Notes on the gravitational S-matrix

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Abstract

We review the basic properties of gravitational scattering at high energies. We discuss to what extent they are compatible with the basic principles of S-matrix theory. Spoiler: they look consistent, but black hole physics is not really used.

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1 Introduction

"There is no S-matrix for the process of black-hole formation and evaporation."
S. Hawking [1]

"We have found no evidence for a lack of harmony between gravity and the basic S-matrix properties."
S. Giddings and R. Porto, "The gravitational S-matrix." [2]

In these notes we consider gravitational theories in asymptotically flat spacetime. In real experiments at particle colliders gravity is not relevant.[1] The reason is that gravitational coupling is too weak. Compared to other couplings however gravitational coupling grows with energy very fast. Therefore in a thought experiment of scattering at ultra-Planckian energies gravity will inevitably become important. We therefore will consider a thought experiment where scattering takes place at very high energies and gravity *is* important.

Famously there is a tension between gravity and quantum mechanics. In the context of high-energy collisions by 'quantum mechanics' we understand the existence of a unitary S-matrix acting on multi-particle states of gravitons. We assume $D > 4$ here, in $D = 4$ asymptotic states are not multi-particle states and a more general framework should be used. By 'gravitational EFT' we mean general relativity plus possibly higher derivative corrections. It is probably the most successful classical theory in physics and we will use it again below.

As we review below gravitational EFT allows us to make nontrivial predictions about scattering amplitudes [4, 5], which are not obviously consistent with the basic principles of S-matrix theory: analyticity, unitarity and crossing. These principles are well-established in the context of QFT, but we assume that they hold for gravity as well. There is therefore a nontrivial cross-check to be made: can we accomodate the expected properties of the gravitational S-matrix into the framework of S-matrix theory?

As of 2024, we believe the answer to this question is 'yes'. No problems have been identified to the best of our knowledge. In fact, at the level of $2 \to 2$ scattering amplitude gravity obeys the usual logic of low-energy EFTs [6] (heavy states decouple, producing higher derivative operators controlled by the mass of states that were integrated out whether these are electrons, strings or black holes).

On the other hand, it is not a very satisfactory 'yes' because in classical GR black holes play a central role and generate tension with quantum mechanics [1]. At present, black holes have not been incorporated in the S-matrix bootstrap approach to gravitational scattering in any substantial way. One concrete proposal of doing that was given in [7], but implementing it seems beyond the current technology.

Why do we believe that basic principles hold in quantum gravity? The main evidence comes from string theory. In the AdS/CFT correspondence a harmony between gravity and quantum mechanics is clearly achieved. Similarly, our knowledge of perturbative string theory amplitudes is fully consistent with the basic principles. Nonperturbatively, the situation is less clear. Known ways to define the gravitational S-matrix holographically involve taking a subtle large N limit [8]:

[1]See [3] for the discussion of high-energy collisions in scenarios with large extra dimensions where gravity is relevant. Recall that in the presence of large extra dimensions we have for the Planck lengths $\ell_d^{d-2} = \ell_D^{d-2} \left(\frac{\ell_D}{\ell_{KK}}\right)^{D-d}$, where $D - d$ dimensions are compactified. Given $\ell_{KK} \gg \ell_D$, we have $\ell_d \ll \ell_D$.

either the flat space limit of the AdS/CFT correspondence, or the decompactification limit in the BFSS matrix model or matrix string theory

$$\hat{S} = \lim_{N\to\infty} S_N \ , \tag{1.1}$$

where the limit is taken at the level of matrix elements. Either way, unitarity $\hat{S}\hat{S}^\dagger = 1$ is not manifest in the limit, therefore the information paradox cannot be really addressed. Interesting new results however can be derived [9]. Finally, black holes violate global symmetries and fuel the swampland program [10], but quantitative derivation of such effects from first principles is missing, see [11] for the recent discussion.

2 Gravitational features

We will be interested in $2 \to 2$ scattering of mass m scalars in $D > 4$. The scattering amplitude is a function of *two* variables. One will always be the overall energy of the process $s > 0$. The other one is conveniently chosen to be:

- transferred momentum $t < 0$;
- scattering angle θ defined through $t = -\frac{s-4m^2}{2}(1-\cos\theta)$;
- impact parameter b, which is the Fourier dual of $\sqrt{-t}$ and controls separation of wavepackets in the transverse space.[2] Equivalently, this corresponds to large J partial waves: to leading order in $sb^2 \gg 1$, we have $J = \frac{\sqrt{s}b}{2}$.

It is important to keep in mind that physics looks quite different if we study $T(s,t)$, $T(s,\theta)$, or $T(s,b)$. We will be mostly interested in high-energy scattering so m will not play much role in the discussion and we can as well set it to zero.

