The algebraic approach: when, how, and why?

Key point: Algebras are a tool for <u>thinking</u> about QFT.

Goal: Add it to your toolkit.

I. Fundamentals

II. Relative entropy & the ANEC

III. Semiclassical black holes

	Bootstrap	AQFT
Key object	Correlators	Field operators
Key aspect	OPE expansion	Structures in spacetimes
Key tools	Crossing, analyticity more powerful	Modular flow, analytic operator theory more theory

What is a quantum field?

Basic: An object in correlators

e.g. $$\langle \phi(x)\,\phi(y) \rangle = \frac{C}{|x-y|^{2\Delta}}$$

We <u>want</u> fields to be interpreted as operators:

$$\langle \phi(x)\,\phi(y) \rangle = \langle \underset{\substack{\uparrow \\ \mathcal{H}}}{\Omega} | \underbrace{\hat{\phi}(x)\,\hat{\phi}(y)}_{\text{act on } \mathcal{H}} | \Omega \rangle$$

Not possible: $\| \hat{\phi}(x) |\Omega\rangle \|^2 = \infty$

In Lorentz sig:

$$\langle\phi(x)\phi(y)\rangle = \lim_{\varepsilon\to 0} \frac{C}{\left(-(x_0-y_0-i\varepsilon)^2+(\vec{x}-\vec{y})^2\right)^{2\Delta}}$$

$$\langle\phi(f)\,\phi(g)\rangle \equiv \lim_{\varepsilon\to 0}\int f(x)\,g(y)\,\frac{C}{\cdots}\,.$$

bumps

C_0^∞

<u>Finite</u> even for $f=g \Rightarrow \|\hat{\phi}[f]|\Omega\rangle\|^2<\infty$

$\Rightarrow \hat{\phi}[f]$ as an operator

Indeed, "good" Lorentzian correlators are distributions s.t. $\exists!\ \mathcal{H}, |\Omega\rangle$ with

$$\left.\begin{aligned} &\int f_1(x_1)\dots f_n(x_n)\langle\phi(x_1)\dots\phi(x_n)\rangle \\ &= \\ &\langle\Omega|\tilde{\phi}[f_1]\dots\tilde{\phi}[f_n]|\Omega\rangle \end{aligned}\right\} \text{GNS construction}$$

<u>Lesson I:</u> Fields smeared in Lorentz can be interpreted as ops.

<u>Upshot:</u> In any QFT, should exist ops localized to small sets.

Ops in spacetime

To a region $\mathcal{O}$, associate

$$\underbrace{A_0(\mathcal{O}) = \langle \phi[f] \mid \underbrace{f \subset\subset \mathcal{O}}_{\text{supp}} \rangle}_{\text{"polynomial algebra"}}$$

Convenient to construct $A_0(\mathcal{O})$ from <u>observable</u> fields

$$[A_0(\mathcal{O}_1), A_0(\mathcal{O}_2)] = 0$$

$A_0(\mathcal{O})$ "poly. alg. of observables"

Remark: $A_0(\mathcal{O})$ is a ***-algebra**:

- polynomials
- adjoint $\leftarrow$ $\phi[f] \in A_0(\mathcal{O})$
  $\phi^\dagger[f] \in A_0(\mathcal{O})$

Question: Is $A_0(\mathcal{O})$ everything?

No! Should be a completed $A(\mathcal{O})$ with:

$$A(\mathcal{O}) \supseteq A_0(\mathcal{O})$$

$$A(\mathcal{O}) \supseteq A_0(D(\mathcal{O}))$$

... other limits

One guess:

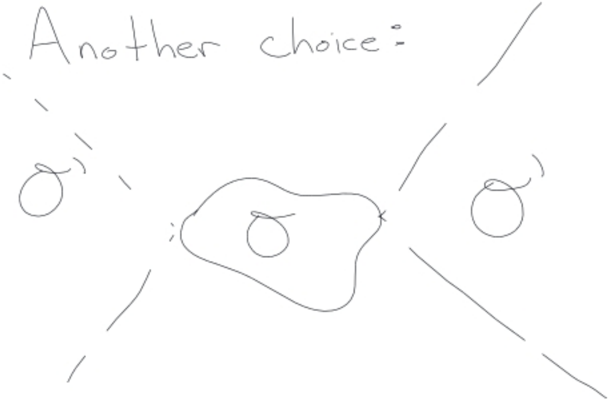
$$A(\mathcal{O}) \equiv \overline{A_0(\mathcal{O})}$$

some topology

"(ultra) weak"

