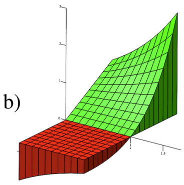
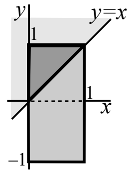
1. a) $\int_0^1\int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_{-1}^1\int_0^1 x^2dx\,dy + \int_0^1\int_{-1}^1 y^2dy\,dx = 2\int_0^1 x^2dx + 2\int_0^1 y^2dy = \frac43$.

b) $\int_{-1}^1\int_0^1 y\,e^{xy}dx\,dy = \int_{-1}^1 \left[e^{xy}\right]_0^1 dy = \int_{-1}^1 \left[e^y-1\right] dy = e-e^{-1}-2$ (≈0.35).

b) 

c) Como el integrando tiene dos expresiones en R, debemos hallar dos integrales:

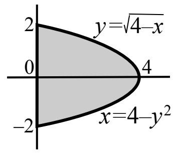
$\int_0^1\int_x^1 (y-x)\,dy\,dx + \int_0^1\int_{-1}^x (x-y)\,dy\,dx = \int_0^1 \left[\frac{(x-1)^2}{2}+\frac{(x+1)^2}{2}\right] dx = \int_0^1 (x^2+1)dx = \frac43$.



d) $\int_{-1}^1\int_0^1 (xy)^2\cos x^3\,dx\,dy = \int_{-1}^1 y^2\left[\frac13 \operatorname{sen} x^3\right]_0^1 dy = \frac23 \operatorname{sen} 1 \int_0^1 y^2dy = \frac29 \operatorname{sen} 1$.

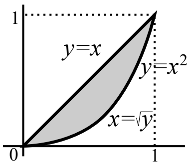
2. a) $\int_1^2\int_1^2 \log(xy)\,dx\,dy = \int_1^2 \log x\,dx + \int_1^2 \log y\,dy = 2[2\log 2-1]$, pues $\int_1^2 \log s\,ds = s\log s\big]_1^2 - \int_1^2 1\,ds = 2\log 2-1$.

b) $\int_{-2}^2\int_0^{4-y^2} x^3y\,dx\,dy = \frac14\int_{-2}^2 \underbrace{y\,(4-y^2)^4}_{\text{impar}}\,dy = 0$. O bien $\int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} x^3y\,dy\,dx = \int_0^4 0\,dx = 0$.



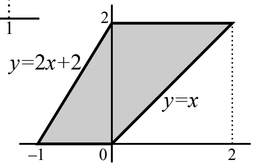
c) $\int_0^1\int_{x^2}^x xy\,dy\,dx = \int_0^1 x\left[\frac{y^2}{2}\right]_{x^2}^x dx = \frac12\int_0^1 [x^3-x^5]\,dx = \frac12\left[\frac14-\frac16\right] = \frac1{24}$.

O bien $\int_0^1\int_y^{\sqrt y} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_y^{\sqrt y} dy = \frac12\int_0^1 [y^2-y^3]\,dy = \frac12\left[\frac13-\frac14\right] = \frac1{24}$.



d) $\int_0^2\int_{y/2-1}^y e^{x-y}\,dx\,dy = \int_0^2 \left[1-e^{-1-y/2}\right]dy = 2\left[1-\frac1e+\frac1{e^2}\right]$, o más largo:

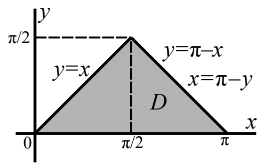
$\int_{-1}^0\int_0^{2x+2} e^{x-y}\,dy\,dx + \int_0^2\int_x^2 e^{x-y}\,dy\,dx = \int_{-1}^0 \left[e^x-e^{-x-2}\right]dx + \int_0^2 \left[1-e^{x-2}\right]dx$.



e) Mejor: $\int_0^{\pi/2}\int_y^{\pi-y} \operatorname{sen} x\,dx\,dy = \int_0^{\pi/2} \Big[\cos y - \underbrace{\cos(\pi-y)}_{=-\cos y}\Big]\,dy = \Big[2\operatorname{sen} y\Big]_0^{\pi/2} = 2$.

Peor: $\int_0^{\pi/2}\int_0^x \operatorname{sen} x\,dy\,dx + \int_{\pi/2}^{\pi}\int_0^{\pi-x} \operatorname{sen} x\,dy\,dx = \int_0^{\pi/2} x\operatorname{sen} x\,dx + \int_{\pi/2}^{\pi} (\pi-x)\operatorname{sen} x\,dx$

$= -\big[x\cos x\big]_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx - \big[(\pi-x)\cos x\big]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\cos x\,dx = 2$.

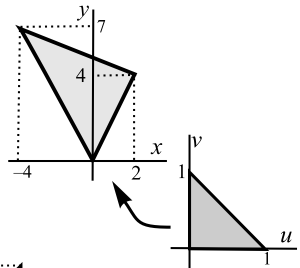


f) $\int_{-4}^0\int_{-7x/4}^{5-x/2} x\,dy\,dx + \int_0^2\int_{2x}^{5-x/2} x\,dy\,dx = 5\int_{-4}^0\left[x+\frac{x^2}{4}\right] + 5\int_0^2\left[x-\frac{x^2}{2}\right] = -\frac{40}{3}+\frac{10}{3} = -10$.

Con un cambio lineal podríamos llevar el triángulo a uno más sencillo; por ejemplo, el que lleva $(1,0)$ a $(2,4)$ y $(0,1)$ a $(-4,7)$, es decir, el dado por la matriz:

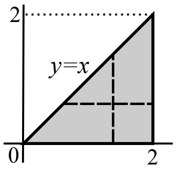
$\mathbf{A} = \begin{pmatrix}2&-4\\4&7\end{pmatrix}$, o sea, $\begin{pmatrix}x\\y\end{pmatrix} = \mathbf{A}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}2u-4v\\4u+7v\end{pmatrix}$; $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix}2&-4\\4&7\end{vmatrix} = 30$;

$\int_0^1\int_0^{1-u} 60(u-2v)\,dv\,du = 60\int_0^1 \left[u(1-u)-(1-u)^2\right]du = -10$.



