1,2,\dots$ y $\begin{cases} T'+\lambda T=0 \\ T'(0)=0 \end{cases}\to T_n=\{\cos\frac{nt}{2}\}$.

$u(x,t)=\sum_{n=1}^{\infty}c_n\cos\frac{nt}{2}\operatorname{sen}\frac{nx}{2}\to u(x,0)=\sum_{n=1}^{\infty}c_n\operatorname{sen}\frac{nx}{2}=\begin{cases} 2\operatorname{sen} x,\ x\in[0,\pi] \\ 0,\ x\in[\pi,2\pi] \end{cases}\to$

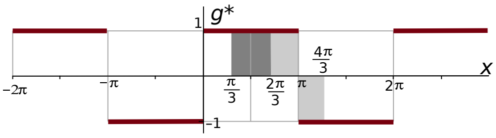
$c_n=\frac{2}{2\pi}\int_0^{\pi}2\operatorname{sen} x\operatorname{sen}\frac{nx}{2}dx=\frac{1}{\pi}\int_0^{\pi}\Big[\cos\big(\frac{n}{2}-1\big)x-\cos\big(\frac{n}{2}+1\big)x\Big]dx\underset{n\neq 2}{=}-\frac{8\operatorname{sen}\frac{n\pi}{2}}{\pi(n^2-4)}=\begin{cases} 0,\ n=2m \\ \frac{8}{\pi}\frac{(-1)^m}{(2m-1)^2-4},\ n=2m-1 \end{cases}$.

Además, $c_2=\frac{1}{\pi}\int_0^{\pi}[1-\cos 2x]dx=1$. $u(x,t)=\cos t\operatorname{sen} x+\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2-4}\cos\frac{(2m-1)t}{2}\operatorname{sen}\frac{(2m-1)x}{2}\xrightarrow{t=\pi}-\operatorname{sen} x$.

9 $\boxed{\begin{array}{l} u_{tt}-u_{xx}=0\,,\ x\in[0,\pi]\,,\ t\in\mathbf{R} \\ u(x,0)=0\,,\ u_t(x,0)=1 \\ u(0,t)=u(\pi,t)=0 \end{array}}$

a] g^* extensión impar y 2π-periódica.

$u\left(\frac{5\pi}{6},\frac{\pi}{2}\right)=\frac{1}{2}\int_{\pi/3}^{4\pi/3}g^*=\frac{1}{2}\int_{\pi/3}^{2\pi/3}ds=\boxed{\frac{\pi}{6}}$.



b] $\begin{cases} X''+\lambda X=0 \\ X(0)=X(\pi)=0 \end{cases}$, $\lambda_n=n^2$, $X_n=\{\operatorname{sen} nx\}_{n=1,2,\ldots}$, $\begin{cases} T''+\lambda T=0 \\ T(0)=0 \end{cases}\rightarrow T_n=\{\operatorname{sen} nt\}$.

La serie $u=\sum_{n=1}^{\infty}c_n\operatorname{sen} nt\operatorname{sen} nx$ debe cumplir $u_t(x,0)=\sum_{n=1}^{\infty}nc_n\operatorname{sen} nx=1\rightarrow c_n=\frac{2}{n\pi}\int_0^{\pi}\operatorname{sen} nx\,dx=\frac{2[1-(-1)^n]}{\pi n^2}$.

$\left[=\frac{4}{\pi},0,\frac{4}{9\pi},0,\frac{4}{25\pi},\ldots\right]$

La solución es, por tanto, la serie $u(x,t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\operatorname{sen}(2n-1)t\operatorname{sen}(2n-1)x$.

$u\left(\frac{5\pi}{6},\frac{\pi}{2}\right)=\frac{4}{\pi}\left[\operatorname{sen}\frac{\pi}{2}\operatorname{sen}\frac{5\pi}{6}+\frac{1}{9}\operatorname{sen}\frac{5\pi}{2}\operatorname{sen}\frac{3\pi}{2}+\cdots\right]=\frac{4}{\pi}\left[\frac{1}{2}-\frac{1}{9}+\cdots\right]\approx\boxed{\frac{14}{9\pi}}\approx 0{,}495$. [El exacto $\frac{\pi}{6}\approx 0.524$].

10 $\boxed{\begin{array}{l} u_{tt}-u_{xx}=t\operatorname{sen} x\,,\ x\in[0,\pi],\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=u(0,t)=u(\pi,t)=0 \end{array}}$

a) $\begin{cases} X''+\lambda X=0 \\ X(0)=X(\pi)=0 \end{cases}$, $X_n=\{\operatorname{sen} nx\}_{n=1,2,\ldots}$ [y $T''+\lambda T=0$]. $u=\sum_{n=1}^{\infty}T_n(t)\operatorname{sen} nx$

$\rightarrow\ \sum_{n=1}^{\infty}\left[T_n''+n^2T_n\right]\operatorname{sen} nx=t\operatorname{sen} x$, $\sum_{n=1}^{\infty}T_n(0)\operatorname{sen} nx=0$ y $\sum_{n=1}^{\infty}T_n'(0)\operatorname{sen} nx=0\ \Rightarrow\ T_n(0)=T_n'(0)=0\ \forall n$.

La única solución no nula la proporciona $\begin{cases} T_1''+T_1=t\rightarrow T_1=c_1\cos t+c_2\operatorname{sen} t+t\ \ (T_{1p}=At+B \text{ o a ojo}) \\ T_1(0)=T_1'(0)=0 \end{cases}$

Imponiendo los datos: $T_1(0)=c_1=0$, $T_1'(0)=-c_2+1=0$. La solución es $\boxed{u(x,t)=(t-\operatorname{sen} t)\operatorname{sen} x}$.

b) i) Debemos extender $F(t,x)=t\operatorname{sen} x$ impar y 2π-periódica en x a todo **R**. Pero F ya lo es, así que $F^*=F$.

Por tanto: $u(x,t)=\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\tau\operatorname{sen} s\,ds\,d\tau=\frac{1}{2}\int_0^t\tau\big(\underbrace{\cos[x-(t-\tau)]-\cos[x+(t-\tau)]}_{=\operatorname{sen} x\,\operatorname{sen}(t-\tau)}\big)d\tau=\operatorname{sen} x\,(t-\operatorname{sen} t)$.

pues $\int_0^t\tau\operatorname{sen}(t-\tau)\,d\tau=\tau\cos(t-\tau)\Big]_0^t-\int_0^t\cos(t-\tau)\,d\tau=t+\operatorname{sen}(t-\tau)\Big]_0^t=t-\operatorname{sen} t$.

ii) Haciendo $w=u-t\operatorname{sen} x$ obtenemos un problema con $f=F=0$, $g(x)=-\operatorname{sen} x=g^*(x)$ (impar y 2π-periódica).

Por tanto es: $w=-\frac{1}{2}\int_{x-t}^{x+t}\operatorname{sen} s\,ds=\frac{1}{2}\left[\cos(x+t)-\cos(x-t)\right]=-\operatorname{sen} x\operatorname{sen} t$, $u=t\operatorname{sen} x-\operatorname{sen} x\operatorname{sen} t$.

11 $\boxed{\begin{array}{l} u_{tt}-u_{xx}=0\,,\ x\in\left[0,\frac{\pi}{2}\right],\ t\in\mathbf{R} \\ u(x,0)=u_t(x,0)=u_x(0,t)=0\,,\ u\left(\frac{\pi}{2},t\right)=t \end{array}}$

$\xrightarrow{w=u-t}\begin{cases} w_{tt}-w_{xx}=0 \\ w_t(x,0)=-1,\ w(x,0)=w_x(0,t)=w\left(\frac{\pi}{2},t\right)=0 \end{cases}$.

