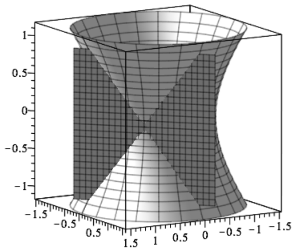
**1.** $\boxed{\mathbf{r}(u,v)=(2u,\,u^2+v,\,v^2)}$ . a) $\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2u&0\\0&1&2v\end{vmatrix}=2(2uv,-2v,1)\xrightarrow{u=0,\,v=1}2(0,-2,1)$ .

Plano tangente: $(0,-2,1)\cdot(x,\,y-1,\,z-1)=0\ \to\ z=2y-1$ .

b) $u=\frac{x}{2}$ , $v=y-\frac{x^2}{4}$ , $z=\left(y-\frac{x^2}{4}\right)^2$ . $z_x(0,1)=0$ , $z_y(0,1)=2$ , $^{\nwarrow}\, z=1+0(x-1)+2(x-1)$ .

**2.** $\boxed{\mathbf{r}(u,v)=(\mathrm{ch}\,u\cos v,\,\mathrm{ch}\,u\,\mathrm{sen}\,v,\,\mathrm{sh}\,u)}$ $\begin{matrix}u\in\mathbf{R}\\ v\in[0,2\pi]\end{matrix}$ $\quad x^2+y^2-z^2=\mathrm{ch}^2u-\mathrm{sh}^2u=1$ . $\quad x^2+y^2=\mathrm{ch}^2u$ curvas de nivel.

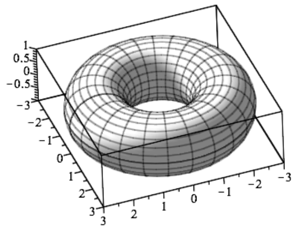



$$\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \mathrm{sh}\,u\cos v&\mathrm{sh}\,u\,\mathrm{sen}\,v&\mathrm{ch}\,u\\ -\mathrm{ch}\,u\,\mathrm{sen}\,v&\mathrm{ch}\,u\cos v&0\end{vmatrix}=\mathrm{ch}\,u(-\mathrm{ch}\,u\cos v,-\mathrm{ch}\,u\,\mathrm{sen}\,v,\,\mathrm{sh}\,u)\xrightarrow{u=0,\,v=\pi/4}-\left(\tfrac{1}{\sqrt2},\tfrac{1}{\sqrt2},0\right).$$

$\mathbf{r}(0,\frac{\pi}{4})=\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)$ . Plano tangente: $-\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)\cdot\left(x-\frac{1}{\sqrt2},\,y-\frac{1}{\sqrt2},\,z\right)=0$ , $x+y=\sqrt2$ .

Partiendo de $F(x,y,z)=x^2+y^2-z^2=1$ . $\nabla F=(2x,2y,-2z)\big|_{(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0)}=(\sqrt2,\sqrt2,0)^{\uparrow}$

**3.** $\boxed{\mathbf{r}(\theta,\phi)=\big((2+\cos\phi)\cos\theta,\,(2+\cos\phi)\,\mathrm{sen}\,\theta,\,\mathrm{sen}\,\phi\big)}$ , $\theta,\phi\in[0,2\pi]$ .




$$\mathbf{r}_\theta\times\mathbf{r}_\phi=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-(2+\cos\phi)\,\mathrm{sen}\,\theta&(2+\cos\phi)\cos\theta&0\\-\mathrm{sen}\,\phi\cos\theta&-\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta&\cos\phi\end{vmatrix}=(2+\cos\phi)\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-\mathrm{sen}\,\theta&\cos\theta&0\\-\mathrm{sen}\,\phi\cos\theta&-\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta&\cos\phi\end{vmatrix},$$

$=(2+\cos\phi)(\cos\theta\cos\phi,\,\mathrm{sen}\,\theta\cos\phi,\,\mathrm{sen}\,\phi)$ . $\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|=2+\cos\phi$ .

Área $=\iint_S dS=\iint_D\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|\,d\theta\,d\phi=\int_0^{2\pi}\!\!\int_0^{2\pi}(2+\cos\phi)\,d\theta\,d\phi=8\pi^2$ .

**4.** $\boxed{\iint_S(x^2+y^2)\,dS}$ , $x^2+y^2+z^2=4\ \to\ \mathbf{r}(u,v)=(2\,\mathrm{sen}\,u\cos v,\,2\,\mathrm{sen}\,u\,\mathrm{sen}\,v,\,2\cos u)$ , $\|\mathbf{r}_u\times\mathbf{r}_v\|=4\,\mathrm{sen}\,u$