3. $\int_0^2\int_y^2 e^{x^2}dx\,dy = \int_0^2\int_0^x e^{x^2}dy\,dx = \int_0^2 x\,e^{x^2}dx = \frac12 e^{x^2}\big]_0^2 = \frac12\left[e^4-1\right]$.

(La integral inicial no tiene primitiva elemental).

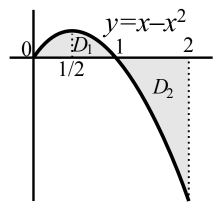


4. El recinto aparece dividido en dos regiones: $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$.

$\int_0^1\int_0^{x-x^2} (x^2+2xy^2+2)\,dy\,dx + \int_1^2\int_{x-x^2}^0 (x^2+2xy^2+2)\,dy\,dx$

$= \int_0^1 \left[(x^2+2)(x-x^2)+\frac{2x}{3}(x-x^2)^3\right]dx - \int_1^2 \left[(x^2+2)(x-x^2)+\frac{2x}{3}(x-x^2)^3\right]dx$

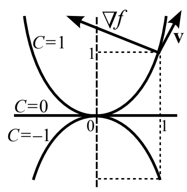
$= \int_0^1 \left(2x-2x^2+x^3-\frac{x^4}{3}-2x^5+2x^6-\frac{2x^7}{3}\right)dx - \int_1^2 (\cdot)\,dx = \frac{27}{70}+\frac{1249}{210} = \frac{19}{3}$.

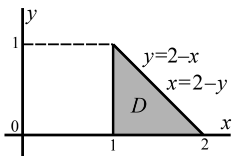


5. a] $\boxed{f(x,y)=\frac{y}{x^2}} = 0, 1, -1 \;\rightarrow\; y=0,\ y=x^2,\ y=-x^2$ (parábolas).

$\nabla f = \left(\frac{-2y}{x^3}, \frac{1}{x^2}\right)\Big|_{(1,1)} = (-2,1)$. $\Delta f = \frac{6y}{x^4}$. $D_{\mathbf v}f(1,1) = (-2,1)\cdot\left(\frac35,\frac45\right) = -\frac25$.

b] $\iint_D f = \int_1^2\int_0^{2-x} \frac{y}{x^2}\,dy\,dx = \frac12\int_1^2 \frac{4-4x+x^2}{x^2}\,dx = \left[-\frac2x-2\ln|x|\right]_1^2 + \frac12 = \frac32-2\ln 2$.

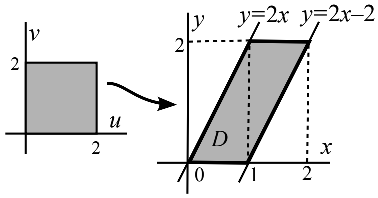




Algo más corto que: $\int_0^1\int_1^{2-y} \frac{y}{x^2}\,dx\,dy = \int_0^1 y\left[1-\frac{1}{2-y}\right]dy = \int_0^1 \left[y+1-\frac{2}{2-y}\right]dy = \frac12+1+2\ln 2$.

6. i) Para no hacer 2 integrales, es mejor integrar primero respecto a x :

$$\int_0^2\int_{y/2}^{y/2+1}(2x-y)^3\,dx\,dy=\int_0^2\tfrac18\big[(2x-y)^4\big]_{y/2}^{y/2+1}dy=\int_0^2 2\,dy=\boxed{4}\,.$$



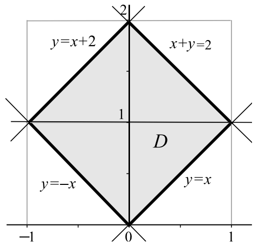
Más largo: $\int_0^1\int_0^{2x}(2x-y)^3dy\,dx+\int_1^2\int_{2x-2}^{2}(2x-y)^3dy\,dx$

$$=\int_0^1 4x^4dx+\int_1^2\big[4-4(x-1)^4\big]\,dx=\tfrac45+4-\tfrac45=4\,.$$

ii) Con cambio: $\begin{matrix}u=2x-y\\ v=y\end{matrix}$, $\begin{matrix}x=\frac{u+v}{2}\\ y=v\end{matrix}$, $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/2&1/2\\0&1\end{vmatrix}=\frac12$. $\frac12\int_0^2\int_0^2 u^3du\,dv=1\cdot\big[\frac14u^4\big]_0^2=\boxed{4}$.

7. $\displaystyle\int_{-1}^{0}\int_{-x}^{x+2}e^{y-x}\,dy\,dx+\int_0^1\int_x^{2-x}e^{y-x}\,dy\,dx=\int_{-1}^{0}\big[e^2-e^{-2x}\big]\,dx+\int_0^1\big[e^{2-2x}-1\big]dx$

$$=e^2-1+\tfrac12\big[e^{-2x}\big]_{-1}^0-\tfrac12\big[e^{2-2x}\big]_0^1=\boxed{e^2-1}\,.$$

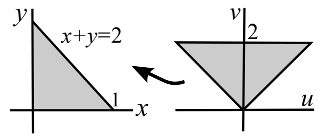


$$\int_0^1\int_{-y}^{y}e^{y-x}\,dx\,dy+\int_1^2\int_{y-2}^{2-y}e^{y-x}\,dx\,dy=\int_0^1\big[e^{2y}-1\big]\,dy+\int_1^2\big[e^2-e^{2y-2}\big]\,dy=\cdots=\boxed{e^2-1}\,.$$

Despejando $\begin{cases}x=\frac12(v-u)\\ y=\frac12(u+v)\end{cases}$, $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-1/2&1/2\\1/2&1/2\end{vmatrix}=-\frac12$, $|J|=\frac12$.

Los lados pasan a ser $u,v=0,2$. La integral se convierte en: $\frac12\int_0^2\int_0^2 e^u\,du\,dv=\frac12\int_0^2\big[e^2-1\big]\,dv=\boxed{e^2-1}$.