Del formulario y los datos nulos obtenemos si $w=XT$:

$\begin{cases} X''+\lambda X=0 \\ X'(0)=X(\pi/2)=0 \end{cases}\rightarrow\lambda_n=(2n-1)^2$, $X_n=\{\cos(2n-1)x\}$, $\begin{cases} T''+(2n-1)^2T=0 \\ T(0)=0 \end{cases}\rightarrow T_n=\{\operatorname{sen}(2n-1)t\}$, $n=1,2,\ldots$

Probamos $w(x,t)=\sum_{n=1}^{\infty}c_n\operatorname{sen}(2n-1)t\cos(2n-1)x$. Del último dato:

$w_t(x,0)=\sum_{n=1}^{\infty}(2n-1)c_n\cos(2n-1)x=-1\ \rightarrow\ c_n=-\frac{4}{\pi(2n-1)}\int_0^{\pi/2}\cos(2n-1)x\,dx=-\frac{4\operatorname{sen}(2n-1)x}{\pi(2n-1)^2}\Big]_0^{\pi}=\frac{4(-1)^n}{\pi(2n-1)^2}\ \rightarrow$

$\boxed{u=t+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\operatorname{sen}(2n-1)t\cos(2n-1)x}$.

No sabemos resolverlo con D'Alembert. Por los datos debe extenderse par respecto a 0 e impar respecto a L. Resulta una g^* de periodo $4L$, que es precisamente el periodo de estos cosenos impares.

12 $\boxed{\begin{array}{l} u_{tt}+2u_t-5u_{xx}=0\,,\ x\in[0,\pi]\,,\ t\in\mathbf{R} \\ u(x,0)=0,\ u_t(x,0)=g(x)\,,\ u(0,t)=u(\pi,t)=0 \end{array}}$

$u=X(x)T(t)\ \rightarrow\ \frac{T''+2T}{5T}=\frac{X''}{X}=-\lambda\ \rightarrow\ \begin{cases} X''+\lambda X=0 \\ T''+2T'+5\lambda T=0 \end{cases}$

De los datos de contorno se deduce $x(0)=X(\pi)=0$, y por tanto: $\lambda_n=n^2$, $X_n=\{\operatorname{sen} nx\}$, $n=1,2,\ldots$

Para esos λ: $T=e^{-t}\left[c_1\cos\left(\sqrt{5n^2-1}\,t\right)+c_2\operatorname{sen}\left(\sqrt{5n^2-1}\,t\right)\right]$, pues $\mu^2+2\mu+5n^2=0\rightarrow\mu=-1\pm i\sqrt{5n^2-1}$.

Imponiendo $u(x,0)=0$: $c_1=0$, $T_n=\left\{e^{-t}\operatorname{sen}\left(\sqrt{5n^2-1}\,t\right)\right\}$. $u(x,t)=\sum_{n=1}^{\infty}c_n\,e^{-t}\operatorname{sen}\left(\sqrt{5n^2-1}\,t\right)\operatorname{sen} nx$.

Imponemos el dato que falta: $u_t(x,t)=\sum_{n=1}^{\infty}c_n\,e^{-t}\left[\sqrt{5n^2-1}\,\cos\left(\sqrt{5n^2-1}\,t\right)-\operatorname{sen}\left(\sqrt{5n^2-1}\,t\right)\right]\operatorname{sen} nx\ \rightarrow$

i) $u_t(x,0)=\sum_{n=1}^{\infty}c_n\sqrt{5n^2-1}\,\operatorname{sen} nx=g(x)\ \rightarrow\ c_n=\frac{2}{\pi\sqrt{5n^2-1}}\int_0^{\pi}g(x)\operatorname{sen} nx\,dx$.

ii) En el caso de ser $g(x)=2\operatorname{sen} x$, todos los $c_n=0$ excepto $c_1\sqrt{4}=2\ \rightarrow\ u(x,t)=e^{-t}\operatorname{sen} 2t\operatorname{sen} x$.

13 $\boxed{\begin{array}{l} u_{tt}-u_{rr}-\frac{2u_r}{r}=0,\ r\le 1,\ t\ge 0 \\ u(r,0)=0,\ u_t(r,0)=\frac{1}{r}\operatorname{sen}\pi r \\ u(1,t)=0 \end{array}}$

i) $rR''+2R'+\lambda rR=0$, R acot., $R(1)=0\rightarrow\lambda_n=n^2\pi^2$, $R_n=\left\{\frac{\operatorname{sen} n\pi r}{r}\right\}$

$T''+\lambda T=0$, $T(0)=0\rightarrow T_n=\{\operatorname{sen} n\pi t\}$, $u=\sum_{n=1}^{\infty}b_n\operatorname{sen} n\pi t\,\frac{\operatorname{sen} n\pi r}{r}$.

$u_t(r,0)=\sum_{n=1}^{\infty}n\pi b_n\frac{\operatorname{sen} n\pi r}{r}=\frac{\operatorname{sen}\pi r}{r}\ \rightarrow\ u=\frac{\operatorname{sen}\pi t\operatorname{sen}\pi r}{\pi r}$.

ii) $v=ur\rightarrow\begin{cases} v_{tt}-v_{rr}=0,\ r\le 1 \\ v_t(r,0)=\operatorname{sen} n\pi r\equiv G(r) \\ v(r,0)=v(0,t)=v(1,t)=0 \end{cases}\rightarrow\ u=\frac{1}{2r}\int_{r-t}^{r+t}G^*(s)\,ds$ G^* extensión impar de G respecto a 0 y 1.

Como $\operatorname{sen}\pi r$ es impar respecto a esos puntos, $G^*(r)=\operatorname{sen}\pi r$, $u=\frac{1}{2r}\int_{r-t}^{r+t}\operatorname{sen}\pi s\,ds=\frac{\operatorname{sen}\pi t\operatorname{sen}\pi r}{\pi r}$.

[Hasta aquí el control].

14 a) $\boxed{\begin{array}{l}\Delta u=0,\ (x,y)\in(0,\pi)\times(0,\pi)\\ u(\pi,y)=5+\cos y,\ u(0,y)=u_y(x,0)=u_y(x,\pi)=0\end{array}}$ $Y''+\lambda Y=0$, $Y'(0)=Y'(\pi)=0\to Y_n=\{\cos ny\}$, $n=0,1,\dots$ $X_n''-n^2X_n=0$, $X(0)=0\to X_0=\{x\}$, $X_n=\{\operatorname{sh} nx\}$, $n\geq 1$.

$$u(x,y)=c_0\,x+\sum_{n=1}^{\infty}c_n\operatorname{sh}nx\cos ny\to u(x,\pi)=c_0\pi+\sum_{n=1}^{\infty}c_n\operatorname{sh}n\pi\cos ny=5+\cos y\to u=\tfrac{5x}{\pi}+\tfrac{\operatorname{sh}x}{\operatorname{sh}\pi}\cos y\,.$$

b) $\boxed{\begin{array}{l}\Delta u=y\cos x,\ (x,y)\in(0,\pi)\times(0,1)\\ u_x(0,y)=u_x(\pi,y)=u_y(x,0)=u_y(x,1)=0\end{array}}$ $u=\sum_{n=0}^{\infty}Y_n(y)\cos nx\to \begin{array}{l}Y_1''-Y_1=y\\ Y_1'(0)=Y_1'(1)=0\end{array}\to Y_1=\frac{e^y-e^{1-y}}{1+e}-y$

Como es de Neumann aparece (al resolver $Y_0''=0+$ c.c.) una C arbitraria: $u=C+\big[\frac{e^y-e^{1-y}}{1+e}-y\big]\cos x$.