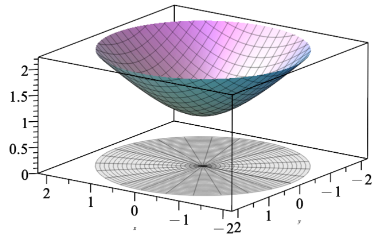
$$\iint_S(x^2+y^2)\,dS=16\int_0^{2\pi}\!\!\int_0^{\pi}\mathrm{sen}^3u\,du\,dv=32\pi\int_0^{\pi}\mathrm{sen}\,u(1-\cos^2u)\,du=32\pi\left[\tfrac13\cos^3u-\cos u\right]_0^{\pi}=\tfrac{128}{3}\pi.$$

Con $\mathbf{r}(x,y)=\left(x,\,y,\,\sqrt{4-x^2-y^2}\right)$ [integrando y recinto simétricos en $z$, basta esta y multiplicar por 2 la integral].

$$\|\mathbf{r}_x\times\mathbf{r}_y\|=\sqrt{(f_x)^2+(f_y)^2+1}=\left[\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1\right]^{1/2}=\frac{2}{\sqrt{4-x^2-y^2}}\ \Rightarrow$$

$$2\iint_S(x^2+y^2)\,dS=4\iint_B\frac{x^2+y^2}{\sqrt{4-x^2-y^2}}\,dx\,dy\underset{\text{polares}\uparrow}{=}4\int_0^{2\pi}\!\!\int_0^2\frac{r^3}{\sqrt{4-r^2}}\,dr\,d\theta=4\pi\int_0^4\frac{s\,ds}{\sqrt{4-s}}=\frac{128}{3}\pi.$$

**5.** $\boxed{f(x,y,z)=z}$ , $z^2=1+x^2+y^2$ . $\mathbf{r}(r,\theta)=\left(r\cos\theta,\,r\,\mathrm{sen}\,\theta,\,\sqrt{1+r^2}\right)$ .
$\theta\in[0,2\pi]$ , $r\in[0,2]$ (es $5=1+r^2$ si $r=2$).




$$\mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\c&s&r/\sqrt{\cdot}\\-rs&rc&0\end{vmatrix}=\left(-\tfrac{r^2c}{\sqrt{\cdot}},-\tfrac{r^2s}{\sqrt{\cdot}},r\right).\ \ \|\mathbf{r}_r\times\mathbf{r}_\theta\|=\frac{r\sqrt{1+2r^2}}{\sqrt{1+r^2}}.$$

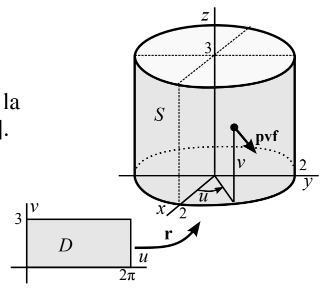
Por tanto: $\iint_S z\,dS=\int_0^{2\pi}\!\!\int_0^2 r\sqrt{1+2r^2}\,dr\,d\theta=\frac{2\pi}{6}\left(1+2r^2\right)^{3/2}\Big]_0^2=\boxed{\frac{26\pi}{3}}$ .

O bien: $\mathbf{r}(x,y)=\left(x,\,y,\,\sqrt{1+x^2+y^2}\right)$ , $(x,y)\in B$ círculo de centro (0,0) y radio 2 .

$\mathbf{r}_x\times\mathbf{r}_y=\left(-\frac{x}{\sqrt{\cdot}},-\frac{y}{\sqrt{\cdot}},1\right)\xrightarrow{\|\ \|}\frac{\sqrt{1+2x^2+2y^2}}{\sqrt{1+x^2+y^2}}$ . $\iint_S z\,dS=\iint_B\sqrt{1+2x^2+2y^2}\,dx\,dy=\cdots$ (usando polares acabamos arriba).

**6.** Parametrización de $x^2+y^2=4$: $\left.\begin{matrix}x=2\cos u\\y=2\,\mathrm{sen}\,u\\z=v\end{matrix}\right\}$ $\begin{matrix}u\in[0,2\pi]\\v\in[0,3]\end{matrix}$ $\quad\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2s&2c&0\\0&0&1\end{vmatrix}=2(\cos u,\,\mathrm{sen}\,u,\,0)$ .

a) área de $S=\iint_S 1\,dS=\iint_D\|\mathbf{r}_u\times\mathbf{r}_v\|\,du\,dv=\int_0^{2\pi}\!\!\int_0^3 2\,dv\,du=12\pi$ [claro, longitud de la base por la altura].

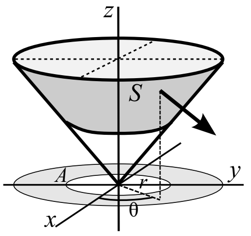



b) i) $\boxed{f(x,y,z)=x^2}$ $\iint_S f\,dS=\int_0^{2\pi}\!\!\int_0^3 8\cos^2u\,dv\,du=12\int_0^{2\pi}(1+\cos2v)\,dv=24\pi$ .

ii) $\boxed{\mathbf{f}(x,y,z)=(xz,\,yz,\,2)}$ $\mathbf{f}\big(\mathbf{r}(u,v)\big)=2(v\cos u,\,v\,\mathrm{sen}\,u,\,1)$ .