Famously, there are situations where the limit $m \to 0$ is subtle [12] but we will not consider those. Secondly, for the collisions of black holes in nature mass is very important [13].

2.1 Schwarzschild radius and impact parameters

We will be performing various thought experiments and it is useful to keep in mind what are the relevant physical parameters. By default, we set $c = \hbar = 1$. The Planck scale is defined as $G_N = \frac{1}{M_P^{D-2}} = \ell_P^{D-2}$.

Probably the most important one is the Schwarzschild radius corresponding to the center-of-mass collision energy

$$R_S^{D-3} = \frac{16\pi G_N \sqrt{s}}{(D-2)\Omega_{D-2}} \ , \tag{2.1}$$

where $\Omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$. The basic physical picture about gravity states that if energy is confined within the Schwarzschild radius a black hole is formed (gravity is *strong*) [14,15].[3] Let us also quote

[2] This only makes sense if $b \gg \frac{\hbar}{\sqrt{s}}$ (Compton wavelength).

[3] An $O(1)$ amount of energy escapes to infinity in the form of gravity waves in the black hole formation process.

the black hole entropy

$$S_{BH} = \frac{\text{Area}}{4G_N} = \frac{\Omega_{D-2} R_S^{D-2}}{4G_N} = \frac{4\pi}{D-2}\sqrt{s}R_S \sim (\sqrt{s})^{1+\frac{1}{D-3}} \ . \tag{2.2}$$

This suggests that a natural space to think about gravity is the *impact parameter space* $T(s,b)$ (in this process energy $\sqrt{s}$ is confined within the region of size b), and it is the space where most easily the predictions based on gravitational EFT are made. This is not where crossing symmetry of the amplitude acts naturally, and therefore we can already sense a possibility of interesting interplay between gravitational EFT and basic principles!

To meaningfully talk about impact parameter we want it to be bigger than the Compton wavelength $b > \frac{\hbar}{\sqrt{s}}$, or $sb^2 \gg 1$. The same applies if we have some non-locality in the theory, e.g. a string scale ℓ_s.

2.2 Massless pole

Probably the most familiar feature of gravity is a pull at large distances. This is encoded in the massless pole as $t \to 0$ so that (we dropped terms $O(m^2/s)$)

$$T_{tree}(s,t) \simeq -\frac{8\pi G_N s^2}{t} + \dots \ , \tag{2.3}$$

This is a non-perturbative statement as $t \to 0$ at fixed s.[4]

A reader might be puzzled by what has just happened: is not gravity strongly coupled when $s \gg M_P^2$ and we cannot make any predictions? Quantum corrections come in integer powers of $G_N \sim \frac{1}{M_P^{D-2}}$ and by simple reasoning that spin two particles run in the loop we can only get one extra power of s per loop. Therefore to construct a dimensionless parameter we need something else and this is the role taken by t. We can therefore construct a dimensionless parameter

$$\Lambda = G_N s (-t)^{\frac{D-4}{2}}. \tag{2.4}$$

The statement of (2.3) is then that the regime $\Lambda \to 0$ is controlled by the IR physics in gravity even when $s \gg M_P^2$.

To make it less mysterious, it is useful to Fourier transform the result above and define (this is what is called the Breit frame)

$$T_{tree}(s,b) \equiv \frac{1}{2s}\int d^{D-2}\vec{q} e^{i\vec{q}\vec{b}} T_{tree}(s,-\vec{q}^2) = \frac{\Gamma(\frac{D-4}{2})}{\pi^{(D-4)/2}}\frac{G_N s}{b^{D-4}}, \tag{2.5}$$

which is nothing but (2.4) with $t \to \frac{1}{b^2}$. Therefore the statement simply becomes the one of *clustering* at large enough impact parameters as $b \to \infty$: we effectively get weakly coupled scattering at arbitrary high energies. This is the physical meaning of the statement that (2.3) is true nonperturbatively.[5] We can also rewrite the result as follows

$$T_{tree}(s,b) \sim (\sqrt{s}b)\Big(\frac{R_S}{b}\Big)^{D-3}. \tag{2.6}$$

We thus see that we can have at the same time $T_{tree}(s,b) \gg 1$, while $\frac{R_S}{b} \ll 1$. This will be relevant for the eikonal scattering that we discuss next.

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[4] In $D > 4$.

[5] In $D = 4$ things do not cluster because $T_{tree}(s,b) \sim \log b \to \infty$ and similarly (2.3) fails and one gets instead $(\frac{\mu_{IR}^2}{-t})^{1-iG_N s}$ [4].