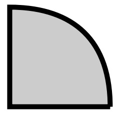
c) $\boxed{\begin{array}{l}u_{xx}+u_{yy}+6u_x=0\text{ en }(0,\pi)\times(0,\pi)\\ u_y(x,0)=u_y(x,\pi)=u_x(0,y)=0,\ u(\pi,y)=\cos 4y\end{array}}$ $u=XY\to\frac{X''+6X'}{X}=-\frac{Y''}{Y}=\lambda$, $\left\{\begin{array}{l}Y''+\lambda Y=0,\ Y'(0)=Y'(\pi)=0\\ X''+6X'-\lambda X=0,\ X'(0)=0\end{array}\right.$

$\to\lambda_n=n^2$, $Y_n=\{\cos ny\}$, $n=0,1,\dots\to$ $X''+6X'-n^2X=0$, $X=c_1e^{(\sqrt{9+n^2}-3)x}+c_2e^{-(\sqrt{9+n^2}+3)x}\ \underset{X'(0)=0}{\longrightarrow}$

$X_0=\{1\}$; $X_n=\Big\{(\sqrt{9+n^2}+3)e^{(\sqrt{9+n^2}-3)x}+(\sqrt{9+n^2}-3)e^{-(\sqrt{9+n^2}+3)x}\Big\}$, $n\geq 1$. $u=\sum_{n=0}^{\infty}c_nX_n(x)\cos ny\to$

$u(\pi,y)=\sum_{n=0}^{\infty}c_nX_n(\pi)\cos ny=\cos 4y\to c_4=\frac{1}{X_4(\pi)}$ y resto cero $\to u=\frac{4e^{2x}+e^{-8x}}{4e^{2\pi}+e^{-8\pi}}\cos 4y$.

15 $\boxed{\begin{array}{l}u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\ r<1,\ 0<\theta<\frac{\pi}{2}\\ u(1,\theta)=f(\theta),\ u(r,0)=u(r,\frac{\pi}{2})=0\end{array}}$ $u=R\Theta\to\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta(\frac{\pi}{2})=0\end{array}\right.$, $\lambda_n=4n^2$, $\Theta_n=\{\operatorname{sen}2n\theta\}$, $n=1,2,\dots$



Además: $r^2R''+rR'-\lambda R=0\to\mu^2=4n^2$, $R=c_1r^{2n}+c_2r^{-2n}\xrightarrow{\text{acotada}}R_n=\{r^{2n}\}$.

$u(r,\theta)=\sum_{n=1}^{\infty}c_n\,r^{2n}\operatorname{sen}2n\theta$ debe cumplir el dato $u(1,\theta)=\sum_{n=1}^{\infty}c_n\operatorname{sen}2n\theta=f(\theta)\to c_n=\frac{4}{\pi}\int_0^{\pi/2}f(\theta)\operatorname{sen}2n\theta\,d\theta$.

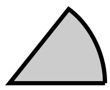
Si $f(\theta)=\operatorname{sen}2\theta$ no hay que hacer integrales y basta mirar: $c_1=0$ y resto nulos. $\boxed{u(r,\theta)=r^2\operatorname{sen}2\theta}$ $[=2xy]$.

Si $f(\theta)=\cos\theta$ sí hay que integrar. El primer término (único que se pide) lo da:

$c_1=\frac{4}{\pi}\int_0^{\pi/2}\cos\theta\operatorname{sen}2\theta\,d\theta=\frac{8}{\pi}\int_0^{\pi/2}\cos^2\theta\operatorname{sen}\theta\,d\theta=-\frac{8}{3\pi}\cos^3\theta\Big]_0^{\pi/2}=\frac{8}{3\pi}\to\boxed{u(r,\theta)=\frac{8}{3\pi}r^2\operatorname{sen}2\theta+\cdots}$.

[No costaría mucho dar todos los $c_n=\frac{4}{\pi}\int_0^{\pi/2}\cos\theta\operatorname{sen}2n\theta\,d\theta=\frac{2}{\pi}\int_0^{\pi/2}[\operatorname{sen}(2n+1)\theta+\operatorname{sen}(2n-1)\theta]\,d\theta=\cdots=\frac{8n}{(4n^2-1)\pi}$].

16 $\boxed{\begin{array}{l}\Delta u=0,\ r<2,\ 0<\theta<\frac{\pi}{4}\\ u(2,\theta)=f(\theta),\ u(r,0)=u_\theta(r,\frac{\pi}{4})=0\end{array}}$ Haciendo $u=R\Theta$: $\Theta''+\lambda\Theta=0$, $\Theta(0)=\Theta'(\frac{\pi}{4})=0\Rightarrow$ $\lambda_n=2^2(2n-1)^2$, $\Theta_n(\theta)=\{\operatorname{sen}(4n-2)\theta\}$, $n=1,2,\dots$



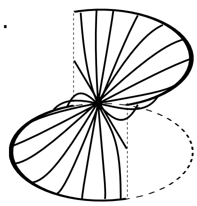
Y la ecuación radial es: $r^2R''+rR'-\lambda_nR=0$, $R=c_1r^{4n-2}+\frac{c_2}{r^{4n-2}}\xrightarrow{R\,acot.}R_n=\{r^{4n-2}\}$.

Imponemos a $u(r,\theta)=\sum_{n=1}^{\infty}c_nr^{4n-2}\operatorname{sen}(4n-2)\theta$, el dato final: $u(2,\theta)=\sum_{n=1}^{\infty}c_n2^{4n-2}\operatorname{sen}(4n-2)\theta=f(\theta)$.

En el caso i) es claro que $c_n=0$, si $n\neq 2$, y que $2^6c_2=8$. $\boxed{u(r,\theta)=\frac{1}{8}r^6\operatorname{sen}6\theta}$. En ii) hay que desarrollar:

$c_n=\frac{2}{2^{4n-2}\pi/4}\int_0^{\pi/4}\pi\operatorname{sen}(4n-2)\theta\,d\theta=\Big[-\frac{8\cos(4n-2)\theta}{(4n-2)2^{4n-2}}\Big]_0^{\pi/4}$. $\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\frac{1}{(2n-1)2^{4n-4}}r^{4n-2}\operatorname{sen}(4n-2)\theta}$.

17 $\boxed{\begin{array}{l}\Delta u=0,\ r<1\\ u(1,\theta)=f(\theta)\end{array},\ f(\theta)=\left\{\begin{array}{l}1,\ 0\leq\theta\leq\pi\\ 0,\ \pi<\theta<2\pi\end{array}\right.}$ El principio del máximo ya da la superior: $0\leq\boxed{u\leq 1}$. Necesitamos la solución para la otra.



Con **Poisson**: $u=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\phi)\,d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$, $u(\frac{1}{2},\frac{\pi}{2})=\frac{3}{8\pi}\int_0^{\pi}\frac{d\phi}{\frac{5}{4}-\cos(\frac{\pi}{2}-\phi)}=\frac{3}{2\pi}\int_0^{\pi}\frac{d\phi}{5-4\sin\phi}\equiv I$

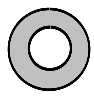
$s=\tan\frac{\phi}{2}\to I=\frac{3}{5\pi}\int_0^{\infty}\frac{ds}{u^2-\frac{8}{5}s+1}=\frac{1}{\pi}\int_0^{\infty}\frac{5/3\,ds}{1+(\frac{5u-4}{3})^2}=\frac{1}{2}+\frac{1}{\pi}\arctan\frac{4}{3}>\frac{1}{2}+\frac{1}{4}>\frac{2}{3}$.

[Sería un buen intento acotar el integrando, pero no basta: $\frac{1}{5}\leq\frac{1}{5-4\sin\phi}\leq 1\to\frac{3}{10}\leq I\leq\frac{3}{2}$].