$$\iint_S\mathbf{f}\cdot d\mathbf{S}=\iint_D\mathbf{f}\big(\mathbf{r}(u,v)\big)\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv=\int_0^{2\pi}\!\!\int_0^3 4v\,dv\,du=36\pi.$$

**7.** $\boxed{z^2=x^2+y^2}$ . a) $\left.\begin{matrix} x=r\cos\theta \\ y=r\,\text{sen}\,\theta \\ z=r \end{matrix}\right\}$ $\begin{matrix} r\in[1,2] \\ \theta\in[0,2\pi] \end{matrix}$ $\mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & s & 1 \\ -r\,s & r\,c & 0 \end{vmatrix}=(-r\cos\theta,-r\,\text{sen}\,\theta,r)$ . [apunta hacia interior]




$\|\mathbf{r}_r\times\mathbf{r}_\theta\|=\sqrt{2}\,r$ , área $=2\pi\int_1^2\|\cdot\|=3\pi\sqrt{2}$ .

$\iint_S \mathbf{f}\cdot d\mathbf{S}=-\iint_D(r\cos\theta,r\,\text{sen}\,\theta,1)\cdot(-r\cos\theta,-r\,\text{sen}\,\theta,r)\,dr\,d\theta=\left[\,\mathbf{f}(x,y,z)=(x,y,1)\,\right]$

$=\int_0^{2\pi}\int_1^2[r^2-r]\,dr\,d\theta=2\pi\left[\frac{r^3}{3}-\frac{r^2}{2}\right]_1^2=\boxed{\frac{5}{3}\pi}$ . [es $>0$ pues $\mathbf{f}$ apunta también hacia exterior]

De otra forma: $\mathbf{r}(x,y)=\left(x\,,y\,,\sqrt{x^2+y^2}\,\right)$ , $\mathbf{r}_x\times\mathbf{r}_y=\left(-x\,(x^2+y^2)^{-1/2},-y\,(x^2+y^2)^{-1/2},1\right)$ .

$-\iint_A(x,y,1)\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dx\,dy=-\iint_A\left[1-\sqrt{x^2+y^2}\,\right]dx\,dy=\int_0^{2\pi}\int_1^2[r^2-r]\,dr\,d\theta=\cdots$ como antes.

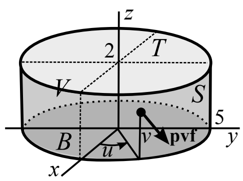
b) $\text{rot}\,\mathbf{f}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 1 \end{vmatrix}=(0,0,0)$ y $\mathbf{f}\in C^1(\mathbf{R}^3)$ $\Rightarrow$ $\mathbf{f}$ es conservativo. [De hecho la $U=xy+z$ se ve a ojo].
Su integral sobre toda línea cerrada es $0$ , sobre esa circunferencia en particular.

**8.** $\boxed{\mathbf{F}(x,y,z)=(x^2+y^2+z^2)(x,y,z)}$ $\text{div}\,\mathbf{F}=2x^2+2y^2+2z^2+3(x^2+y^2+z^2)=5(x^2+y^2+z^2)=5\rho^2$ . $\text{rot}\,\mathbf{F}=\mathbf{0}$ .

$\iiint_V \text{div}\,\mathbf{F}\,dx\,dy\,dz=\int_0^1\int_0^{2\pi}\int_0^\pi 5\rho^4\,\text{sen}\,\phi\,d\phi\,d\theta\,d\rho=2\pi\left[-\cos\phi\right]_0^\pi=4\pi$ .

$\mathbf{r}=(\text{sen}\,\phi\cos\theta,\text{sen}\,\phi\,\text{sen}\,\theta,\cos\phi)$ , $\phi\in[0,\pi]$
$\mathbf{r}_\phi\times\mathbf{r}_\theta=\text{sen}\,\phi\;\mathbf{r}(\phi,\theta)$ , $\theta\in[0,2\pi]$ , $\iint_{\partial V}\underset{=1}{\mathbf{F}\cdot\mathbf{n}}\,dS=\int_0^{2\pi}\int_0^\pi\text{sen}\,\phi\,d\phi\,d\theta=4\pi$ (= superficie de S).

**9.** $\boxed{\mathbf{f}(x,y,z)=4x\,\mathbf{i}+4y\,\mathbf{j}+z^2\,\mathbf{k}}$ en el cilindro $x^2+y^2\le 25$ , $0\le z\le 2$ . $\text{div}\,\mathbf{f}=8+2z$ .




$\iiint_V\text{div}\,\mathbf{f}=$ [cilíndricas] $=\int_0^{2\pi}\int_0^5\int_0^2 r(8+2z)\,dz\,dr\,d\theta=2\pi\cdot\frac{25}{2}(16+4)=\boxed{500\pi}$ .