"(ultra) strong"

Another choice:



$$A(\mathcal{O}) \equiv A_0(\mathcal{O}')'$$

$$= \left\{ \underset{\text{bounded}}{T} \;\middle|\; [T, a'] = 0 \quad a' \in A_0(\mathcal{O}') \right\}$$

Either is fine. Both are closed in nice tops.;

both "von Neumann algebras" (vNA)

ultraweak topology

$*$-algebra is von Neumann if

$$\left(\begin{array}{c} T_n \in A \\ tr(\rho T_n) \to tr(\rho T) \\ \text{forall } \rho \end{array} \right) \Rightarrow \left(T \in A \right)$$

Equiv defs:

- strong, weak, ultrastrong

- A is vN iff $A = W''$

  $*$-algebra

  - A is vN $\iff A = (A')'$

- $T \in A$ Hermitian $\Rightarrow \underbrace{f(T)}_{f:\mathbb{R}\to\mathbb{C} \text{ bounded}} \in A$

Lesson 2: Regions of spacetime naturally carry vVA's.

Upshot: Topological & analytic props.
can be used to study QFT.

Local physics (& modular flow)

Subsystems on a lattice

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Physics of $|\Psi\rangle_{AB}$ in A controlled by

$$\rho_A \equiv \mathrm{tr}_B |\Psi\rangle\langle\Psi| \qquad \Bigg| \qquad \mathcal{H} = \bigoplus_\alpha \mathcal{H}^L_\alpha \otimes \mathcal{H}^R_\alpha$$

Advantage: If ρ_A is full-rank

$$\rho_A = e^{-\overbrace{(-\log \rho_A)}^{K_A}}$$

$K_A \geq 0$ Hermitian

K_A is a "Hamiltonian" for which ρ_A is thermal.

Conceptual: Statistical mech of K_A ← "modular Hamiltonian"

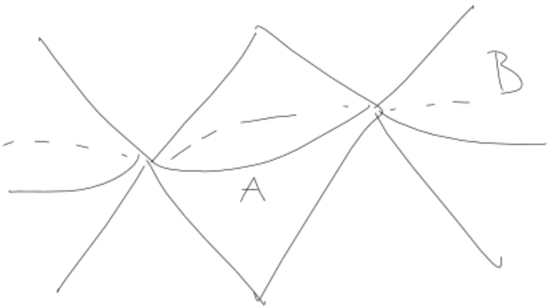
$\updownarrow$

info theory of ρ_A

e.g. vN entropy

$$S(\rho_A) = -\mathrm{tr}(\rho_A \log \rho_A)$$

Spacetime:



$$\rho_A \otimes \rho_B^{-1}$$

States have infinite UV ent. in $A:B \Rightarrow$

ρ_A is unnormalizable

$$\mathrm{tr}(\rho \log \rho)$$

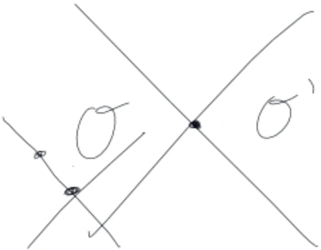
$\mathcal{H}, \quad A$

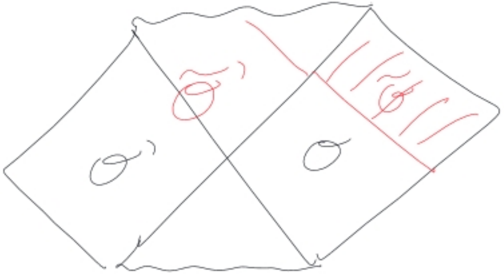
$$A' = \{ T \in \mathcal{B}(\mathcal{H}) \mid [T, a] = 0 \quad a \in A \}$$

$$Z(A) = A \cap A'$$

$$A' \supseteq Z(A)$$

$$A'' \supseteq Z(A') \supseteq Z(A)$$





$$A(O') \overset{?}{\supset} Z(A(O))$$

$$A(O') = A(O)'$$