Con la serie de 4.3 es más largo: $u=\frac{a_o}{2}+\sum_{n=1}^{\infty}r^n[a_n\cos n\theta+b_n\operatorname{sen}n\theta]=\cdots=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n-1}}{2n-1}\operatorname{sen}(2n-1)\theta$

$\to I=\frac{1}{2}+\frac{2}{\pi}\big[\frac{1}{2}-\frac{1}{3}\frac{1}{2^3}+\frac{1}{5}\frac{1}{2^5}-\cdots\big]>\frac{1}{2}+\frac{1}{\pi}\big[1-\frac{1}{12}\big]>\frac{1}{2}+\frac{11}{48}=\frac{35}{48}>\frac{2}{3}$ [$I=\frac{1}{2}+\frac{2}{\pi}\arctan\frac{1}{2}$ otro valor exacto de I].

18 a) $\boxed{\begin{array}{l}\Delta u=0,\ 1<r<2\\ u(1,\theta)=1+\operatorname{sen}2\theta,\ u_r(2,\theta)=0\end{array}}$ $u=R\Theta\to\Theta''+\lambda\Theta=0$ y $r^2R''+rR'-\lambda R=0$. $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta\ 2\pi\text{-periódica}\end{array}\right.\to\lambda_n=n^2$, $\Theta_n=\{\operatorname{sen}n\theta,\cos n\theta\}$, $n=0,1,\dots$



Las soluciones de las ecuaciones de Euler para estos λ_n, utilizando ya que $u_r(2,\theta)=R'(2)\Theta(\theta)=0$, son:

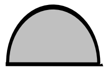
$r^2R''+rR'-n^2R=0\to\mu=\pm n$, $\begin{array}{l}R_0=c_1+c_2\ln r\\ R_n=c_1r^n+c_2r^{-n}\end{array}\xrightarrow{R'(2)=0}R_0=\{1\}$, $R_n=\{r^n+2^{2n}r^{-n}\}$, $n=1,2,\dots$

Probamos entones una solución de la forma: $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[r^n+2^{2n}r^{-n}][a_n\cos n\theta+b_n\operatorname{sen}n\theta]$.

Imponiendo el dato que falta: $u(1,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[1+2^{2n}][a_n\cos n\theta+b_n\operatorname{sen}n\theta]=1+\operatorname{sen}2\theta$ (ya desarrollada).

Sólo son no nulos: $\frac{a_0}{2}=1$ y $[1+16]b_2=1$. Por tanto la solución única es: $\boxed{u(r,\theta)=1+\frac{1}{17}\big[r^2+\frac{16}{r^2}\big]\operatorname{sen}2\theta}$.

18 b) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<1\,,\ 0<\theta<\pi\\ u_r(1,\theta)=4\cos^3\theta\,,\ u_\theta(r,0)=u_\theta(r,\pi)=0\end{array}}$ **Neumann**. $\Theta''+\lambda\Theta=0\,,\ \Theta'(0)=\Theta'(\pi)=0\ \Rightarrow$ $\lambda_n=n^2,\ \Theta_n=\{\cos n\theta\}\,,\ n=0,1,\dots\ \big[\{1\}$ entre ellas$\big]$.



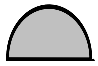
$r^2R''+rR'-\lambda_nR=0$. Si $n=0$, $R=c_1+c_2\ln r\xrightarrow{\text{R ac.}}R_0=\{1\}$. Si $n>0$: $R=c_1r^n+c_2r^{-n}\xrightarrow{\text{R ac.}}R_n=\{r^n\}$.

Imponemos a $u(r,\theta)=a_0+\sum_{n=1}^{\infty}a_nr^n\cos n\theta$ el dato final: $u_r(1,\theta)=0+\sum_{n=1}^{\infty}na_n\cos n\theta=4\cos^3=3\cos\theta+\cos 3\theta\ \to$

a_0 cualquiera, $a_1=3$, $3a_3=1$ y demás $a_n=0$. Las infinitas soluciones son $\boxed{u=C+3r\cos\theta+\frac{1}{3}r^3\cos 3\theta}$.

c) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<1\,,\ 0<\theta<\pi\\ u(1,\theta)=\theta^2,\ u_\theta(r,0)=u_\theta(r,\pi)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta'(0)=\Theta'(\pi)=0\end{array}\right.\to\ \lambda_n=n^2\,,\ \Theta_n=\{\cos n\theta\}\,,\ n=0,1,\dots$

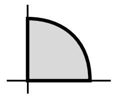
$r^2R''+rR-n^2R=0$ y acotado $\to R_n=\{r^n\}$.



$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\,r^n\cos n\theta\ \xrightarrow{u(1,\theta)=\theta^2}\ \boxed{u(r,\theta)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\,r^n\cos n\theta}$, pues $a_0=\frac{2}{\pi}\int_0^\pi\theta^2d\theta=\frac{2\pi^2}{3}$,

$a_n=\frac{2}{\pi}\int_0^\pi\theta^2\cos n\theta\,d\theta=\frac{2\theta^2}{\pi n}\operatorname{sen} n\theta\Big]_0^\pi-\frac{4}{n\pi}\int_0^\pi\theta\operatorname{sen} n\theta\,d\theta=\frac{4\theta}{\pi n^2}\cos n\theta\Big]_0^\pi-\frac{4}{n^2\pi}\int_0^\pi\cos n\theta\,d\theta=\frac{4(-1)^n}{n^2}$.

d) $\boxed{\begin{array}{l}\Delta u=4\operatorname{sen}2\theta\,,\ \theta\in\big(0,\frac{\pi}{2}\big),\ r<1\\ u(1,\theta)=\operatorname{sen}4\theta\,,\ u(r,0)=u\big(r,\frac{\pi}{2}\big)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta(\frac{\pi}{2})=0\end{array}\right.\to\ \lambda_n=4n^2,\ \Theta_n=\{\operatorname{sen}2n\theta\}\,,\ n=1,2,\dots$



Llevamos a la ecuación: $\sum_{n=1}^{\infty}R_n(r)\operatorname{sen}2n\theta\to\sum_{n=1}^{\infty}\Big[R_n''+\frac{R_n'}{r}-\frac{4n^2R_n}{r^2}\Big]\operatorname{sen}2n\theta=4\operatorname{sen}2\theta$ (desarrollada).

Del dato de contorno: $\sum_{n=1}^{\infty}R_n(1)\operatorname{sen}2n\theta=\operatorname{sen}4\theta$, (también desarrollada). Además soluciones acotadas en $r=0$.

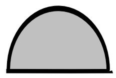
Únicos $R_n\not\equiv 0$ (solución única) salen de: $\left\{\begin{array}{l}r^2R_1''+rR_1'-4R_1=4r^2\\ R_1\text{ acotada},\ R_1(1)=0\end{array}\right.$. $c_1r^2+c_2r^{-2}+R_{1p}$. La R_{1p} se puede hallar:

Con la f.v.c.: $|W|=\begin{vmatrix}r^2 & r^{-2}\\ 2r & -2r^{-3}\end{vmatrix}=-\frac{4}{r}$, $R_{1p}=r^{-2}\int\frac{r^2\cdot 4}{-4/r}dr-r^2\int\frac{r^{-2}\cdot 4}{-4/r}dr=r^2\ln r-\frac{1}{4}r^2$, o probando

$R_{1p}=Ar^2\ln r\ \big(Ase^{2s}\big)\to 4Ar^2=4r^2$, $R_1=c_1r^2+c_2r^{-2}+r^2\ln r\xrightarrow{c.c}c_2=0$ y $c_1+0=0$.