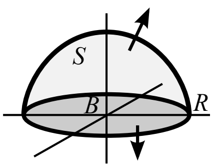
$S$ dada por $\mathbf{r}=(5\cos\theta,5\,\text{sen}\,\theta,z)$ , $\mathbf{r}_\theta\times\mathbf{r}_z=5(\cos\theta,\text{sen}\,\theta,0)$ , $\theta\in[0,2\pi]$ , $z\in[0,2]$ .

Sobre ella: $\mathbf{f}(\mathbf{r}(\theta,z))=(20\cos\theta,20\,\text{sen}\,\theta,z^2)$ , $\iint_S\mathbf{f}\cdot d\mathbf{S}=\int_0^{2\pi}\int_0^2 100\,dz\,d\theta=400\pi$ .

En la tapa superior $T$ : $\mathbf{f}\cdot\mathbf{n}=(4x,4y,4)\cdot(0,0,1)=2\;\rightarrow\;\iint_{B_5}4=\pi\,5^2\,4=100\pi$ .

En base $B$ : $\mathbf{f}\cdot\mathbf{n}=(4x,4y,0)\cdot(0,0,-1)=0\;\rightarrow\;\iint_{B_5}0=0$ . Por tanto, $\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}=400\pi+100\pi+0=\boxed{500\pi}$ .

**10.** $\boxed{\mathbf{f}(x,y,z)=(\,y\,,-x,1)}$ Flujo $\Phi=\Phi_S+\Phi_B$ a través de la superficie cerrada $\partial V=S\cup B$
(hemisferio superior de la superficie esférica de radio $R$ unido al círculo que es su base):




Parametrización $\mathbf{r}(\phi,\theta)$ de $S$ : $\left.\begin{matrix} x=R\,\text{sen}\,\phi\cos\theta \\ y=R\,\text{sen}\,\phi\,\text{sen}\,\theta \\ z=R\cos\phi \end{matrix}\right\}$ $\begin{matrix} \phi\in\left[0,\frac{\pi}{2}\right] \\ \theta\in[0,2\pi] \end{matrix}$ $\begin{matrix} \mathbf{r}_\phi=(R\,c\phi\,c\theta,R\,c\phi\,s\theta,-R\,s\phi) \\ \mathbf{r}_\theta=(-R\,s\phi\,s\theta,R\,s\phi\,c\theta,0) \end{matrix}$

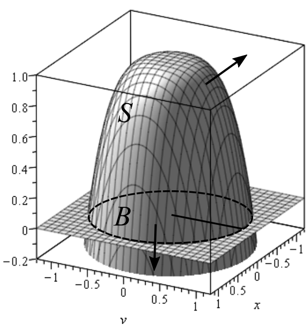
$\mathbf{r}_\phi\times\mathbf{r}_\theta=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R\,c\phi\,c\theta & R\,c\phi\,s\theta & -R\,s\phi \\ -R\,s\phi\,s\theta & R\,s\phi\,c\theta & 0 \end{vmatrix}=R^2(s^2\phi\,c\theta,s^2\phi\,s\theta,s\phi\,c\phi)=R^2\,\text{sen}\,\phi\,(\text{sen}\,\phi\cos\theta,\text{sen}\,\phi\,\text{sen}\,\theta,\cos\phi)$
$=R\,\text{sen}\,\phi\,\mathbf{r}(\phi,\theta)$ (apunta hacia afuera).

$\Phi_S=\iint_S\mathbf{f}\cdot d\mathbf{S}=\int_0^{\pi/2}\int_0^{2\pi}R^2\,\text{sen}\,\phi\cos\phi\,d\theta\,d\phi=2\pi R^2\left[\frac{1}{2}\,\text{sen}^2\phi\right]_0^{\pi/2}=\pi\,R^2$ .

Para $B$ es $\mathbf{n}=(0,0,-1)$ , $\mathbf{f}\cdot\mathbf{n}=-1$ $\Rightarrow$ $\Phi_B=\iint_B\mathbf{f}\cdot\mathbf{n}\,dS=-\iint_B dS=-\pi R^2$ [$=-$área]. $\Phi=\Phi_S+\Phi_B=\boxed{0}$ .

Comprobamos el teorema de Gauss. Como $\nabla\cdot\mathbf{f}=0$ : $\iiint_V\nabla\cdot\mathbf{f}\,dV=\iiint_V 0\,dV=\boxed{0}=\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}$ .

**11.** $\boxed{\mathbf{f}(x,y,z)=(x,1,y)}$ $z=1-\left(x^2+y^2\right)^2$ corta $z=0$ en la circunferencia unidad.




$\text{div}\,\mathbf{f}=1$ . Cilíndricas: $\iiint_V\text{div}\,\mathbf{f}=\int_0^{2\pi}\int_0^1\int_0^{1-r^4}r\,dz\,dr\,d\theta=2\pi\int_0^1\left(r-r^5\right)dr=\boxed{\frac{2\pi}{3}}$ .

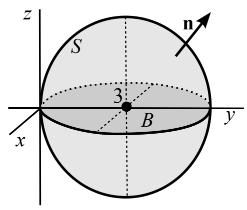
Sobre la base $B$ : $(x,y,0)$ , $\iint_\bigcirc(x,1,y)\cdot(0,0,-1)\,dx\,dy=-\iint_\bigcirc y\,dx\,dy=0$ (impar).