8. $\begin{cases}u=y-x\\ v=y+x\end{cases}\Leftrightarrow\begin{cases}x=(v-u)/2\\ y=(u+v)/2\end{cases}$. $J=\begin{vmatrix}-1/2&1/2\\1/2&1/2\end{vmatrix}=-\frac12$. $\begin{matrix}x=0\ \to\ u=v\\ y=0\ \to\ u=-v\\ x+y=2\ \to\ v=2\end{matrix}$



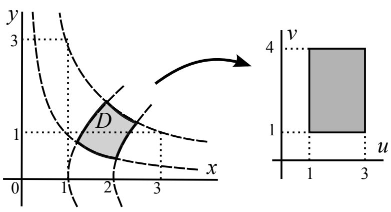
Luego, $\displaystyle\iint_D e^{(y-x)/(y+x)}\,dx\,dy=\frac12\int_0^2\int_{-v}^{v}e^{u/v}du\,dv=\int_0^2 v(e-\tfrac1e)\,dv=e-\tfrac1e\,.$

9. $u=xy$, $v=x^2-y^2\ \to\ \frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}y&x\\2x&-2y\end{vmatrix}=-2(x^2+y^2)\ \to$

$$\iint_D (x^2+y^2)\,dx\,dy=\int_1^3\int_1^4\tfrac12\,dv\,du=3\,.$$

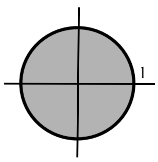
↘ [Casualidad que coincida casi con el jacobiano].

[Despejar x,y en función de u,v es complicado].



10. a) $\displaystyle\int_{-1}^1 x^3\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\,dx=2\int_{-1}^1 \underset{\text{impar}}{x^3\sqrt{1-x^2}}\,dx=0\,.\ \int_0^1 r^4\,dr\int_0^{2\pi}\cos^3\theta\,d\theta=\frac15\int_0^{2\pi}\cos\theta(1-\text{sen}^2\theta)\,d\theta=0\,.$

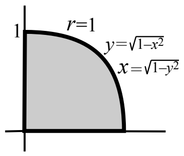
[integral de función impar en recinto simétrico]



b) $\displaystyle\int_0^1 r^5dr\int_0^{2\pi}\cos^4\theta\,d\theta=\frac1{24}\int_0^{2\pi}\Big(1+2\cos2\theta+\tfrac12[1+\cos4\theta]\Big)\,d\theta=\frac1{24}\cdot2\pi\cdot\frac32=\frac\pi8\,.$

$$\int_{-1}^1 x^4\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\,dx=4\int_0^1 x^4\sqrt{1-x^2}\,dx=4\int_0^{\pi/2}\text{sen}^4t\cos^2t\,dt=\cdots=\frac\pi8\,.$$

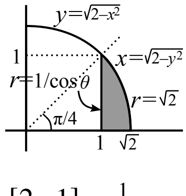
11. a) i) Cartesianas: $\displaystyle\iint_D f=\int_0^1\int_0^{\sqrt{1-x^2}}x^2y\,dy\,dx=\frac12\int_0^1x^2(1-x^2)\,dx=\frac12\Big[\frac13-\frac15\Big]=\frac1{15}\,.$



Más largo: $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}}x^2y\,dx\,dy=\frac13\int_0^1 y\,(1-y^2)^{3/2}\,dy=-\frac1{15}\,(1-y^2)^{5/2}\Big]_0^1=\frac1{15}\,.$

ii) En polares: $\displaystyle\iint_D f=\int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\ \text{sen}\,\theta\,dr\,d\theta=\big[\tfrac15r^5\big]_0^1\big[-\tfrac13\cos^3\theta\big]_0^{\pi/2}=\frac15\cdot\frac13=\frac1{15}\,.$

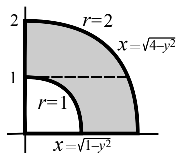
b) i) $\displaystyle\int_1^{\sqrt2}\int_0^{\sqrt{2-x^2}}x\,dy\,dx=\int_1^{\sqrt2}x\sqrt{2-x^2}\,dx=-\frac13(2-x^2)^{3/2}\Big]_1^{\sqrt2}=\frac13\,,$ o bien



$$\int_0^1\int_1^{\sqrt{2-y^2}}x\,dx\,dy=\int_0^1\Big[\frac{x^2}2\Big]_1^{\sqrt{2-y^2}}dy=\int_0^1\frac{1-y^2}{2}\,dy=\frac12-\frac16=\frac13\,.$$

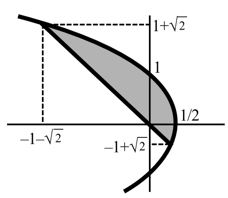
ii) $\displaystyle\int_0^{\pi/4}\int_{1/\cos\theta}^{\sqrt2}r^2\cos\theta\,dr\,d\theta=\frac13\int_0^{\pi/4}\Big[2\sqrt2\,\cos\theta-\frac1{\cos^2\theta}\Big]d\theta=\frac13\big[2\sqrt2\ \text{sen}\,\theta-\tan\theta\big]_0^{\pi/4}=\frac13[2-1]=\frac13\,.$

c) ii) En polares: $\displaystyle\iint_D f\,dx\,dy=\int_0^{\pi/2}\int_1^2 r\frac{r\cos\theta}{r}\,dr\,d\theta=\Big[\frac{r^2}2\Big]_1^2\big[\text{sen}\,\theta\big]_0^{\pi/2}=\frac32\,.$



i) $\displaystyle\int f\,dx=\sqrt{x^2+y^2}\,,\ \int_0^1\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}}f\,dx\,dy+\int_1^2\int_0^{\sqrt{4-y^2}}f\,dx\,dy=\int_0^1 1\,dy+\int_1^2\big[2-y\big]\,dy=3-\frac{4-1}2=\frac32\,.$

d) $r+r\cos\theta=1$, $x^2+y^2=(1-x)^2$, $x=\frac{1-y^2}2$ que corta $y=-x$ si $y=-1\pm\sqrt2\ \to$

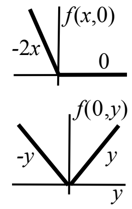


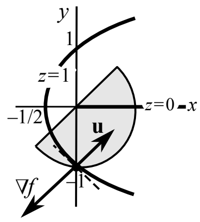
i) $\displaystyle A=\frac12\int_{1-\sqrt2}^{1+\sqrt2}\big(1-y^2+2y\big)\,dy=\sqrt2+\frac12\Big[y^2-\frac13y^3\Big]_{1-\sqrt2}^{1+\sqrt2}=\frac43\sqrt2\,.$

$\tan\frac{3\pi}8=1+\sqrt2$

ii) $\displaystyle\int_{-\pi/4}^{3\pi/4}\int_0^{1/(1+\cos\theta)}r\,dr\,d\theta=\frac12\int_{-\pi/4}^{3\pi/4}\frac{d\theta}{(1+\cos\theta)^2}=\Big[u=\tan\frac\theta2\ \cdots\Big]=\frac14\Big[\tan\frac\theta2+\frac13\tan^3\frac\theta2\Big]_{-\pi/4}^{3\pi/4}\overset{\downarrow}{=}\frac43\sqrt2\,.$

12. $\boxed{f(x,y)=\sqrt{x^2+y^2}-x}$ a) $f(x,0)=|x|-x$, $f(0,y)=|y|$ no derivables $\Rightarrow \nexists$ **parciales** en $(0,0)$ $\Rightarrow$ f **no es diferenciable**.





