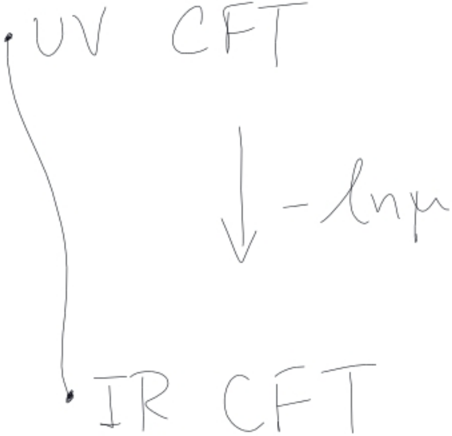
Gradient Properties of RG Flows

arXiv: 2402.17817

with William Pannell

Relativistic
Unitary
Perturbative
QFT



$$\beta^I = \mu \frac{d\lambda^I}{d\mu} = \underset{\substack{\nearrow \\ \text{anti-symm}}}{Q^I{}_J}\, \lambda^J$$

Monotonicity theorems

Riemannian

- Strongest $\beta^I = G^{IJ}\partial_J A$
- Strong $\mu \frac{dA}{d\mu} > 0$
- Weak $A_{UV} > A_{IR}$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi^i + \frac{1}{4!} \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_l$$

$$\beta_{\underbrace{ijkl}_{I}} = \mu \frac{d\lambda_{ijkl}}{d\mu}$$

We want to see if

$$\beta_I = G_{IJ} \partial_J A$$

If G_{IJ} is symm., then at any point:

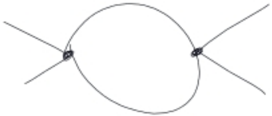
$$G_{IJ} = \delta_{IJ}$$

$$\beta_I = \partial_I A \Longrightarrow \partial_I \beta_J = \partial_I \partial_J A$$

One loop

$$\beta_{ijkl} = \lambda_{ijmn}\lambda_{mnkl} + \text{perms}$$

$$=$$

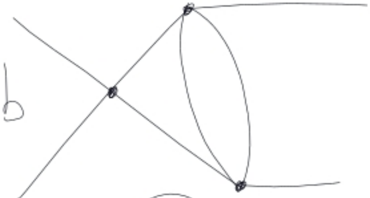


$$A = \lambda_{ijkl}\lambda_{klmn}\lambda_{mnij} =$$

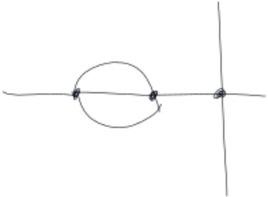
$$G = \delta_{ijkljmnpq} =$$

Two loops

${}^2\beta = {}^2b$



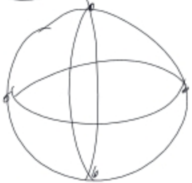
$+ \; {}^2e$



${}^2A = {}^2a_1$

$+ \; {}^2a_2$

$+ \; {}^2a_3$

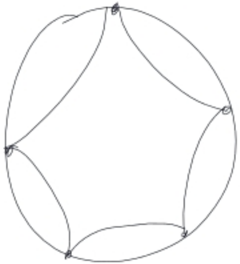


${}^2G = {}^2g$

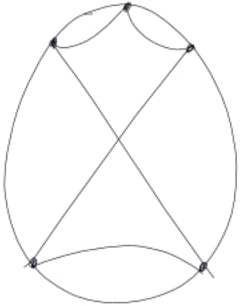
	1	2	3	4	5	6
β	1	2	7	23	110	571
A	1	3	5	17	42	177
G	1	1	7	18	97	453
Eqs.	1	3	10	36	164	819

Three loops

3



1



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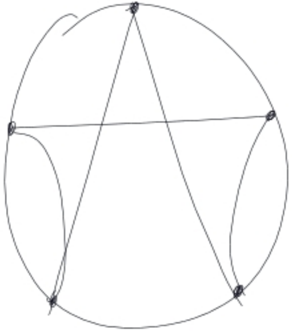
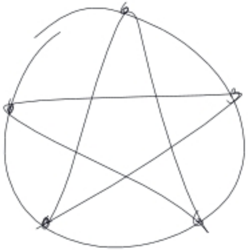
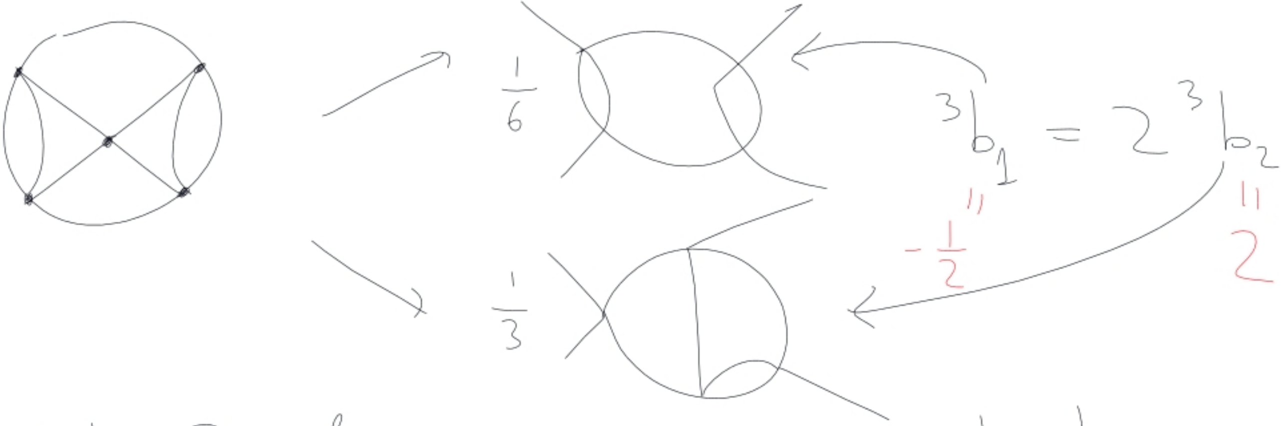