Sobre la tapa $S$ : $\mathbf{r}(x,y)=\left(x,y,1-(x^2+y^2)^2\right)\rightarrow\mathbf{r}_x\times\mathbf{r}_y=\left(4x(),4y(),1\right)$ (exterior).

$\iint_\bigcirc\left(4x^2(x^2+y^2)+\underset{\text{impar}}{4y(x^2+y^2)}+\underset{\text{impar}}{y}\right)dx\,dy=\int_0^{2\pi}\int_0^1\underset{2r^5(1+\cos 2\theta)}{\underset{\|}{4r^5\cos^2\theta}}\,dr\,d\theta=\frac{1}{3}\cdot 2\pi=\iiint_{\partial V}\mathbf{f}\cdot d\mathbf{S}$ .

[O bien, $\mathbf{r}(r,\theta)=\left(r\cos\theta,r\,\text{sen}\,\theta,1-r^4\right)$ , $r\in[0,1]$ , $\theta\in[0,2\pi]$ , . . . ].

**12.** $\boxed{\mathbf{f}(x,y,z)=3yz\,\mathbf{i}+2xz\,\mathbf{j}+(z+xy)\,\mathbf{k}}$ $x^2-6x+y^2+z^2=0 \Leftrightarrow (x-3)^2+y^2+z^2=3^2$ [esfera de radio 3 centrada en $(3,0,0)$].




Como $\operatorname{div}\mathbf{f}=1$, según Gauss: $\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}=\iiint_V 1\;dx\,dy\,dz=$ volumen de $V=\frac{4}{3}\pi 3^3=36\pi$.

Calcularlo directamente sería largo, tanto usando esféricas centradas en el punto:

$\mathbf{r}(\phi,\theta)=\big(3+3\,\mathrm{sen}\,\phi\cos\theta,\,3\,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\,3\cos\phi\big)$, $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$,

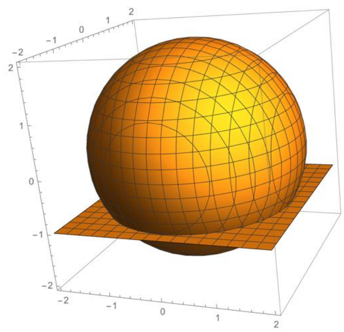
como las parametrizaciones cartesianas: $\big(x,y,\pm\sqrt{6x-x^2-y^2}\,\big)$, $(x,y)\in B$ círculo de radio 3 centrado en $(3,0)$.

**13.** $\boxed{\mathbf{F}(x,y,z)=(y,x,2)}$ Como $\operatorname{div}\mathbf{f}=0$ la $\iiint_V=0$. Calculemos el flujo hacia el exterior del volumen dado. La $\partial V$ está compuesta por dos partes: $S_1$ sobre la esfera y $S_2$ en el plano $z=-1$.

En la esfera, si $\mathbf{r}(\phi,\theta)=2(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)$, la normal exterior es $\mathbf{r}_\phi\times\mathbf{r}_\theta=2\,\mathrm{sen}\,\phi\;\mathbf{r}(\phi,\theta)$.

Los ángulos varían: $0\le 2\pi$, $\cos\phi=-\frac{1}{2}\to 0\le\phi\le\frac{2\pi}{3}$. El flujo sobre la esfera es:

$$8\int_0^{2\pi}\!\!\int_0^{2\pi/3}(\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\mathrm{sen}\,\phi\cos\theta,1)\cdot(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)\,\mathrm{sen}\,\phi\,d\phi\,d\theta$$
$$=8\int_0^{2\pi}\!\!\int_0^{2\pi/3}\big(2\,\mathrm{sen}\,\theta\cos\theta\,\mathrm{sen}^2\phi+\cos\phi\big)\,\mathrm{sen}\,\phi\,d\phi\,d\theta$$
$$=16\pi\int_0^{2\pi/3}\cos\phi\,\mathrm{sen}\,\phi\,d\phi=8\pi\big[\mathrm{sen}^2\phi\big]_0^{2\pi/3}=6\pi\,.$$



El flujo en $z=-1$ es a través del círculo $x^2+y^2\le 3$, con vector normal $-\mathbf{k}$:

$\mathbf{F}\cdot(-\mathbf{k})=-2$, $\iint_{S_2}\mathbf{F}\cdot d\mathbf{s}=-2\iint_{x^2+y^2\le 3}dx\,dy=-2\,\underbrace{\big[\pi(\sqrt{3})^2\big]}_{\text{área del círculo}}=-6\pi$.

Sumado al flujo a través de $S_1$ da cero como asegura el teorema de Gauss.

**14.** $\boxed{\mathbf{F}(x,y,z)=(-y,x,2)}$ El rotacional de $\mathbf{F}$ es: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial_x&\partial_y&\partial_z\\ -y&x&2\end{vmatrix}=2\mathbf{k}$.