$\left\{\begin{array}{l}r^2R_2''+rR_2'-16R_2=0\\ R_2\text{ acotada},\ R_2(1)=1\end{array}\right.\to R_2=c_1r^4+c_2r^{-4}\underset{c.c.}{\to}c_2=0\,,\ c_1=1$. $\boxed{u(r,\theta)=r^2\ln r\operatorname{sen}2\theta+r^4\operatorname{sen}4\theta}$.

e) $\boxed{\begin{array}{l}\Delta u=3\operatorname{sen}\frac{\theta}{2}\,,\ r<1\,,\ \theta\in(0,\pi)\\ u(1,\theta)=u(r,0)=u_\theta(r,\pi)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta'(\pi)=0\end{array}\right.\to\ \Theta_n=\Big\{\operatorname{sen}\frac{(2n-1)\theta}{2}\Big\},\ n=1,2,\dots.$



Llevamos $u(r,\theta)=\sum_{n=1}^{\infty}R_n(r)\Theta_n(\theta)$, a la EDP: $\sum_{n=1}^{\infty}\Big[R_n''+\frac{1}{r}R_n'-\frac{(2n-1)^2}{4r^2}R_n\Big]\operatorname{sen}\frac{(2n-1)\theta}{2}=3\operatorname{sen}\frac{\theta}{2}$ (ya desarrollada).

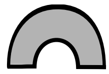
Además debe $u(1,\theta)=\sum_{n=1}^{\infty}R_n(1)\Theta_n(\theta)=0\to R_n(1)=0\ \forall n$ y estar la solución acotada en el origen.

Única $R_n\not\equiv 0$ sale de: $\left\{\begin{array}{l}r^2R_1''+rR_1-\frac{1}{4}R_1=3r^2\\ R_1\text{ acotada},\ R_1(1)=0\end{array}\right.\overset{\bullet}{\to}R_1=c_1r^{1/2}+c_2r^{-1/2}+\frac{4}{5}r^2\xrightarrow{c.c.}\boxed{u(r,\theta)=\frac{4}{5}\big[r^2-r^{1/2}\big]\operatorname{sen}\frac{\theta}{2}}$.

• $\mu(\mu-1)+\mu-\frac{1}{4}=0$, $\mu=\pm\frac{1}{2}$. $R_1=c_1r^{1/2}+c_2r^{-1/2}+R_p$. $R_p=Ar^2\ \big(R_p=Ae^{2s}\big)\to 2A+2A-\frac{1}{4}A=3$, $A=\frac{4}{5}$.

O con la fvc: $\begin{vmatrix}r^{1/2} & r^{-1/2}\\ r^{-1/2}/2 & -r^{-3/2}/2\end{vmatrix}=-r^{-1}$, $R_p=-r^{-1/2}\int\frac{r^{1/2}\cdot 3}{r^{-1}}+r^{1/2}\int\frac{r^{-1/2}\cdot 3}{r^{-1}}=-\frac{6}{5}r^2+2r^2=\frac{4}{5}r^2$.

f) $\boxed{\begin{array}{l}\Delta u=-1/r\,,\ 1<r<3\,,\ 0<\theta<\pi\\ u(1,\theta)=2+\cos\theta,\ u(3,\theta)=u_\theta(r,0)=u_\theta(r,\pi)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta'(0)=\Theta'(\pi)=0\end{array}\right.\to\ \Theta_n=\{\cos n\theta\}\,,\ n=0,1,\dots$



$u(r,\theta)=R_0(r)+\sum_{n=1}^{\infty}R_n(r)\cos n\theta\ \to\ R_0''+\frac{1}{r}R_0'+\sum_{n=1}^{\infty}\Big[R_n''+\frac{1}{r}R_n'-\frac{n^2}{r^2}R_n\Big]\cos n\theta=-\frac{1}{r}$.

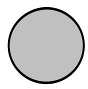
Además: $u(1,\theta)=R_0(1)+\sum_{n=1}^{\infty}R_n(1)\cos n\theta=2+\cos\theta$, $u(3,\theta)=R_0(3)+\sum_{n=1}^{\infty}R_n(3)\cos n\theta=0$. Son no nulos:

$\left\{\begin{array}{l}r^2R_0''+rR_0=-r\overset{\bullet}{\to}R_0=c_1+c_2\ln r-r\xrightarrow{d.i.}R_0(r)=3-r\\ R_0(1)=2\,,\ R_0(3)=0\end{array}\right.$ • $R'=v\to v'=-\frac{v}{r}-\frac{1}{r}$. $v=\frac{C}{r}-\frac{1}{r}\int dr=\frac{C}{r}-1$, $R=K+C\ln r-r$.

O Euler: $\mu=0$ doble. $R=c_1+c_2\ln r+R_p$. $R_p=Ar\ \big(R_p=Ae^s\big)\to A=-1$.

$\left\{\begin{array}{l}r^2R_1''+rR_1'-R_1=0\to R_2=c_1r+c_2r^{-1}\underset{d.i.}{\longrightarrow}R_2(r)=\frac{1}{8}\big[\frac{9}{r}-r\big]\\ R_1(1)=1\,,\ R_1(3)=0\end{array}\right.\ \to\ \boxed{u(r,\theta)=3-r+\frac{1}{8}\big[\frac{9}{r}-r\big]\cos\theta}$.

g) $\boxed{\begin{array}{l}\Delta u=\cos\theta\,,\ r<2\\ u(2,\theta)=\operatorname{sen}2\theta\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta\ \ 2\pi\text{-per.}\end{array}\right.\to\ u=a_0(r)+\sum_{n=1}^{\infty}\big[a_n(r)\cos n\theta+b_n(r)\operatorname{sen}n\theta\big]\ \to$



$\frac{1}{r}\big(ra_0''+a_0'\big)+\sum_{n=1}^{\infty}\Big[\frac{1}{r^2}\big(r^2a_n''+ra_n'-n^2a_n\big)\cos n\theta+\frac{1}{r^2}\big(r^2b_n''+rb_n'-n^2b_n\big)\operatorname{sen}n\theta\Big]=\cos\theta$.

$u(2,\theta)=\operatorname{sen}2\theta\to a_n(2)=0$; $b_n(2)=0$, $n\neq 2$; $b_2(2)=1$ y todas acotadas $\Rightarrow a_{n\neq 1},b_{n\neq 2}\equiv 0$. Y además:

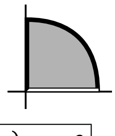
$\left\{\begin{array}{l}r^2a_1''+ra_1'-a_1=r^2\\ \text{acotada y }a_1(2)=0\end{array}\right.\ \underset{3A=1}{\xrightarrow{a_{1p}=Ar^2}}\ a_0=c_1r+c_2r^{-1}+\frac{r^2}{3}\xrightarrow{c.c.}a_1(r)=\frac{r^2}{3}-\frac{2r}{3}$

$\left\{\begin{array}{l}r^2b_2''+rb_2'-4b_2=0\\ \text{acotada y }b_2(2)=1\end{array}\right.\to b_2=c_1r^2+c_2r^{-2}\xrightarrow{c.c.}\begin{array}{l}c_2=0\\ 4c_1=1\end{array}\to b_2(r)=\frac{r^2}{4}$

$\to\ \boxed{u=\frac{1}{3}r(r-2)\cos\theta+\frac{1}{4}r^2\operatorname{sen}2\theta}$.

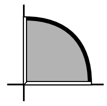
h) $\boxed{\begin{array}{l}\Delta u=8r\cos\theta\,,\ r<1\,,\ 0<\theta<\frac{\pi}{2}\\ u(1,\theta)=u_\theta(r,0)=u\big(r,\frac{\pi}{2}\big)=0\end{array}}$ $u=\sum_{n=1}^{\infty}R_n(r)\cos(2n-1)\theta\ \to\ \sum_{n=1}^{\infty}R_n(1)\cos(2n-1)\theta=0\,,\ R_n'(1)=0$,

y $\sum_{n=1}^{\infty}\Big[R_n''+\frac{1}{r}R_n'-\frac{(2n-1)^2}{r^2}R_n\Big]\cos(2n-1)\theta=8r\cos\theta$.