2



1



$$\underset{-\frac{1}{2}}{\underset{\|}{{}^3b_1}} = 2\,\underset{2}{\underset{\|}{{}^3b_2}}$$

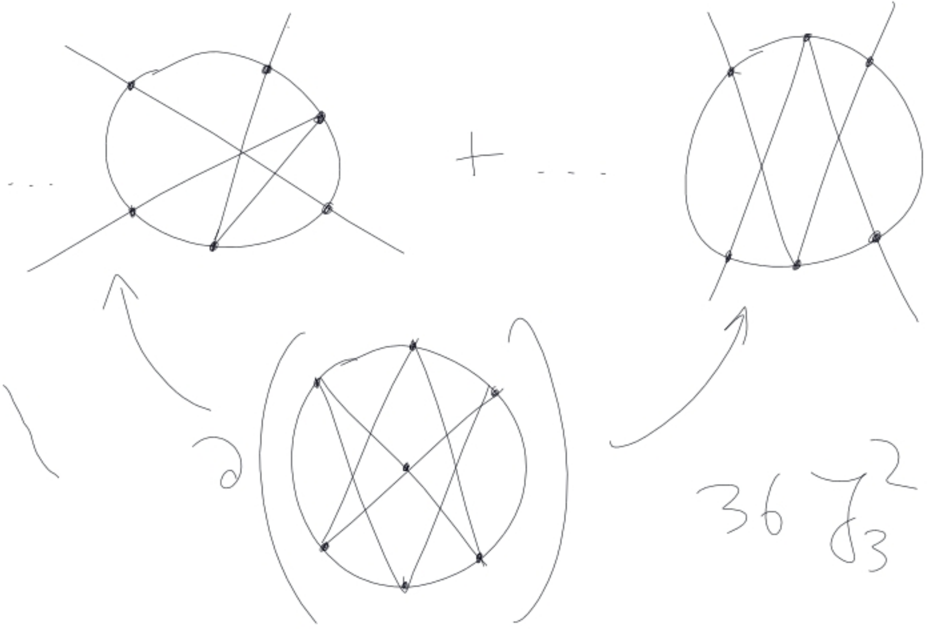
At 3 loops metric cannot be globally flat.

Four loops and beyond

4 : 4 equations ✓

5 : 37 equations ✓

$5\beta \supset$



6 loops:

$$\begin{tikzcd} & 229 \ \checkmark \\ 234 \arrow[ur] \arrow[dr] & \\ & 5 \ \times \end{tikzcd}$$

Options:

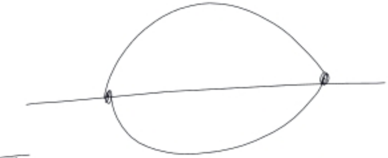
(A) Someone is wrong

[✓ (B)] RG flow is gradient

(C) None of the above

In dim-reg

$$\phi_B = Z^{1/2}\,\phi_R$$



$$(Z^{1/2})^T Z^{1/2} = (Z^{1/2})^T \underbrace{O^T O}_{\mathbb{1}} Z^{1/2} = (O Z^{1/2})^T (O Z^{1/2})$$

Key: $T^{\mu}{}_{\mu} = \beta^{i} \mathcal{O}_{i} + \partial^{\mu} J_{\mu}$

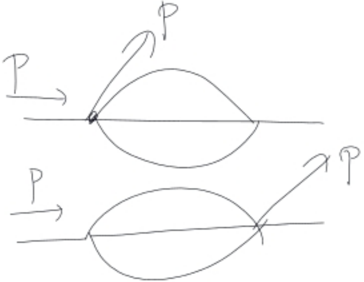
$$J_{\mu} = S_{ij} \phi_{i} \partial_{\mu} \phi_{j}$$

$$S_{ij} = - S_{ji}$$

$$T^{\mu}{}_{\mu} = B^{i} \mathcal{O}_{i} \, , \quad B^{i} = \beta^{i} - (S \lambda)^{i}$$

$$(N_I)_{ij}\,\phi_i\,\partial_\mu\lambda^I\,\partial^\mu\phi_j$$

$$(N_I)_{ij}=\sum_n\frac{(N_I^{(n)})_{ij}}{\varepsilon^n}$$



$$S_{ij}=(N_I^{(1)})_{ij}\,\lambda^I$$

$${}^5S_2 = \frac{259}{4608}$$

$$192\,{}^5S_3 + 384\,{}^5S_4 = 31 - 36\zeta_3$$

5 loops 4 ς-diagrams

6 loops 19 ς-diagrams

$\Gamma = \gamma + \varsigma$

$B = \beta - \varsigma\lambda$

$\beta \to \beta + \omega\lambda$ $\gamma \to \gamma + \varsigma\omega$

$\varsigma \to \varsigma + \omega$

$$\beta \longrightarrow \beta - \omega\lambda$$

$$S \longrightarrow S + \omega$$

$$J_\mu = S_{ij}\, \phi_i\, \partial_\mu \phi_j$$

$$B = \beta - (S\lambda)$$

$$\langle T^\mu{}_\mu\, \mathcal{O}_I \rangle \qquad (N_I)_{ij}\, \partial_\mu \lambda^I(x)\, \phi_i\, \partial_\mu \phi_j$$

$$\langle T^\mu{}_\mu\, \partial^\mu J_\nu \rangle$$