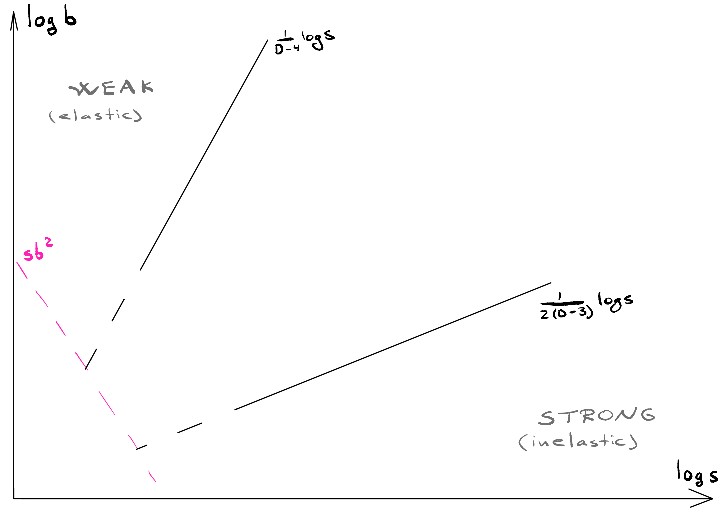

Figure 1: From weak to strong coupling in the impact parameter plane

2.3 Eikonal scattering

The eikonal approximation is a relativistic version of the WKB approximation. The scattering phase shift in this regime is large and dynamics is semi-classical. The leading order effect is exponentiation of the tree-level result.

We consider the following ansatz for the amplitude

$$T_{eik}(s,t) = 2is \int d^{d-2}\vec{b} e^{-i\vec{q}\vec{b}} \Big(1 - e^{2i\delta_{tree}(s,|\vec{b}|)} \Big), \tag{2.7}$$

where we take

$$2\delta_{tree}(s,|\vec{b}|) = T_{tree}(s,b), \tag{2.8}$$

from the previous section. The key point is that the number of gravitons exchanged is $\sim \delta_{tree}$, a typical momentum of each graviton is $\frac{1}{b}$ which is very small. Finally, the UV divergencies are short distance effects and we do not expect them to play any role at large impact parameters. We expect this to be true in any theory of gravity and this is true in string theory.

There are various ways to motivate this result [12]. Perhaps, the simplest one is to notice that it describes a very universal dynamical process. We consider a Lorentz frame where we can take one particle to be so energetic that it backreacts on the geometry. A gravitational field of a relativistic particle is given by the Aichelburg-Sexl shockwave

$$ds^2_{AS} = -dx^+dx^- - 4\frac{\Gamma(\frac{D-4}{2})G_N P_-}{\pi^{(D-4)/2} b^{D-4}} \delta(x^-)(dx^-)^2 + \sum_{i=1}^{D-2}(dx^i)^2, \quad b = \sqrt{\sum_{i=1}^{D-2}(x^i)^2}\ , \tag{2.9}$$

which corresponds to the stress tensor $T_{--} = -P_- \delta(x^-)\delta^{D-2}(\vec{x})$.

We then consider a probe particle that propagates through the shockwave [16]. We can solve the classical dynamics in this background. The leading order effect is the Shapiro time delay

$\Delta x^+ = 4\frac{\Gamma(\frac{D-4}{2})G_N P_-}{\pi^{(D-4)/2} b^{D-4}}$ which acts on the wavefunction as a phase $\psi \to e^{-i\Delta x^+ p_+}$. As a result one gets (2.8) using $s = 4P_- p_+$.

We therefore see that the Born result of the previous section can be extended using simple semi-classical computation. It is instructive to compute the integral above in the Regge limit $s \to \infty$ and t - fixed. This was done in [17] and after some trivial manipulations one gets at large s the following integral

$$T_{eik}(s,t) \simeq 2e^{-i(D-1)\pi/4} s(-t)^{\frac{2-D}{2}} \lambda^{(D-1)/2} \int_0^\infty du u^{(D-3)/2} \Big(e^{i\lambda(u+\frac{u^{D-4}}{D-4})} - 1 \Big) , \tag{2.10}$$

where the dimensionless parameter λ is given by

$$\lambda^{D-3} = \frac{2\Gamma(\frac{D-2}{2})}{\pi^{(D-4)/2}} \Lambda, \quad \Lambda = G_N s (-t)^{\frac{D-4}{2}} \ . \tag{2.11}$$