Sobrevive $\left\{\begin{array}{l}r^2R_1''+rR_1'-R_1=8r^3\\ R_1\text{ acotada},\ R_1(1)=0\end{array}\right.\underset{R_p=Ar^3}{\longrightarrow}R_1=c_1r+c_2r^{-1}+r^3\xrightarrow{cc}\begin{array}{l}c_2=0\\ c_1+c_2+1=0,\ c_1=-1\end{array}$. $\boxed{u(r,\theta)=(r^3-r)\cos\theta}$.

19 a) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<2\,,\ \theta\in\left(0,\frac{\pi}{2}\right)\\ u_r(2,\theta)+ku(2,\theta)=8\cos 2\theta\\ u_\theta(r,0)=u_\theta\left(r,\frac{\pi}{2}\right)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta'(0)=\Theta'\left(\frac{\pi}{2}\right)=0\end{array}\right.$, $\lambda_n=4n^2$, $\Theta_n=\{\cos 2n\theta\}$, $n=0,1,2,\dots$



Y además: $r^2R''+rR'-\lambda R=0\to$ $\begin{array}{l}R_0=c_1+c_2\ln r\overset{R\text{ acotado}}{\longrightarrow}R_0=\{1\}\\ R_{2n}=c_1r^{2n}+c_2r^{-2n}\longrightarrow R_{2n}=\{r^{2n}\}\end{array}$ $\to u=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_{2n}\,r^{2n}\cos 2n\theta$.

Imponiendo el último dato de contorno: $k\frac{a_0}{2}+\sum_{n=1}^{\infty}a_{2n}\left[2n2^{2n-1}+k2^{2n}\right]\cos 2n\theta=8\cos 2\theta$.

i) Para $k=1$, todos los $a_{2n}=0$, excepto $a_2[4+4]=8$, y la solución (única) es: $\boxed{u(r,\theta)=r^2\cos 2\theta}$.

ii) Para $k=0$ (es problema de Neumann), a_0 queda libre, $a_2[4+0]=8$, y los demás $a_{2n}=0$.

En este caso existen las infinitas soluciones: $\boxed{u(r,\theta)=C+2r^2\cos 2\theta}$.

b) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<1,\ \theta\in(0,\pi)\\ u(1,\theta)+ku_r(1,\theta)=4\,\text{sen}\,\frac{3\theta}{2}\\ u(r,0)=u_\theta(r,\pi)=0\end{array}}$ $\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta'(\pi)=0\end{array}\right.\to\lambda_n=\frac{(2n-1)^2}{4}$, $\Theta_n=\left\{\text{sen}\,\frac{2n-1}{2}\theta\right\}$, $n=1,2,\dots\to$

$r^2R''+rR-\lambda_nR=0\to R=c_1r^{n-\frac12}+c_2r^{-n+\frac12}\overset{R\text{ acotada}}{\longrightarrow}R_n=\left\{r^{n-\frac12}\right\}\to$

$u(r,\theta)=\sum_{n=1}^{\infty}c_n\,r^{n-\frac12}\,\text{sen}\,\frac{2n-1}{2}\theta$, $u_r=\sum_{n=1}^{\infty}c_n(n-\frac12)r^{n-\frac32}\,\text{sen}\,\frac{2n-1}{2}\theta\overset{\text{dato final}}{\longrightarrow}\sum_{n=1}^{\infty}\left[1+kn-\frac{k}{2}\right]c_n\,\text{sen}\,\frac{2n-1}{2}\theta=4\,\text{sen}\,\frac{3\theta}{2}$

i) si $k=2$, $\sum_{n=1}^{\infty}2nc_n\,\text{sen}\,\frac{2n-1}{2}\theta=4\,\text{sen}\,\frac{3\theta}{2}\to c_2=1$ y todos los demás $c_n=0\to$ $\boxed{u=r^{3/2}\,\text{sen}\,\frac{3\theta}{2}}$

ii) si $k=-2$, $\sum_{n=1}^{\infty}2\left[1-n\right]c_n\,\text{sen}\,\frac{2n-1}{2}\theta=4\,\text{sen}\,\frac{3\theta}{2}\to c_2=-2$, c_1 cualquiera y los demás $c_n=0$.

Hay infinitas soluciones $\boxed{u=C\,r^{1/2}\,\text{sen}\,\frac{\theta}{2}-2\,r^{3/2}\,\text{sen}\,\frac{3\theta}{2}}$ [En este caso la fórmula de Green no permite probar la unicidad y el problema no tiene sentido físico].

20 $\boxed{\begin{array}{l}u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}+4u=0\,,\ r<1\,,\ 0<\theta<\pi\\ u(1,\theta)=\text{sen}\,\frac{\theta}{2}\,,\ u(r,0)=u_\theta(r,\pi)=0\end{array}}$ $u=R\Theta\to R''\Theta+\frac{R'\Theta}{r}+\frac{R\Theta''}{r^2}+4R\Theta=0\to\frac{r^2R''}{R}+\frac{rR'}{R}+4r^2=-\frac{\Theta''}{\Theta}=\lambda$

$\to\left\{\begin{array}{l}\Theta''+\lambda\Theta=0\\ \Theta(0)=\Theta'(\pi)=0\end{array}\right.\to\lambda_n=\frac{(2n-1)^2}{4}$, $\Theta_n=\left\{\text{sen}\,\frac{2n-1}{2}\theta\right\}$, $n=1,2,\dots$ y $r^2R''+rR'+(4r^2-\lambda_n)R=0$.

Parecida a Bessel. Para quitar el 4: $s=\sqrt{4}\,r=2r\to R'=2\frac{dR}{ds}$, $R''=4\frac{d^2R}{ds^2}\to s^2\frac{d^2R}{ds^2}+s\frac{dR}{ds}+(s^2-\lambda_n)R=0$,

que es Bessel con $p=n-\frac12$, cuyas soluciones acotadas en $r=0$ son: $\{J_{n-\frac12}(s)\}=\{J_{n-\frac12}(2r)\}=R_n$ (funciones elementales)

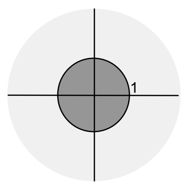
Probamos: $u=\sum_{n=1}^{\infty}c_nJ_{n-\frac12}(2r)\,\text{sen}\,\frac{2n-1}{2}\theta\to\sum_{n=1}^{\infty}c_nJ_{n-\frac12}(2)\,\text{sen}\,\frac{2n-1}{2}\theta=\text{sen}\,\frac{\theta}{2}\to c_1=\frac{1}{J_{\frac12}(2)}$ y los demás $c_n=0$

$\to u=\frac{1}{J_{\frac12}(2)}J_{\frac12}(2r)\,\text{sen}\,\frac{\theta}{2}$, que podemos escribir en términos de funciones elementales.

Como (salvo constante) $J_{\frac12}(2r)=\frac{\text{sen}\,2r}{\sqrt{2r}}$ $\left[\frac{\cos 2r}{\sqrt{2r}}\right.$ no acotada en $\left.r=0\right]$ y $J_{\frac12}(2)=\frac{\text{sen}\,2}{\sqrt2}$, $\boxed{u(r,\theta)=\frac{\text{sen}\,2r}{\text{sen}\,2\,\sqrt{r}}\,\text{sen}\,\frac{\theta}{2}}$

21 $\boxed{\begin{array}{l}u_{rr}+\frac1r u_r+\frac{1}{r^2}u_{\theta\theta}=\frac{2\,\text{sen}\,\theta}{1+r^2}\\ u(1,\theta)=1\,,\ u\text{ acotada}\end{array}}$ Separando variables en la homogénea: $\Theta''+\lambda\Theta=0$ y $r^2R''+rR'-\lambda R=0$. Las autofunciones son $\Theta_n=\{\cos n\theta,\text{sen}\,n\theta\}$, $n=0,1,\dots$ [Θ 2π-periódica].