Volvemos a calcular el vector normal exterior en esféricas $\mathbf{r}(\phi,\theta)=2(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)$:

$$\mathbf{r}_\phi=2(\cos\phi\cos\theta,\mathrm{sen}\,\phi\cos\theta,-\mathrm{sen}\,\phi),\quad \mathbf{r}_\theta=2(-\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi\,\mathrm{sen}\,\theta,0),$$
$$\mathbf{r}_\phi\times\mathbf{r}_\theta=4\,\mathrm{sen}\,\phi\,(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)=2\,\mathrm{sen}\,\phi\,\mathbf{r}\,.$$

En $z>-1$, es $0\le\theta\le 2\pi$ mientras que $0\le\phi\le\frac{2\pi}{3}$. Y el flujo a través de la superficie esférica (hacia el exterior) es:

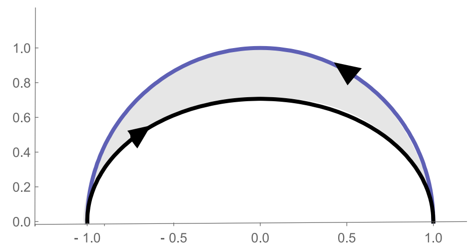
$$\iint_S\operatorname{rot}\mathbf{F}\cdot d\mathbf{S}=\int_0^{2\pi}\!\!\int_0^{2\pi/3}2\mathbf{k}\cdot 2\mathbf{r}\;\mathrm{sen}\,\phi\,d\phi\,d\theta=16\pi\int_0^{2\pi/3}\mathrm{sen}\,\phi\cos\phi\,d\phi=-8\pi\big[\cos^2\phi\big]_0^{2\pi/3}=6\pi\,.$$

En $\partial S$, circunferencia en $z=-1$ con centro en $(0,0,-1)$ y radio $\sqrt{3}$ (pues $x^2+y^2=4-1=3$) la integral de línea de $\mathbf{F}$ es (en el sentido correcto para que el vector normal vaya hacia el exterior):

$$\mathbf{c}(t)=\big(\sqrt{3}\cos t,\sqrt{3}\,\mathrm{sen}\,t,-1\big),\ \mathbf{c}'(t)=\big(-\sqrt{3}\,\mathrm{sen}\,t,\sqrt{3}\cos t,0\big),\ \int_0^{2\pi}3\,(-s,c,2)\cdot(-s,c,0)\,dt=3\int_0^{2\pi}dt=6\pi\,.$$

**15.** $\boxed{\mathbf{F}(x,y,z)=\big(y,2x+z,\mathrm{e}^x\big)}$ Para calcular el flujo del rotacional:

$$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial_x&\partial_y&\partial_z\\ y&2x+z&\mathrm{e}^x\end{vmatrix}=\big(-1,-\mathrm{e}^x,1\big)\,.$$




El vector normal a la superficie es $\mathbf{k}$ y $(\nabla\times\mathbf{F})\cdot\mathbf{k}=1$. Entonces el flujo es el área (medio círculo, $\frac{\pi}{2}$, menos la mitad de la elipse, $\frac{\pi}{2\sqrt{2}}$):

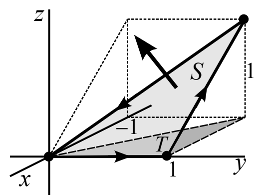
$$\iint_D dx\,dy=\int_{-1}^{1}\int_{\frac{1}{\sqrt{2}}\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\,dx=\Big(1-\tfrac{1}{\sqrt{2}}\Big)\int_{-1}^{1}\sqrt{1-x^2}\,dx=\Big(1-\tfrac{1}{\sqrt{2}}\Big)\tfrac{\pi}{2}$$
(la integral puede hacerse con el cambio usual, $x=\mathrm{sen}\,t$).

La circulación en el círculo: $\int_0^{\pi}(\mathrm{sen}\,t,2\cos t,\mathrm{e}^{\cos t})\cdot(-\mathrm{sen}\,t,\cos t,0)\,dt=\int_0^{\pi}\big(2\cos^2 t-\mathrm{sen}^2 t\big)dt=\pi-\frac{1}{2}\pi=\frac{1}{2}\pi$.

En la elipse (sentido opuesto): $\int_\pi^0\big(\frac{1}{\sqrt{2}}\mathrm{sen}\,t,2\cos t,\mathrm{e}^{\cos t}\big)\cdot\big(-\mathrm{sen}\,t,\frac{1}{\sqrt{2}}\cos t,0\big)\,dt=\frac{1}{\sqrt{2}}\int_\pi^0\big(2\cos^2 t-\mathrm{sen}^2 t\big)dt=\frac{-\pi}{2\sqrt{2}}$.

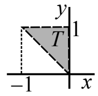
Y la suma de ambas nos da el valor del flujo (área) calculado arriba.