[Que f es continua en $\mathbf{R}^2$ es obvio, pues lo es la raíz para valores positivos].

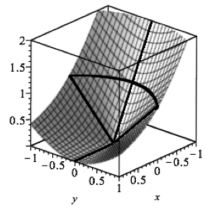
b) $f=1 \Rightarrow x^2+y^2=x^2+2x+1$, $x=\frac{1}{2}[y^2-1]$ $\big[$parábola con $x'(-1)=-1\big]$.

$$\nabla f=\Big(\frac{x}{\sqrt{x^2+y^2}}-1\,,\ \frac{y}{\sqrt{x^2+y^2}}\Big)\,,\ \nabla f(0,-1)=(-1,-1)\ \Rightarrow\ \overline{u}=\Big(\frac{1}{\sqrt{2}}\,,\ \frac{1}{\sqrt{2}}\Big)\,,$$

pues la derivada direccional el mínima en sentido opuesto al gradiente.

$\big[$En polares: $\nabla f=f_r\,\mathbf{e_r}+\frac{1}{r}f_\theta\,\mathbf{e}_\theta=(\cos\theta-1,\operatorname{sen}\theta)\big]$.

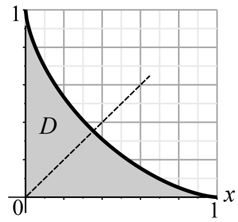
[También se deduce del hecho de que el gradiente es perpendicular a la curva de nivel en el punto y de que apunta hacia donde crece el campo].



dibujo de f con Maple

c) $\displaystyle\int_{-3\pi/4}^{\pi/4}\int_0^1 r^2(1-\cos\theta)\,dr\,d\theta=\frac{1}{3}\big(\pi-[\operatorname{sen}\theta]_{-3\pi/4}^{\pi/4}\big)=\boxed{\frac{\pi-\sqrt{2}}{3}}$. $\big[$En cartesianas las integrales son bastante complicadas$\big]$.

13. $\boxed{x^{2/3}+y^{2/3}\leq 1}$ Como $x^{2/3}+y^{2/3}=1$ es simétrica, tiene $y=\big[1-x^{2/3}\big]^{3/2}$ pendiente 0 en $(1,0)$, corta $y=x$ si $x=\sqrt{2}/4$, . . . la región D es la del dibujo (este con Maple):



Jacobiano: $\displaystyle\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}c^3 & -3rc^2s\\ s^3 & 3rs^2c\end{vmatrix}=3r\operatorname{sen}^2\theta\cos^2\theta\big(\operatorname{sen}^2\theta+\cos^2\theta\big)=3r\operatorname{sen}^2\theta\cos^2\theta$.

Veamos que el conjunto $D^*=[0,1]\times\big[0,\frac{\pi}{2}\big]$ se transforma en nuestro recinto D :

$\theta=0\rightarrow y=0$, $\theta=\frac{\pi}{2}\rightarrow x=0$; $x^{2/3}+y^{2/3}=r^{2/3}\big(\operatorname{sen}^2\theta+\cos^2\theta\big)=r^{2/3}\leq 1$, $r\leq 1$.

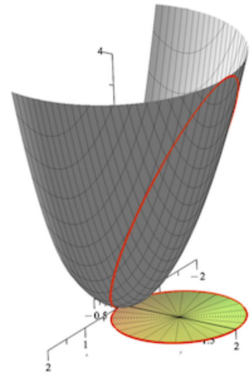
$\big[$Y en D^* es inyectivo, salvo en $(0,0)$, como las polares: $\theta=K$ constante lleva a distintas rectas $y=(\tan^3K)x\big]$.

$$\text{Área}=\iint_D 1=\frac{3}{4}\int_0^{\pi/2}\!\!\int_0^1 r\operatorname{sen}^2 2\theta\,dr\,d\theta=\frac{3}{16}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta=\frac{3\pi}{32}-0=\boxed{\frac{3\pi}{32}}\,.$$

14. z positiva en el rectángulo: $\displaystyle V=\int_0^1\!\!\int_1^2(x^2+y)\,dy\,dx=\int_0^1 x^2\,dx+\int_1^2 y\,dy=\frac{1}{3}+\frac{4-1}{2}=\boxed{\frac{11}{6}}$.

15. Se puede describir el círculo $x^2+(y-1)^2\leq 1$ con polares centradas en $(0,1)$:

$x=\rho\cos\phi$, $y=1+\rho\operatorname{sen}\phi$, con $\rho\leq 1$ y $\phi\in[0,2\pi]$. [sigue siendo $J=\rho$].



Con ellas se tiene $z=\rho^2\cos^2\phi+\rho^2\operatorname{sen}^2\phi+2\rho\operatorname{sen}\phi+1$, y el volumen es:

$$V=\int_0^{2\pi}\!\!\int_0^1\big(\rho^3+\rho+2\rho^2\operatorname{sen}\phi\big)\,d\rho\,d\phi=\int_0^{2\pi}\big(\tfrac{3}{4}+\tfrac{2}{3}\operatorname{sen}\phi\big)\,d\phi=\boxed{\frac{3\pi}{2}}\,.$$

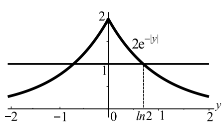
No sale mal en polares usuales: $r^2=2r\operatorname{sen}\theta$, $r=2\operatorname{sen}\theta$, $\theta\in[0,\pi]$. Y es $z=r^2$.