Probamos en ambos casos: $u(r,\theta)=a_0(r)+\sum_{n=1}^{\infty}\left[a_n(r)\cos n\theta+b_n(r)\,\text{sen}\,n\theta\right]\to$



$a_0''+\frac1r a_0'+\sum_{n=1}^{\infty}\left[\left(a_n''+\frac1r a_n'-\frac{n^2}{r^2}a_n\right)\cos n\theta+\left(b_n''+\frac1r b_n'-\frac{n^2}{r^2}b_n\right)\text{sen}\,n\theta\right]=\frac{2}{1+r^2}\,\text{sen}\,\theta\to$

$r^2a_0''+ra_0'=0$; $r^2a_n''+ra_n'-n^2a_n=0$, $n\geq1$; $r^2b_1''+rb_1'-b_1=\frac{2r^2}{1+r^2}$; $r^2b_n''+rb_n'-n^2b_n=0$, $n\geq2$.

$u(1,\theta)=1\Rightarrow a_0(1)=1$ y que las demás se anulan en 1. Sólo tendrán solución no trivial:

$\left\{\begin{array}{l}r^2a_0''+ra_0'=0\\ a_0(1)=1,\ a_0\text{ acotada}\end{array}\right.\to a_0=c_1+c_2\ln r\overset{\text{c.c.}}{\longrightarrow}a_0=1$, para **i]** y para **ii]**.

$\left\{\begin{array}{l}r^2b_1''+rb_1'-b_1=\frac{2r^2}{1+r^2}\\ b_1(1)=0,\ b_1\text{ acotada}\end{array}\right.\to b_1=c_1r+c_2r^{-1}+b_{1p}$. Necesitamos la fvc: $\begin{vmatrix}r & r^{-1}\\ 1 & -r^{-2}\end{vmatrix}=-2r^{-1}$, $f(r)=\frac{2}{1+r^2}$,

$b_{1p}=-r^{-1}\int\frac{r^2+1-1}{1+r^2}+r\int\frac{1}{1+r^2}=\left(r+\frac1r\right)\arctan r-1$.

i] En $r<1$, como $\frac{\arctan r}{r}\overset{r\to0}{\longrightarrow}1$, debe ser $c_2=0$. Imponiendo la otra: $c_1+2\arctan 1-1=0\to$

$b_1=\left(1-\frac{\pi}{2}\right)r+\left(r+\frac1r\right)\arctan r-1$. $u=1+\left[r-\frac{\pi}{2}r+\left(r+\frac1r\right)\arctan r-1\right]\text{sen}\,\theta$.

ii] En el infinito $b_{1p}\sim\frac{\pi}{2}r-1$. Para que b_1 pueda estar acotada debe ser $c_1=-\frac{\pi}{2}$. Además: $-\frac{\pi}{2}+c_2+\frac{\pi}{2}-1=0\to$

$b_1=\frac1r-\frac{\pi}{2}r+\left(r+\frac1r\right)\arctan r-1$. $u=1+\left[\frac1r-\frac{\pi}{2}r+\left(r+\frac1r\right)\arctan r-1\right]\text{sen}\,\theta$ $\left[\frac{\arctan r-\pi/2}{1/r}\underset{r\to\infty}{\longrightarrow}-1\right]$.

22 **Plano** En los apuntes: **Espacio**

$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} r^{-n}[a_n\cos n\theta+b_n\,\text{sen}\, n\theta]$ $\qquad$ $u(r,\theta)=\sum_{n=0}^{\infty} a_n\, r^{-(n+1)}P_n(\cos\theta)$

$a_n=\frac{R^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta\,,\ n=0,1,\dots$ $\qquad$ $a_n=\frac{2n+1}{2}R^{n+1}\int_0^{\pi} f(\theta)\,P_n(\cos\theta)\,\text{sen}\,\theta\, d\theta$

$b_n=\frac{R^n}{\pi}\int_0^{2\pi} f(\theta)\,\text{sen}\, n\theta\, d\theta\,,\ n=1,2,\dots$ $\quad$ $\boxed{f(\theta)=\cos^3\theta}$ $\quad$ $a_n=\frac{2n+1}{2}R^{n+1}\int_{-1}^{1} t^3 P_n(t)\,dt$

$\frac{a_0}{2}+\sum_{n=1}^{\infty}\big[\frac{a_n}{R^n}\cos n\theta+\frac{b_n}{R^n}\,\text{sen}\, n\theta\big]=\frac{\cos 3\theta}{4}+\frac{3\cos\theta}{4}$ $\qquad$ $a_1=3R^2\int_0^1 t^4 dt=\frac{3R^2}{5}\,,\ a_3=7R^4\int_0^1\big(\frac{5t^6}{2}-\frac{3t^4}{2}\big)dt=\frac{2R^4}{5}\ \to$

$\to\ \boxed{u=\frac{3R}{4r}\cos\theta+\frac{R^3}{4r^3}\cos 3\theta}$ $\qquad$ $\boxed{u=\frac{3R^2}{5r^2}\cos\theta+\frac{2R^4}{5r^4}\big(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\big)}$ [o tanteando].

23 a) $\boxed{\begin{array}{l}\Delta u=0\,,\ 1<r<2\\ u(1,\theta)=\cos\theta\,,\ u(2,\theta)=0\end{array}}$ $\xrightarrow[u=R\Theta]{}$ $(\text{sen}\,\theta\,\Theta')'+(\lambda\,\text{sen}\,\theta)\Theta=0 \xrightarrow{ac\ 0,\pi} \lambda_n=n(n+1)\,,\ \Theta_n=\{P_n(\cos\theta)\}$
y $r^2R''+2rR'-\lambda R=0\ \to\ R=c_1r^n+c_2r^{-n-1}\,,\ n=0,1,\dots$

$u(r,2)=R(2)\Theta(\theta)=0\to R(2)=c_1 2^n+c_2 2^{-n-1}=0\,,\ R_n=\{r^n-2^{2n+1}r^{-n-1}\}\,,\ u=\sum_{n=0}^{\infty} a_n\big[r^n-2^{2n+1}r^{-n-1}\big]P_n(\cos\theta)\,.$

Sólo falta el dato: $u(1,\theta)=\sum_{n=0}^{\infty} a_n\big[1-2^{2n+1}\big]P_n(\cos\theta)=\cos\theta=P_1(\cos\theta)\ \to\ a_1=-\frac{1}{7}$ y los demás cero.

Así pues, la solución en nuestra corona esférica es: $\boxed{u(r,\theta)=\frac{1}{7}\big[8r^{-2}-r\big]\cos\theta}$.

b) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<1\\ u_r(1,\theta)=\cos^3\theta\end{array}}$ La $u(r,\theta)=\sum_{n=0}^{\infty} a_nr^nP_n(\cos\theta)$ satisface todo menos $u_r(1,\theta)=\sum_{n=1}^{\infty} na_nP_n(\cos\theta)=\cos^3\theta$

$\to\ a_n=\frac{2n+1}{2n}\int_0^{\pi}\cos^3\theta\,P_n(\cos\theta)\,\text{sen}\,\theta\, d\theta=\frac{2n+1}{2n}\int_{-1}^{1} t^3\,P_n(t)\,dt$ y $\forall a_0$ (es Neumann).

(Desarrollo posible por ser 0 el primer término del desarrollo de $\cos^3\theta$ en estas autofunciones).