**16.** $\boxed{\mathbf{f}(x,y,z)=(yz,e^y,1)}$ $\to$ $\text{rot}\,\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ yz&e^y&1\end{vmatrix}=(0,y,-z)$ . vector normal: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 0&1&0\\ -1&1&1\end{vmatrix}=\mathbf{i}+\mathbf{k}$ .




Triángulo sobre el plano $(1,0,1)\cdot(x,y,z)=x+z=0$ $\to$ $\mathbf{r}(x,y)=(x,y,-x)$ en $T$ , $\mathbf{r}_x\overset{\uparrow}{\times}\mathbf{r}_y$

$$\iint_S \text{rot}\,\mathbf{f}\cdot\mathbf{n}\,dS=\int_{-1}^{0}\int_{-x}^{1}(0,y,x)\cdot(1,0,1)\,dy\,dx=\int_{-1}^{0}(x+x^2)\,dx=\Big[\tfrac{x^2}{2}+\tfrac{x^3}{3}\Big]_{-1}^{0}=\boxed{-\tfrac{1}{6}}.$$




Parametrizamos $\partial S$:

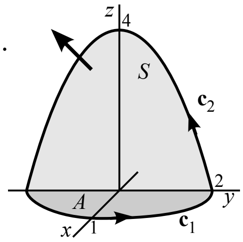
$\mathbf{c}_1(t)=(0,t,0)$ , $t\in[0,1]$ $\quad \mathbf{c}_1'=(0,1,0)$ $\quad \mathbf{f}(\mathbf{c}_1)=(0,e^t,1)$

$\mathbf{c}_2(t)=(-t,1,t)$ , $t\in[0,1]$ $\quad \mathbf{c}_2'=(-1,0,1)$ $\quad \mathbf{f}(\mathbf{c}_2)=(t,e,1)$ $\quad\to$

$\mathbf{c}_3(t)=(-t,t,t)$ , $t\in[1,0]$ $\quad \mathbf{c}_3'=(-1,1,1)$ $\quad \mathbf{f}(\mathbf{c}_3)=(t^2,e^t,1)$

$$\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_0^1 e^t\,dt+\int_0^1(1-t)\,dt+\int_1^0(1-t^2+e^t)\,dt=e-1+1-\tfrac12-1+\tfrac13+1-e=\boxed{-\tfrac16}$$ como debía.

**17.** $\mathbf{f}(x,y,z)=(3,x^2,y)$, $\mathbf{r}(x,y)=(x,y,4-4x^2-y^2)$ $\to$ $\text{rot}\,\mathbf{f}=(1,0,2x)$ , $\mathbf{r}_x\times\mathbf{r}_y=(-f_x,-f_y,1)=(8x,2y,1)$ .




$$\iint_S \text{rot}\,\mathbf{f}\cdot d\mathbf{S}=\iint_A(1,0,2x)\cdot(8x,2y,1)\,dx\,dy=\int_{-2}^{2}\int_0^{\sqrt{1-y^2/4}}10x\,dx\,dy$$

$$=\int_{-2}^{2}5\big[1-\tfrac14y^2\big]dy=20-\tfrac{5}{12}\big[y^3\big]_{-2}^{2}=\boxed{\tfrac{40}{3}}.$$

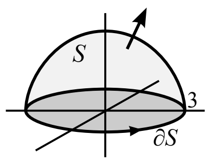
$\mathbf{c}_1(t)=(\cos t,2\,\text{sen}\,t,0)$ , $t\in\big[-\frac{\pi}{2},\frac{\pi}{2}\big]$ $\quad \mathbf{c}_1'=(-\text{sen}\,t,2\cos t,0)$ $\quad \mathbf{f}(\mathbf{c}_1)=(3,\cos^2 t,2\,\text{sen}\,t)$

$\mathbf{c}_2(t)=(0,-t,4-t^2)$ , $t\in[-2,2]$ $\quad \mathbf{c}_2'=(0,-1,-2t)$ $\quad \mathbf{f}(\mathbf{c}_2)=(3,0,-t)$ $\quad\to$

$$\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_{-\pi/2}^{\pi/2}\big[2\cos t(1-\text{sen}^2t)-\underset{\text{impar}}{3\,\text{sen}\,t}\big]dt+\int_{-2}^{2}2t^2dt=2\big[2\,\text{sen}\,t-\tfrac23\,\text{sen}^3t\big]_0^{\pi/2}+\tfrac{32}{3}=\boxed{\tfrac{40}{3}}.$$

[Sale más largo parametrizando $\mathbf{c}_1(t)=\big((1-\frac{t^2}{4})^{1/2},t,0\big)$ , $t\in[-2,2]$ ].