The Regge limit thus corresponds to taking the dimensionless constant $\Lambda \to \infty$. The integral can be evaluated using the saddle point approximation, which gives $u_0 = 1$. Let us notice for future reference that in terms of the impact parameters the location of the saddle, we will call it *the ACV saddle*, corresponds to

$$b_0^{D-3} \sim \frac{G_N s}{\sqrt{-t}}. \tag{2.12}$$

The leading result for the Regge limit $s \to \infty$, $t < 0$ and fixed is thus

$$T_{eik}(s,t) \simeq \frac{2^{\frac{D}{2}} e^{-\frac{1}{4} i\pi(D-2)} s(-t)^{\frac{2-D}{2}} \left(2\pi^{\frac{D-2}{2}} G_N s(-t)^{\frac{D-4}{2}} \Gamma\left(\frac{D}{2}-2\right)\right)^{\frac{D-2}{2(D-3)}}}{\sqrt{D-3}} e^{i\lambda \frac{D-3}{D-4}} \Big(1 + O(1/\sqrt{\lambda}) \Big) . \tag{2.13}$$

Notice that

$$|T_{eik}(s,t)| \sim s^{2-\frac{D-4}{2(D-3)}} \ , \tag{2.14}$$

which in particular means that it grows strictly slower than s^2, something that we will discuss again later.

As we take $D \to 4$ the phase of the amplitude develops a pole $\frac{1}{D-4}$ which we can substitute to an IR regulator $-\log \mu_{IR}$. We then get

$$T_{eik}(s,t) \simeq i\frac{8\pi G_N s^2}{t} \Big(\frac{-t}{\mu_{IR}^2} \Big)^{iG_N s} e^{-2iG_N s(\log(G_N s)-1)}. \tag{2.15}$$

Let us emphasize again the difference between the formulas in this section and the universal $1/t$ pole (2.3). We get the universal pole as we take $t \to 0$ with s fixed, so that the dimensionless coupling $\Lambda \to 0$. Here we instead keep t fixed and send $s \to \infty$ so that $\Lambda \to \infty$. Curiously both regimes are controlled by the leading term in the gravitational EFT (or general relativity) as we argue next.

Similar formulas arise very naturally in CFTs when doing the OPE and focusing on the contribution of double trace operators [18].

Problem: Use CFT eikonal representation to derive a CFT analog of (2.13) and see what is the Regge intercept that it produces.

It is interesting to discuss the expected regime of validity of the eikonal computation. The average number of exchanged gravitons is $\langle N \rangle \sim \delta_{tree} \gg 1$ required for the validity of the semi-classical approximation. It is also interesting to compute the curvature invariant in the region where particles collide. We would like to impose

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \ll \frac{1}{\ell_{Pl}^4} \ . \tag{2.16}$$

A simple estimate $\ell_{Pl}^4 \left(\frac{G_N s}{b^{D-2}} \right)^2 \frac{1}{\Delta u \Delta v}$, where $\Delta u \Delta v$ is the shock width which we take using quantum uncertainty to be minimal $\sim s$, gives, see [15, 19],

$$\frac{\delta_{tree}^{D/2}}{s b^2} \ll 1 \ , \tag{2.17}$$

which in particular implies that

$$b \gg R_S \ . \tag{2.18}$$

2.4 Further corrections: tidal excitations, KK modes, gravity waves and all that

An obvious question is: what does the eikonal computation tell us about the behavior of the actual amplitude $T(s,t)$, or in other words what is the regime of validity of the eikonal computation. This question has been famously analyzed in [5].

First type of correction is the so-called tidal excitations. Imagine that a particle has inner degrees of freedom (it could be hydrogen energy levels, string excitation modes, etc) of characteristic size $\delta l \sim \frac{1}{m_*}$. We then expect that as a probe passes through the shockwave, it experiences a tidal force and it will get excited when $m_* \sim \delta l F$, where F is the tidal force. For the shockwave above $F \sim$ Riemann $\sim \partial_b^2 \delta_{tree}$ therefore we expect tidal inelastic effects to be controlled by the dimensionless parameter (i stands for inelastic effects because it reduces the amplitude when plugged in (2.8))

$$\delta_{tidal}(s,b) = i \frac{G_N s}{m_*^2 b^{D-2}} . \tag{2.19}$$

This is indeed confirmed by an explicit computation in string theory [5], but we expect this result to be very general and does not depend on the nature of compositeness of the scattered particles. Notice that plugging (2.19) into the curvature estimate we get

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \sim m_*^4 \ . \tag{2.20}$$

For example, if we scatter protons there will be hadronic diffractive excitations when $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \sim (\Lambda_{QCD})^4$.