Integrando: $a_1=\frac{3}{2}\int_{-1}^{1} t^4\,dt=\frac{3}{5}$, $a_3=\frac{7}{6}\int_{-1}^{1} t^3\big[\frac{5}{2}t^3-\frac{3}{2}t\big]\,dt=\frac{7}{6}\int_0^1[5t^6-3t^4]\,dt=\frac{2}{15}$ y demás $a_n=0$,
pues para desarrollar un Q_k bastan los k primeros P_n y es $\int_{-1}^{1}=0$ para n par. O mejor tanteando:

$\cos^3\theta=\frac{2}{5}\underbrace{\big(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\big)}_{P_3(\cos\theta)}+\frac{3}{5}\underbrace{\cos\theta}_{P_1(\cos\theta)}\to 3a_3=\frac{2}{5}\,,\ a_1=\frac{3}{5}\ \to\ \boxed{u=C+\frac{3}{15}[3r-r^3]\cos\theta+\frac{1}{3}r^3\cos^3\theta}$.

c) $\boxed{\begin{array}{l}\Delta u=0\,,\ r<3\\ u_r(3,\theta)+u(3,\theta)=\text{sen}^2\theta\end{array}}$ La serie de los apuntes satisface todo excepto el nuevo dato inicial:
$u=\sum_{n=0}^{\infty} a_nr^nP_n(\cos\theta)\ \to\ u_r(3,\theta)+u(3,\theta)=\sum_{n=0}^{\infty} 3^{n-1}(n+3)a_n\,P_n(\cos\theta)=\text{sen}^2\theta$

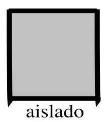
$\to\ a_n=\frac{2n+1}{3^{n-1}2(n+3)}\int_0^{\pi}\text{sen}^2\theta\,P_n(\cos\theta)\,\text{sen}\,\theta\, d\theta=\frac{2n+1}{3^{n-1}2(n+3)}\int_{-1}^{1}(1-t^2)P_n(t)\,dt\ \to\ a_0=\frac{1}{2}\int_{-1}^{1}(1-t^2)dt=\frac{2}{3}\,,$

$a_2=\frac{1}{6}\int_{-1}^{1}(1-t^2)\big(\frac{3}{2}t^2-\frac{1}{2}\big)\,dt=\frac{1}{3}\int_0^1\big(-\frac{1}{2}+2t^2-\frac{3}{2}t^4\big)\,dt=-\frac{2}{45}$. [Demás $a_n=0$, $\int_{-1}^{1}=0$ si n impar].

Pero para esta $f(\theta)$ mejor tanteamos: $1-\cos^2\theta=-\frac{2}{3}\underbrace{\big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\big)}_{P_2(\cos\theta)}+\frac{2}{3}\cdot\underbrace{1}_{P_0(\cos\theta)}\ \to\ a_0=\frac{2}{3}\,,\ 15a_2=-\frac{2}{3}\,.$

Por tanto, $u=\frac{2}{3}-\frac{2}{45}r^2\big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\big)=\frac{2}{3}+\frac{1}{45}r^2-\frac{1}{15}r^2\cos^2\theta=\frac{2}{3}+\frac{1}{45}\big[x^2+y^2-2z^2\big]\,.$

24 a) $\boxed{\begin{array}{l}u_t-\Delta u=0,(x,y)\in(0,\pi)\times(0,\pi),t>0\\ u(x,y,0)=1+\cos x\cos 2y\\ u_x(0,y,t)=u_x(\pi,y,t)=0\\ u_y(x,0,t)=u_y(x,\pi,t)=0\end{array}}$ $u=XYT\to\begin{cases}\left.\begin{array}{l}X''+\lambda X=0\\ X'(0)=X'(\pi)=0\end{array}\right\}X_n=\{\cos nx\}\\ \left.\begin{array}{l}Y''+\mu Y=0\\ Y'(0)=Y'(\pi)=0\end{array}\right\}Y_n=\{\cos my\}\\ T'+(\lambda+\mu)T=0\,,\ T_{nm}=\big\{e^{-(n^2+m^2)t}\big\}\,,\ n,m=0,1,\dots\end{cases}$



aislado

$u=\frac{a_{00}}{4}+\sum_{n=1}^{\infty}\frac{a_{n0}}{2}e^{-n^2t}\cos nx+\sum_{m=1}^{\infty}\frac{a_{0m}}{2}e^{-m^2t}\cos my+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{nm}e^{-(n^2+m^2)t}\cos nx\cos my\Big|_{t=0}=1+\cos x\cos 2y$

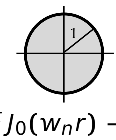
$\boxed{u(x,y,t)=1+e^{-5t}\cos x\cos 2y}\ \xrightarrow{t\to\infty} 1$, valor medio de las temperaturas iniciales.

b) $\boxed{\begin{array}{l}u_{tt}-\Delta u=0\,,\ (x,y)\in(0,\pi)\times(0,\pi),\ t\in\mathbf{R}\\ u(x,y,0)=0\,,\ u_t(x,y,0)=\text{sen}\,3x\,\text{sen}^2 2y\\ u(0,y,t)=u(\pi,y,t)=0\\ u_y(x,0,t)=u_y(x,\pi,t)=0\end{array}}$ $u=XYT\to\begin{cases}\left.\begin{array}{l}X''+\lambda X=0\\ X(0)=X(\pi)=0\end{array}\right\}X_n=\{\text{sen}\, nx\},\ n=1,2,\cdots\\ \left.\begin{array}{l}Y''+\mu Y=0\\ Y'(0)=Y'(\pi)=0\end{array}\right\}Y_n=\{\cos my\},\ m=1,2,\cdots\\ T''+(\lambda+\mu)T=0,\ T(0)=0,\ T_{nm}=\{\text{sen}\,\sqrt{n^2+m^2}\,t\}\end{cases}$

$u=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\,\text{sen}\sqrt{n^2+m^2}\,t\ \text{sen}\, nx\,\cos my\,,\ u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\sqrt{n^2+m^2}\ \text{sen}\, nx\cos my=\frac{1-\cos 4y}{2}\ \text{sen}\,3x$

$\to\ c_{30}\sqrt{9}=\frac{1}{2}\,,\ c_{34}\sqrt{25}=-\frac{1}{2}$ [los demás $c_{nm}=0$], $\boxed{u=\frac{1}{6}\,\text{sen}\,3t\,\text{sen}\,3x-\frac{1}{10}\,\text{sen}\,5t\,\text{sen}\,3x\cos 4y}$.

25 $\boxed{\begin{array}{l}u_t-\big[u_{rr}+\frac{u_r}{r}\big]=0\,,\ r<1,\ t>0\\ u(r,0)=0\,,\ u(1,t)=1\end{array}}$ $\xrightarrow{v=u-1}\begin{cases}v_t-\big[v_{rr}+\frac{1}{r}v_r\big]=0\\ v(r,0)=-1,\ v(1,t)=0\end{cases}\ \to\ \begin{cases}rR''+R'+\lambda rR=0\,,\ T'+\lambda T=0\\ R\text{ acotada},\ R(1)=0\end{cases}$



Problema singular visto en 3.1 (y 3.2): $\lambda_n=w_n^2$ con $J_0(w_n)=0$ y $R_n=\{J_0(w_nr)\}$; $v=\sum_{n=1}^{\infty} c_n\,e^{-\lambda_nt}J_0(w_nr)\ \to$

$\sum_{n=1}^{\infty} c_nJ_0(w_nr)=-1\,,\ c_n=-\frac{2}{J_1^2(w_n)}\int_0^1 rJ_0(w_nr)\,dr\to\ u=1-2\sum_{n=1}^{\infty}\frac{e^{-\lambda_nt}}{w_nJ_1(w_n)}\,J_0(w_nr)\ \Big[\xrightarrow[t\to\infty]{} 1$ en todo el círculo$\Big]$

pues $\int_0^1 rJ_0(w_nr)\,dr=\frac{1}{\lambda_n}\int_0^{w_n} sJ_0(s)\,ds=\frac{1}{w_n}J_1(w_n)$, ya que $[sJ_1]'=sJ_0$.