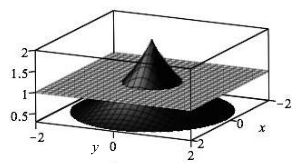
Así pues: $\displaystyle V=\int_0^{\pi}\!\!\int_0^{2\operatorname{sen}\theta}r^3\,dr\,d\theta=\int_0^{\pi}4\operatorname{sen}^4\theta\,d\theta=\int_0^{\pi}(1-\cos 2\theta)^2 d\theta=\boxed{\frac{3\pi}{2}}$.

En ambos órdenes de integración en cartesianas se complica. Por ejemplo:

$$\int_{-1}^{1}\!\!\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}(x^2+y^2)\,dz\,dy\,dx=2\int_0^1\Big[2x^2\sqrt{1-x^2}+\frac{(1+\sqrt{\cdot})^3-(1-\sqrt{\cdot})^3}{3}\Big]dx=\frac{8}{3}\int_0^1(x^2+2)\sqrt{1-x^2}\,dx=\cdots$$

16. $\boxed{g(x,y)=2\,e^{-\sqrt{x^2+y^2}}}$ a) De revolución. $g(0,y)=2e^{-|y|}\Rightarrow g_y(0,0)$ no existe $\Rightarrow g$ **no diferenciable** en $(0,0)$.

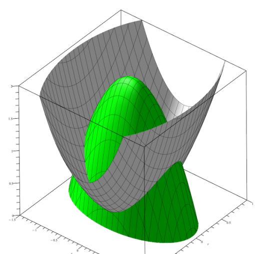




b) $2\,e^{-\sqrt{x^2+y^2}}=1\ \Leftrightarrow\ r=\sqrt{x^2+y^2}=\log 2$. Polares-cilíndricas:

$$V=\int_0^{2\pi}\!\!\int_0^{\ln 2}r\big[2\,e^{-r}-1\big]\,dr\,d\theta=2\pi\big[-2(r+1)\,e^{-r}-\tfrac{1}{2}r^2\big]_0^{\ln 2}=2\pi\big[1-\ln 2-\tfrac{1}{2}(\ln 2)^2\big]\,.$$

17. $\boxed{z=x^2+y^2}$ y $\boxed{z=2-x^2-7y^2}$ se cortan sobre la elipse $x^2+4y^2=1$, $y=\pm\frac{1}{2}\sqrt{1-x^2}$.



El volumen vendrá dado por $\displaystyle V=8\int_0^1\!\!\int_0^{\sqrt{1-x^2}/2}\big(1-x^2-4y^2\big)dy\,dx=\frac{8}{3}\int_0^1\big(1-x^2\big)^{2/3}dx$

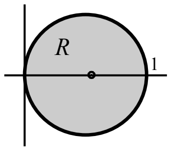
$$\overset{x=\operatorname{sen}t}{=}\ \frac{8}{3}\int_0^{\pi/2}\cos^4 t\,dt=\frac{8}{3}\int_0^{\pi/2}\Big(1+2\cos 2t+\frac{1+\cos 4t}{2}\Big)\,dt=\frac{\pi}{2}\,.$$

O con cambio de las elipses: $x=r\cos\theta$, $y=\frac{r}{2}\operatorname{sen}\theta$, $J=\frac{r}{2}\ \rightarrow\ V=\displaystyle\int_0^{2\pi}\!\!\int_0^1\big(r-r^3\big)dr\,d\theta=\frac{\pi}{2}$.

18. $M = 2\int_0^{\pi/2}\int_0^{\cos\theta} r\cos\theta\, dr\, d\theta = \int_0^{\pi/2}(1-\text{sen}^2\theta)\cos\theta\, d\theta = \frac{2}{3}$.

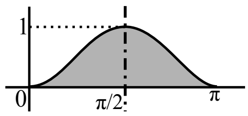
$\bar{x} = \frac{1}{M}\, 2\int_0^{\pi/2}\int_0^{\cos\theta} r^2\cos^2\theta\, dr\, d\theta = \int_0^{\pi/2}\left(1-\text{sen}^2\theta\right)^2\cos\theta\, d\theta = \frac{8}{15}$.

$\bar{y} = 0$ por simetría de R y de la función densidad.



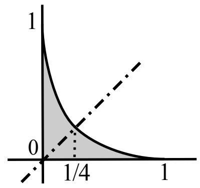
19. a) $\{0 \le y \le \text{sen}^2 x,\ 0 \le x \le \pi\}$. Por simetría, $\bar{x} = \frac{\pi}{2}$.

$$\bar{y} = \frac{1}{\int_0^{\pi}\text{sen}^2 x\, dx}\int_0^{\pi}\int_0^{\text{sen}^2 x} y\, dy\, dx = \frac{2}{\pi}\int_0^{\pi}\frac{\text{sen}^4 x}{2}dx = \frac{3}{8}.$$



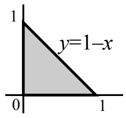
b) $\{\sqrt{x}+\sqrt{y} \le 1,\ x \le 0,\ y \ge 0\}$. $y = (1-\sqrt{x})^2 = 1+x-2\sqrt{x}$.

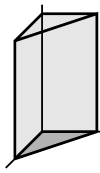
$$\bar{x} = \frac{1}{\int_0^1 (1+x-2\sqrt{x})\, dx}\int_0^1\int_0^{1+x-2\sqrt{x}} x\, dy\, dx = 6\int_0^1\left(x+x^2-2x^{3/2}\right)dx = \frac{1}{5} = \bar{y}, \text{ por simetría.}$$



20. a) $\iiint_V (2x+3y+z)\, dx\, dy\, dz = 2\int_1^2 2x\, dx + \int_{-1}^1 3y\, dy + 2\int_0^1 z\, dz = 2(4-1)+0+1 = 7$.

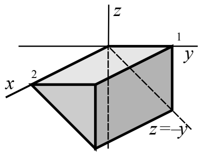
b) $\iiint_V x^2\cos z\, dx\, dy\, dz = \int_0^1\int_0^{1-x}\int_0^{\pi} x^2\cos z\, dz\, dy\, dx = \int_0^1\int_0^{1-x} 0\, dy\, dx = 0$.





c) $\int_0^2\int_0^1\int_{-y}^0 e^y\, dz\, dy\, dx = \int_0^2\int_0^1 y\, e^y\, dy\, dx = 2\int_0^1 y\, e^y\, dy = 2\big[(y-1)\, e^y\big]_0^1 = 2$.