**18.** $\boxed{\mathbf{F}(x,y,z)=-y\,\mathbf{i}+2x\,\mathbf{j}+(x+z)\,\mathbf{k}}$ . $\text{rot}\,\mathbf{f}=(0,-1,3)$ . Superficie $x^2+y^2+z^2=9$ , $z\ge 0$ :



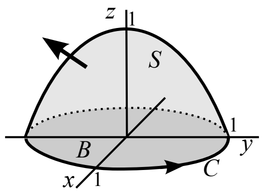

$\mathbf{r}(\phi,\theta)=\big(3\,\text{sen}\,\phi\cos\theta,3\,\text{sen}\,\phi\,\text{sen}\,\theta,3\cos\phi\big)$ , $\phi\in[0,\frac{\pi}{2}]$ , $\theta\in[0,2\pi]$ ,

$\mathbf{r}_\phi\times\mathbf{r}_\theta=9(\text{sen}^2\phi\cos\theta,\text{sen}^2\phi\,\text{sen}\,\theta,\text{sen}\,\phi\cos\phi)$ .

$$\iint_S \text{rot}\,\mathbf{F}\cdot d\mathbf{S}=9\int_0^{2\pi}\int_0^{\pi/2}(3\,\text{sen}\,\phi\cos\phi-\underset{0\swarrow}{\text{sen}^2\phi\,\text{sen}\,\theta})\,d\phi\,d\theta=27\pi\big[\,\text{sen}^2\phi\big]_0^{\pi/2}=\boxed{27\pi}.$$

Además: $\mathbf{c}(t)=(3\cos t,3\,\text{sen}\,t,0)$ , $t\in[0,2\pi]$ $\to$ $\oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}=9\int_0^{2\pi}\big[\,\text{sen}^2t+2\cos^2t\big]dt=9\int_0^{2\pi}\frac{3+\cos t}{2}\,dt=\boxed{27\pi}$ .

**19.** $\boxed{\mathbf{f}(x,y,z)=(x,xy,2z)}$ $\text{rot}\,\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ x&xy&2z\end{vmatrix}=(0,0,y)$ . $\quad S$: $\mathbf{r}(x,y)=(x,y,1-x^2-y^2)$ . $\mathbf{r}_x\times\mathbf{r}_y=(2x,2y,1)$ .




$$\iint_S \text{rot}\,\mathbf{f}\cdot d\mathbf{S}=\iint_B y\,dx\,dy=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}y\,dy\,dx=\boxed{0}$$ [o impar y recinto simétrico].

$\mathbf{c}(t)=(\cos t,\text{sen}\,t,0)$ , $t\in[0,2\pi]$ $\to$ $\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_0^{2\pi}(-\text{sen}\,t\cos t+\cos^2t\,\text{sen}\,t)\,dt=\boxed{0}$ .

$\text{div}\,\mathbf{f}=3+x$ . $\iiint_V\text{div}\,\mathbf{f}=$ [cilíndricas] $=\int_0^{2\pi}\int_0^1\int_0^{1-r^2}r(3+r\cos\theta)\,dz\,dr\,d\theta=6\pi\int_0^1(r-r^3)\,dr=\boxed{\tfrac32\pi}$ .

$$\iint_S\mathbf{f}\cdot d\mathbf{S}=\iint_B\underset{=2(1+xy-y^2)}{(x,xy,2-2x^2-2y^2)\cdot(2x,2y,1)}\,dx\,dy=2\int_0^{2\pi}\int_0^1\big(r+r^3\cos\theta\,\text{sen}\,\theta-r^3\,\text{sen}^2\theta\big)\,dr\,d\theta=\tfrac32\pi.$$

$\mathbf{r}_B(x,y)=(x,y,0)$ , $(x,y)\in B$ , $\iint_B\mathbf{f}\cdot\mathbf{n}\,dS=-\iint_B(x,xy,0)\cdot(0,0,-1)\,dx\,dy=0$ . $\iint_{S^*}=\iint_S+\iint_B=\boxed{\tfrac32\pi}$ .

**20.** a) $u\,\Delta u=u\big(u_{xx}+u_{yy}\big)=\text{div}\big(uu_x,uu_y\big)-\big(u_x^2+u_y^2\big)=\text{div}\big(u\nabla u\big)-\|\nabla u\|^2$ (y casi igual para $n=3$ ).

b) $$\iint_D u\,\Delta u\,dx\,dy=\iint_D\text{div}\big(u\nabla u\big)dx\,dy-\iint_D\text{div}\|\nabla u\|^2dx\,dy\overset{\text{TD}}{=}\oint_{\partial D}u\,\frac{\partial u}{\partial\mathbf{n}}\,ds-\iint_D\|\nabla u\|^2dx\,dy,$$

$\downarrow$ $u\nabla u\cdot\mathbf{n}=u\frac{\partial u}{\partial\mathbf{n}}$ .

c) Usando el teorema de la divergencia en el espacio: $\iiint_V u\,\Delta u\,dx\,dy\,dz=\iint_{\partial V}u\,\frac{\partial u}{\partial\mathbf{n}}\,dS-\iiint_V\|\nabla u\|^2dx\,dy\,dz$ .

d) En una variable: $\int_a^b u\,u''\,dx=u\,u'\big]_a^b-\int_a^b(u')^2dx$ es una simple integración por partes.