Second universal effect is production of gravity waves. As particle pass each other and get deflected they emit gravity waves. The leading effect has been again computed by ACV in [5] with the following result

$$\delta_{GW}(s,b) = i \frac{G_N^3 s^2}{b^{3D-10}} \ . \tag{2.21}$$

If we take $b \sim (G_N^3 s^2)^{1/(3D-10)}$, we get that the curvature invariant in the previous section becomes large as $s \to \infty$ in $D > 6$.[6]

Therefore this regime is not under good theoretical control at asymptotically high energies [19]. However, following [15] we can introduce a smeared amplitude

$$T_{\Delta s}(s,b) = \int_0^\infty \frac{ds'}{\sqrt{2\pi}\Delta s} e^{-(s'-s)^2/(\Delta s)^2} T(s',b). \tag{2.22}$$

The estimate of the curvature in the collision region then changes to $\ell_{Pl}^4 \left(\frac{G_N s}{b^{D-2}}\right)^2 \Delta s$.

An important comment: The actual computation contains an extra $\log s$. This will not be relevant in the Regge limit, but it is important if we want to go to fixed angles. *The ACV conjecture* states that all such corrections get dressed into the ratios of the type we considered above, e.g. $\log R_S/b$, and therefore do not cause any problem.

Finally, we could have a KK scale b_{KK}, however, compared to other scales discussed so far it does not grow with energy and thus does not affect the discussion. The basic picture of the high-energy scattering in gravity therefore takes the following form.

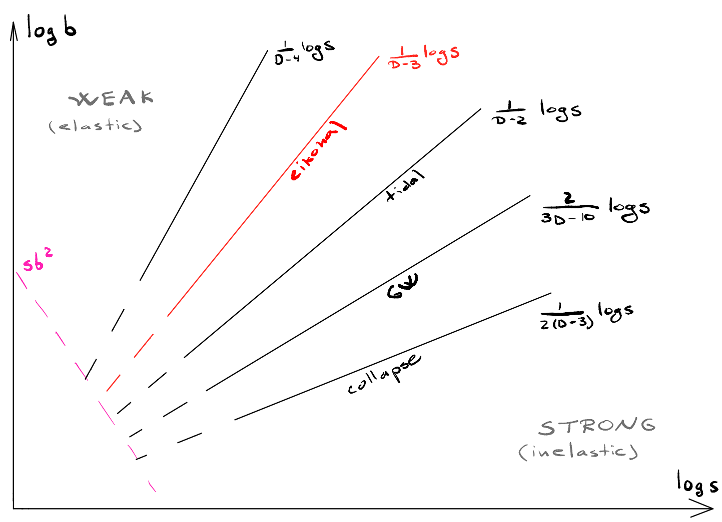

Figure 2: From weak to strong coupling in the impact parameter plane: eikonal, tidal, gravity waves.

From this we see that the eikonal saddle point lies in the regime which is still well controlled by the tree-level phase shift and all inelastic effects go to zero at large $s \to \infty$. It motivates the following conjecture about the Regge limit of gravitational amplitudes.

Quantum Regge growth conjecture: In the UV completions of Einstein gravity the Regge limit of the amplitude is given by (2.13)

$$\lim_{s\to\infty} \frac{T(s,t)}{T_{eik}(s,t)} = 1. \tag{2.23}$$

Notice that on the ACV saddle $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{ACV} \to 0$ at large s. In other words, the Regge limit is an IR limit in gravity.

[6] This is also related to what is called the D'Eath bound which predicts a breakdown of the PM expansion (perturbation theory in R_S/b) in the certain regimes. This conjecture has been tested in the soft limit.

Finally, if we consider the collapse region $b \sim R_S$ we get the following bound on smearing [15] (to keep curvatures small and make semi-classical reasoning applicable)

$$\frac{\Delta s}{R_S^2 M_P^4} \ll 1 \ . \tag{2.24}$$

The conclusion is that smearing is essential to apply the semi-classical approximation in the collapse region.

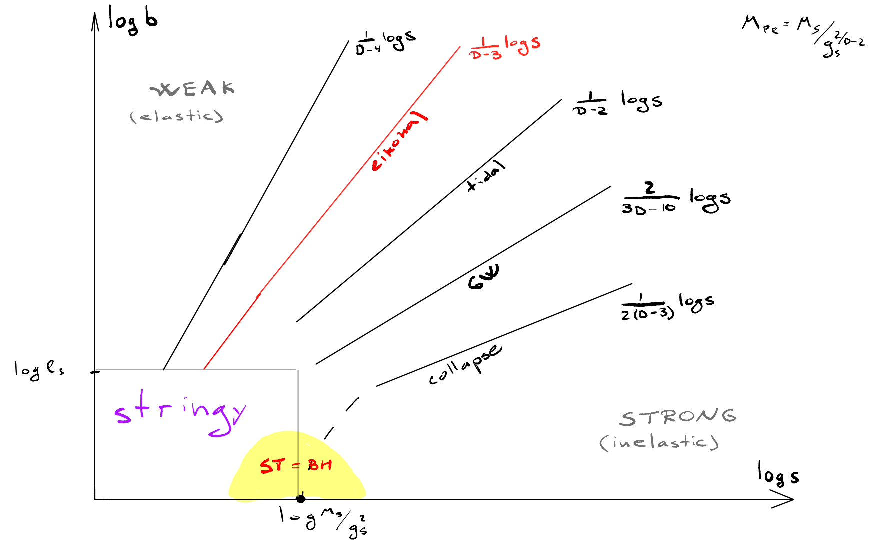

Figure 3: From weak to strong coupling in the impact parameter plane: eikonal, tidal, gravity waves, stringy gravity.