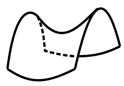
O bien: $\int_0^2\int_{-1}^0\int_{-z}^1 e^y\, dy\, dz\, dx = 2\int_{-1}^0\left[e-e^{-z}\right]dz = 2\,e + 2\left[e^{-z}\right]_{-1}^0 = 2$.

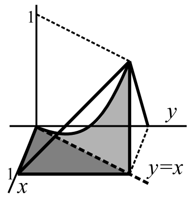


d) $\iiint_V xy^2z^3\, dx\, dy\, dz = \int_0^1\int_0^x\int_0^{xy} xy^2z^3\, dz\, dy\, dx = \frac{1}{4}\int_0^1\int_0^x x^5y^6\, dy\, dx = \frac{1}{28}\int_0^1 x^{12}dx$

$= \frac{1}{364}$.

$[\,z=xy$ era un 'paraboloide hiperbólico' (silla de montar)$\,]$.



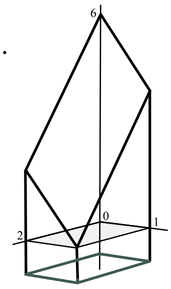


21. El plano tangente a $x^2+y^2+z=4$ es $\nabla F(1,1,2)\cdot(x-1,y-1,z-2) = 2(x-1)+2(y-1)+(z-2) = 0$.

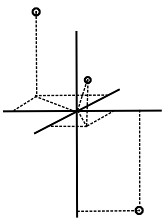
Es decir, $z = 6-2x-2y$.

La integral es $\iiint_V xy\, dx\, dy\, dz = \int_0^2\int_0^1\int_{-1}^{6-2x-2y} xy\, dz\, dy\, dx = \int_0^2\int_0^1 xy(7-2x-2y)\, dy\, dx$

$= \int_0^2\left[\frac{7}{2}xy^2 - x^2y^2 - \frac{2}{3}xy^3\right]_0^1 dx = \int_0^2\left(\frac{17}{6}x - x^2\right)dx = 3$.



22.



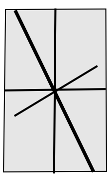
	cartesianas	polares	esféricas
P	$(0,2,-4)$	$(2,\frac{\pi}{2},-4)$	$(2\sqrt{5},\frac{\pi}{2},\pi-\arctan\frac{1}{2})$
Q	$(-2,-2\sqrt{3},3)$	$(4,\frac{4\pi}{3},3)$	$(5,\frac{4\pi}{3},\arctan\frac{4}{3})$
R	$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1)$	$(1,\frac{\pi}{4},1)$	$(\sqrt{2},\frac{\pi}{4},\frac{\pi}{4})$

23. $\{x=0,\ z=-2y\}$

$\{\theta=\frac{\pi}{2},\ z=-2r$ ó $\theta=-\frac{\pi}{2},\ z=2r\}$

$\{\theta=\frac{\pi}{2},\ \phi=\frac{5\pi}{6}$ ó $\theta=-\frac{\pi}{2},\ \phi=\frac{\pi}{6}\}$

recta por el origen

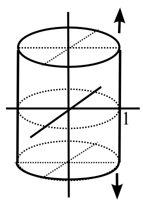


$\{x^2+y^2=1\}$

$\{r=1\}$

$\{\rho\,\text{sen}\,\theta=1\}$

cilindro vertical

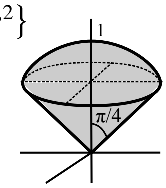


$\{x^2+y^2 \le z^2 \le 1-x^2-y^2\}$

$\{r \le z \le \sqrt{1-r^2}\}$

$\{\phi \le \frac{\pi}{4},\ \rho \le 1\}$

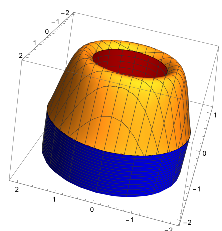
volumen entre cono y esfera



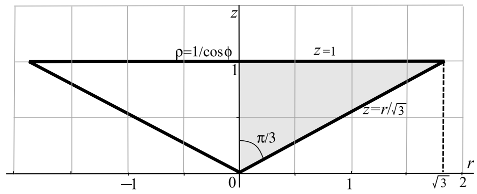
24. $\boxed{\iiint_V \frac{y^2}{x^2+y^2}\, dx\, dy\, dz}$ En cilíndricas, la función es $f(r,\theta,z)=\text{sen}^2\theta$, es $J=r$ y los límites de la región son $1\le\rho\le2$, $0\le\theta\le2\pi$, $-2\le z\le\text{sen}\, r^2$,

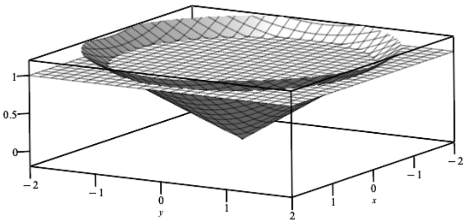
La integral es: $\iiint_V f = \int_1^2\int_0^{2\pi}\int_{-2}^{\text{sen}\, r^2} r\,\text{sen}^2\theta\, dz\, d\theta\, dr$

$= \left[\int_1^2\left(2r+r\,\text{sen}\, r^2\right)dr\right]\left[\int_0^{2\pi}\text{sen}^2\theta\, d\theta\right] = \frac{\pi}{2}\,(6-\cos 4+\cos 1)$.



25. En cilíndricas la superficie cónica $3z^2=x^2+y^2$ pasa a ser $z=\pm\frac{r}{\sqrt{3}}$.
Y esta superficie corta el plano $z=1$ cuando $r=\sqrt{3}$.
V es el cono generado al rotar el triángulo respecto al eje z.