2.5 Fixed angles

As we do the Fourier transform we get *the ACV saddle*

$$q = -\frac{\partial 2\delta_{tree}}{\partial b}, \quad t = -q^2, \tag{2.25}$$

which becomes at high energies and using $\theta \simeq \frac{2q}{\sqrt{s}}$

$$\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)} \left(\frac{R_S}{b_{ACV}}\right)^{D-3} \ . \tag{2.26}$$

The key point is that if we take $\theta \ll 1$ and fixed we get that the dominant scattering comes from very large impact parameters at high energies. As we increase the energy this $b_{ACV}(s, \theta)$ grows with energy as $\sqrt{s}$ and eventually we enter into the regime where the amplitude is very inelastic and is controlled by the gravity waves emission.

We can therefore estimate the behavior of the amplitude in this regime as follows

$$|T(s,\theta)| \sim e^{-\delta_{GW}} = e^{-G_N^3 s^2 b_{ACV}^{3D-10}} = e^{-S_{BH}(s) c_0 \theta^{\frac{3D-10}{D-3}}}, \quad \theta \ll 1. \tag{2.27}$$

As we increase the scattering angle inelasticity grows.

High-energy fixed-angle suppression hypothesis: In gravitational theories the amplitude at high energies decays exponentially fast

$$|T(s, \cos\theta)| \simeq e^{-S_{BH}(s) f(\theta)}, \tag{2.28}$$

where $f(\theta) \sim \theta^{\frac{3D-10}{D-3}} \to 0$ at small angles (modulo $\log \theta$). For spherical shells in AdS a version of this was confirmed in [11].

Notice that at small θ the physics of this formula has nothing to do with black holes. It is however expected that when we enter the collapse region $\theta = O(1)$, and $b \sim R_S$ it does.[7]

Problem: Derive the small angle asymptotic of $f(\theta)$ in gravity exaclty. This requires understanding $\log s$ in the LO computation and how it becomes something like $\log \theta$.

3 Bootstrap Test: graviton pole from dispersion relations

In the previous section we reviewed very general properties of the gravitational scattering amplitude $T(s,t)$ and ended up concluding that the Regge limit of the amplitude is controlled by simple physics. It is interesting next to put this conclusion to test.

3.1 Unitarity

We can consider scattering of identical particles $A, A \to A, A$ in which case the partial wave expansion takes the form

$$T(s,t) = \sum_{J=0, J-\text{even}}^{\infty} n_J^{(d)} f_J(s) P_J^{(d)}\Big(1 + \frac{2t}{s-4m^2}\Big), \tag{3.1}$$

or of non-identical scalars $A, B \to A, B$ in which case we get

$$T(s,t) = \frac{1}{2}\sum_{J=0}^{\infty} n_J^{(d)} f_J(s) P_J^{(d)}\Big(1 + \frac{2t}{s-4m^2}\Big), \tag{3.2}$$

where the conventions can be found in [20]. Convergence properties of the partial wave expansion are the following [21]:

- in $D > 7$ it converges absolutely;
- in $D = 6, 7$ it is convergent in the sense that the limit $\lim_{J_{max}\to\infty} \sum_{J=0}^{J_{max}}$ exists;
- in $D = 4, 5$ it converges in the sense of distributions, see [22].

In particular, it means that we can consider smeared amplitudes [23]

$$T_\psi(s) \equiv \int_0^{q_0} dq\; q[\psi(q) T(s, -q^2)]. \tag{3.3}$$

[7] To the best of my knowledge this is not everyone's expectation in the community. For example, D. Gross argues that scattering amplitudes never decay faster than $e^{-\#\sqrt{s}\log s}$ even in gravity (private communication).

such that

$$\psi_{\mathsf{a},\mathsf{b}}(q) \overset{q\to 0}{\sim} q^{\mathsf{a}}, \quad \mathsf{a} > 0,$$
$$\psi_{\mathsf{a},\mathsf{b}}(q) \overset{q\to q_0}{\sim} (q_0 - q)^{\mathsf{b}}, \quad \mathsf{b} \geq 0, \tag{3.4}$$

And apply the partial wave expansion to them.[8] The swapping property should be checked, as in [25], but the result is that it holds.

Let us define the full partial wave scattering amplitude

$$S_J(s) \equiv 1 + i\frac{(s-4m^2)^{\frac{d-3}{2}}}{\sqrt{s}} f_J(s). \tag{3.5}$$

Unitarity states that

$$|S_J(s)| \leq 1, \quad s \geq 4m^2. \tag{3.6}$$

Equivalently, we can rewrite it as follows

$$2\text{Im} f_J(s) \geq \frac{(s-4m^2)^{\frac{d-3}{2}}}{\sqrt{s}} |f_J(s)|^2 \ . \tag{3.7}$$

This can be solved in terms of the phase shifts $\delta_J(s)$

$$S_J(s) = e^{2i\delta_J(s)}, \quad f_J(s) = \frac{\sqrt{s}}{(s-4m^2)^{\frac{d-3}{2}}} i(1 - e^{2i\delta_J(s)}) \ , \tag{3.8}$$

with the unitarity constraint being $\text{Im}[\delta_J(s)] \geq 0$.

To relate to the semi-classical discussion in the impact parameter space we notice that the classical relation is

$$J = \frac{\sqrt{s}b}{2} \gg 1 \ . \tag{3.9}$$

Recall that this definition of b (sometimes called b_J) is different from the one used in the previous section in the Breit frame

$$b_J = b \cos\frac{\theta}{2} \ . \tag{3.10}$$

We then get

$$\delta_{J=\frac{\sqrt{s}b}{2}}(s) \simeq \delta(s,b), \qquad \sqrt{s}b \gg 1. \tag{3.11}$$

We therefore see that the amplitude (2.10) describes purely elastic scattering $\text{Im}\delta(s,b) = 0$.

One can also project it to partial waves using the following formula [26]

$$S_J(s) = \sqrt{s} \int_0^\infty J_{d-3+2J}(\tilde{b}\sqrt{s}) e^{2i\delta(s,\tilde{b})} d\tilde{b} \,, \tag{3.12}$$

and check that unitarity is satisfied at large but finite s and J. This computation was erroneously done in [17] with the opposite conclusion.

[8]Curiously, smeared amplitudes were considered by Martin in 1966 [24] for precise the same reason they are used in the modern S-matrix bootstrap.

3.2 Dispersion relations and the graviton pole

One can carefully carry out the arguments above together with the assumption of *subexponentiality* in the upper half-plane in s

$$|T(s,t)| \leq e^{Cs^{\beta}}, \quad \beta < 1 \tag{3.13}$$

to conclude that [21]

$$\lim_{|s|\to\infty} \frac{T(s,t)}{s^2} = 0. \tag{3.14}$$

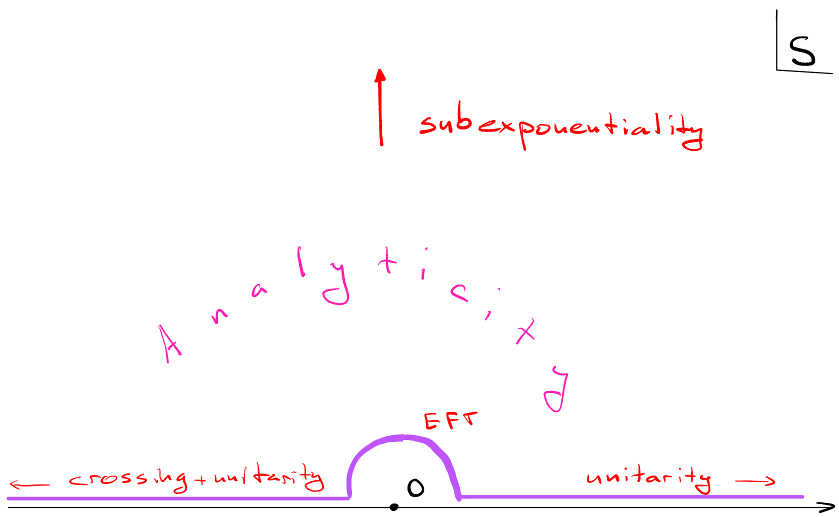

Figure 4: Basic principle of ACU imply the twice-subtracted dispersion relation.

This means that we can write down twice-subtracted dispersion relations which have been explored meticulously in the recent years. In particular, we can write the following relationship between the graviton pole (2.3) and the nonperturbative Regge limit that we conjectured so far

$$\frac{8\pi G_N}{q^2} = \frac{2}{\pi} \int_{s_0}^{\infty} \frac{ds'}{(s')^3}\, T_s(s', t = -q^2), \quad q \to 0, \tag{3.15}$$

where an important factor of 2 in the RHS comes from *crossing*. The factors of s canceled on both sides because we are doing two subtractions.