La integral en cilíndricas se puede hacer de dos formas sencillas:

$$\iiint_V z=\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r/\sqrt{3}}^{1}\overset{J\downarrow}{r}\,z\,dz\,dr\,d\theta=\pi\int_0^{\sqrt{3}}\big[r-\tfrac{1}{3}r^3\big]dr=\pi\big[\tfrac{3}{2}-\tfrac{3}{4}\big]=\boxed{\tfrac{3\pi}{4}}.$$

$$\iiint_V z=\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{3}z} r\,z\,dr\,dz\,d\theta=\pi\int_0^1 3z^3\,dz=\boxed{\tfrac{3\pi}{4}}.$$

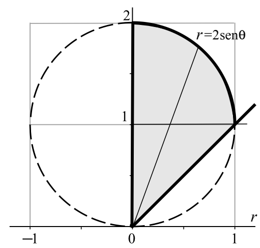
En esféricas, el plano se complica, pasando a ser $\rho=1/\cos\phi$.
Aunque el ángulo ϕ varía simplemente de 0 a $\frac{\pi}{3}$ $\big[\tan\frac{\pi}{3}=\sqrt{3}\big]$.

$$\iiint_V z=\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1/\cos\phi}\rho^3\,\mathrm{sen}\,\phi\cos\phi\,d\rho\,d\phi\,d\theta=\frac{2\pi}{4}\int_0^{\pi/3}\frac{\mathrm{sen}\,\phi}{\cos^3\phi}\,d\phi$$
$$=\tfrac{\pi}{4}\big[\cos^{-2}\phi\big]_0^{\pi/3}=\tfrac{\pi}{4}\big[\tfrac{1}{1/4}-\tfrac{1}{1}\big]=\boxed{\tfrac{3\pi}{4}}.$$

En cartesianas es calculable, pero claramente más largo (la base D es el círculo $x^2+y^2=3$):

$$\iiint_V z\,dz\,dy\,dx=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{\sqrt{x^2+y^2}/3}^{1}z\,dz\,dy\,dx=\tfrac{2}{3}\int_0^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\big[3-x^2-y^2\big]\,dy\,dx=\tfrac{4}{9}\int_0^{\sqrt{3}}(3-x^2)^{3/2}\,dx$$
$$\underset{x=\sqrt{3}\,\mathrm{sen}\,t\uparrow}{=}4\int_0^{\pi/2}\cos^4 t\,dt=\int_0^{\pi/2}\big(1+2\cos 2t+\tfrac{1+\cos 4t}{2}\big)\,dt=\boxed{\tfrac{3\pi}{4}}.$$

26. a) $y^2-2y+x^2=0\ \rightarrow\ y=1+\sqrt{1-x^2}$, $x=\sqrt{2y-y^2}$.

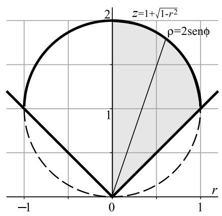


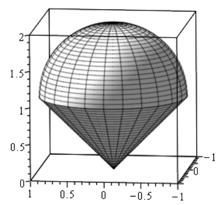
i) $\displaystyle\int_0^1\int_x^{1+\sqrt{1-x^2}}x\,dy\,dx=\int_0^1\big[x-x^2+x(2-x^2)^{1/2}\big]dx=\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{3}(1-x^2)^{3/2}\Big]_0^1=\boxed{\tfrac{1}{2}}.$

$\displaystyle\int_0^1\int_0^y x\,dx\,dy+\int_1^2\int_0^{\sqrt{2y-y^2}}x\,dx\,dy=\tfrac{1}{2}\int_0^1 y^2dy+\tfrac{1}{2}\int_1^2(2y-y^2)\,dy=\tfrac{1}{6}+\tfrac{1}{2}\big[y^2-\tfrac{y^3}{3}\big]_1^2=\boxed{\tfrac{1}{2}}.$

ii) $r^2=2r\,\mathrm{sen}\,\theta$. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^{2\,\mathrm{sen}\,\theta}r^2\cos\theta\,dr\,d\theta=\tfrac{8}{3}\int_{\pi/4}^{\pi/2}\mathrm{sen}^3\theta\cos\theta\,d\theta=\tfrac{2}{3}\,\mathrm{sen}^4\theta\Big]_{\pi/4}^{\pi/2}=\boxed{\tfrac{1}{2}}.$

b) El cono $x^2+y^2=z^2$ y la esfera $x^2+y^2=2z-z^2$ se cortan si $z=0\,,1$.
V es el sólido de revolución del primer dibujo respecto al eje z.





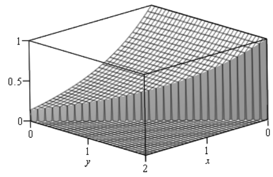
i) Cilíndricas. La esfera es $z^2-2z-r^2=0$, $z=1+\sqrt{1-r^2}$ y el cono $z=r$.

$$\iiint_V 1=\int_0^{2\pi}\int_0^1\int_r^{1+\sqrt{1-r^2}}r\,dz\,dr\,d\theta=2\pi\int_0^1\big[r-r^2+r(1-r^2)^{1/2}\big]dr$$
$$=2\pi\big[\tfrac{1}{2}r^2-\tfrac{1}{3}r^3-\tfrac{1}{3}(1-r^2)^{3/2}\big]_0^1=\boxed{\pi}.$$ [Con el orden $dr\,dz$ aparecen dos integrales].

ii) En esféricas, la esfera toma la forma $\rho^2=2\rho\cos\phi$, $\rho=2\cos\phi$.

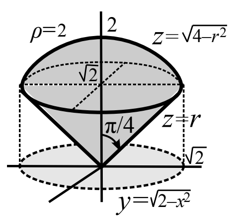
$$\iiint_V 1=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi}\rho^2\,\mathrm{sen}\,\phi\,d\rho\,d\phi\,d\theta=\tfrac{2\pi}{3}\int_0^{\pi/4}8\cos^3\phi\,\mathrm{sen}\,\phi\,d\phi=\tfrac{4\pi}{3}\big[-\cos^4\phi\big]_{\pi/4}^{\pi/2}=\tfrac{4\pi}{3}\big(1-\tfrac{1}{4}\big)=\boxed{\pi}.$$

27. a) $\displaystyle\int_0^2\int_0^{2-x}\int_0^{e^{-x}}z\,dz\,dy\,dx=\int_0^2\int_0^{2-x}\tfrac{1}{2}e^{-2x}dy\,dx=\int_0^2\big(1-\tfrac{x}{2}\big)e^{-2x}dx=$ (partes)
$\displaystyle=\big(\tfrac{x}{4}-\tfrac{1}{2}\big)e^{-2x}\Big]_0^2-\tfrac{1}{4}\int_0^2 e^{-2x}dx=\tfrac{1}{2}+\tfrac{1}{8}\big[e^{-2x}\big]_0^2=\boxed{\tfrac{1}{8}(3+e^{-4})}.$



[Pedía cartesianas. Algo más corto era haciendo $\int_0^2\int_0^{2-y}\int_0^{e^{-x}}z\,dz\,dx\,dy$].

b) El cono $z=\sqrt{x^2+y^2}$ y la esfera $x^2+y^2+z^2=4$ piden hacer la integral en esféricas

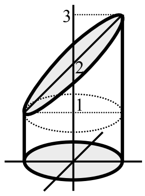


$$\iiint_V z=2\pi\int_0^{\pi/4}\int_0^2\rho^3\,\mathrm{sen}\,\phi\cos\phi\,d\rho\,d\phi=\pi\big[\tfrac{1}{4}\rho^4\big]_0^2\big[\mathrm{sen}^2\phi\big]_0^{\pi/4}=\boxed{2\pi}.$$

En cilíndricas: $\displaystyle 2\pi\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}}rz\,dz\,dr=2\pi\int_0^{\sqrt{2}}r\big(2-r^2\big)\,dr=2\pi\big[r^2-\tfrac{r^4}{4}\big]_0^{\sqrt{2}}=2\pi.$

En cartesianas: $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}}z\,dz\,dy\,dx=\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\big(2-x^2-y^2\big)\,dy\,dx=\cdots$

28. a) $V=\iiint_V 1=\int_0^{2\pi}\int_0^1\int_0^{r\,\mathrm{sen}\,\theta+2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 (r^2\,\mathrm{sen}\,\theta+2r)\,dr\,d\theta=\int_0^{2\pi}\left[\frac{\mathrm{sen}\,\theta}{3}+1\right]d\theta=2\pi$.

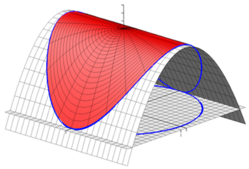


Peor en cartesianas: $V=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{y+2} dz\,dy\,dx=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(y+2)\,dy\,dx=4\int_{-1}^1\sqrt{1-x^2}\,dx$

$=[\text{par, } x=\mathrm{sen}\,t\,]=8\int_0^{\pi/2}\cos^2 t\,dt=4\int_0^{\pi/2}(1+\cos 2t)\,dt=2\pi$.

[Sin integrar: volumen de cilindro de altura 1 más medio volumen de cilindro de altura 2].

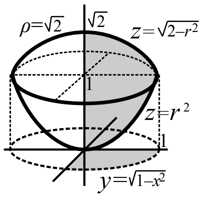
b) Sobre $x^2+y^2\le 1$ la gráfica de $z=1-x^2$ está por encima del plano $z=0$:



$V=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2} dz\,dy\,dx=2\int_{-1}^1(1-x^2)^{3/2}dx\underset{x=\mathrm{sen}\,t}{=}4\int_0^{\pi/2}\cos^4 t\,dt=\frac{3\pi}{4}$.

Mejor cilíndricas: $V=\int_0^{2\pi}\int_0^1\int_0^{1-r^2\cos^2\theta} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\big(r-r^3\cos^2\theta\big)dr\,d\theta=\int_0^{2\pi}\frac{3-\cos 2\theta}{8}\,d\theta=\frac{3\pi}{4}$.

c) Cilíndricas: $V=2\pi\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr=2\pi\int_0^1\big(r\sqrt{2-r^2}-r^3\big)\,dr=\frac{\pi}{6}\big(8\sqrt{2}-7\big)$.



O bien: $V=\int_0^{2\pi}\int_0^1\int_0^{\sqrt{z}} r\,dr\,dz\,d\theta+\int_0^{2\pi}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-z^2}} r\,dr\,dz\,d\theta$

$=\pi\int_0^1 z\,dz+\pi\int_1^{\sqrt{2}}(2-z^2)\,dz=\frac{\pi}{2}+2\pi\big(\sqrt{2}-1\big)-\frac{\pi}{3}\big(2\sqrt{2}-1\big)=\frac{\pi}{6}\big(8\sqrt{2}-7\big)$.

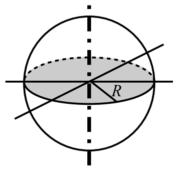
Esféricas: $V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}}\rho^2\,\mathrm{sen}\,\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cos\phi/\mathrm{sen}^2\phi}\rho^2\,\mathrm{sen}\,\phi\,d\rho\,d\phi\,d\theta$ [de $\rho\cos\phi=\rho^2\,\mathrm{sen}^2\phi$]

$=2\pi\,\frac{2\sqrt{2}}{3}\big[-\cos\phi\big]_0^{\pi/4}+\frac{2\pi}{3}\int_{\pi/4}^{\pi/2}\frac{\cos^3\phi}{\mathrm{sen}^5\phi}\,d\phi=\frac{4\pi}{3}(\sqrt{2}-1)+\frac{2\pi}{3}\int_{1/\sqrt{2}}^1\frac{1-t^2}{t^5}\,dt=\frac{\pi}{6}\big(8\sqrt{2}-7\big)$.

En cartesianas salen difíciles integrales: $V=4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx=\cdots$

29. $\iiint_V f\,dx\,dy\,dz=\int_0^{2\pi}\int_0^{\pi}\int_1^2\frac{\rho^2\,\mathrm{sen}\,\phi}{\rho^3}d\rho\,d\phi\,d\theta=2\pi\big[\log\rho\big]_1^2\int_0^{\pi}\mathrm{sen}\,\phi\,d\phi=\boxed{4\pi\log 2}$.

30. $M=\frac{4}{3}\pi R^3\sigma$, con σ densidad constante.



$I_z=\iiint_V(x^2+y^2)\sigma\,dx\,dy\,dz=\sigma\int_0^{2\pi}\int_0^{\pi}\int_0^R\rho^4\,\mathrm{sen}^3\phi\,d\rho\,d\phi\,d\theta=\frac{2\pi R^5\sigma}{5}\int_0^{\pi}(1-c^2)s\,d\phi=\frac{8\pi R^5\sigma}{15}=\frac{2MR^2}{5}$.