Indeed, it is clear that any finite energy integral cannot reproduce the pole through the dispersion relations. Therefore dispersion relations (derived using bootstrap constraints) provide a nontrivial relationship between the two facts that we argued for using gravitational EFT (graviton pole, and the Regge limit).

Let us therefore test if the relationship above holds. We plug for the imaginary part

$$T_s(s, t = -q^2) = 2s(2\pi)^{\frac{D-2}{2}} \int_0^\infty db b^{D-3} (qb)^{\frac{4-D}{2}} J_{\frac{D-4}{2}}(bq) 2\sin^2 \delta_{tree}(s,b). \tag{3.16}$$

Model	Pointwise	Smeared
Born	$\lvert s\rvert^{2-\frac{d-7}{2(d-4)}}$	$\lvert s\rvert^{2-\min(1,\frac{\mathsf{a}}{d-4},\frac{\mathsf{b}+\frac{d-5}{2}}{d-4})}$
Eikonal+tidal$\big\rvert_{d>5}$	$\lvert s\rvert^{2-\frac{d-4}{2(d-3)}}$	$\lvert s\rvert^{2-\min(1,\frac{\mathsf{a}}{d-4},\frac{\mathsf{b}+\frac{d-1}{2}}{d-2})}$
Eikonal+GW$\big\rvert_{d=5}$	$\lvert s\rvert^{2-\frac{1}{5}}$	$\lvert s\rvert^{2-\min(1,\mathsf{a},\frac{2\mathsf{b}+3}{5})}$

Table 1: Summary of the asymptotic Regge bounds derived in this paper for the elastic $2 \to 2$ scattering amplitudes in a gravitational theory in $D > 4$. Pointwise bounds refer to $\lim_{|s|\to\infty} |T(s,t)| \lesssim s^{\#}$ for $t < 0$. Smeared bounds refer to the Regge bound $\lim_{|s|\to\infty} |T_{\psi_{\mathsf{a},\mathsf{b}}}(s)| \lesssim s^{\#}$ on the scattering amplitude smeared over the transferred momenta (3.3), where the smearing function $\psi_{\mathsf{a},\mathsf{b}}(q)$ satisfies (3.4). Different models refer to the different large impact parameter ansätze that have been used to estimate the amplitude.

The relevant integral takes the form

$$2\int_{s_0}^{\infty} \frac{ds'}{s'^2} \sin^2\left(\frac{\Gamma\left(\frac{D-4}{2}\right)}{2\pi^{\frac{D-4}{2}}} \frac{G_N s'}{b^{D-4}}\right) \simeq 2\frac{\Gamma\left(\frac{D-4}{2}\right)}{2\pi^{\frac{D-4}{2}}} G_N b^{4-D} \int_0^{\infty} \frac{d\tilde{x}}{\tilde{x}^2} \sin^2(\tilde{x}) \tag{3.17}$$

$$= \pi \frac{\Gamma\left(\frac{D-4}{2}\right)}{2\pi^{\frac{D-4}{2}}} G_N b^{4-D} \ . \tag{3.18}$$

where we changed variable to $\tilde{x} = \frac{\Gamma\left(\frac{D-4}{2}\right)}{2\pi^{\frac{D-4}{2}}} \frac{G_N s'}{b^{D-4}}$ and the finite energy s_0-dependent correction is subleading at large b and does not contribute to the $1/t$ pole. We therefore get

$$\frac{8\pi G_N}{q^2} = 4(2\pi)^{\frac{D-2}{2}} \int_0^{\infty} db b^{D-3} (qb)^{\frac{4-D}{2}} J_{\frac{D-4}{2}}(bq) \left(\frac{\Gamma\left(\frac{D-4}{2}\right)}{2\pi^{\frac{D-4}{2}}} G_N b^{4-D}\right). \tag{3.19}$$

This is indeed a true identity. The integral converges for $D > 5$ and one can repeat the whole argument in a careful tauberian manner [27] by smearing in q and making the RHS non-negative such that $D > 4$ works.

It would be interesting to explore in the same way universal gravitational loop corrections. Recently this has been done for stringy amplitudes [28].

3.3 Crossing symmetry

These arguments suggest that there exists a class of amplitudes that on one hand satisfy dispersion relations and crossing symmetry, and, on the other hand, reproduce the eikonal behavior of the amplitude.

Problem: Construct such amplitudes. Only the full crossing-symmetry is nontrivial, the $s-u$ crossing symmetry is manifest in the dispersion relations (and in fact has been used by ACV).

In the current implementation of the primal bootstrap the eikonal physics is not correctly reproduced.

3.4 Eikonal in CFTs

In principle we can try repeating the same exercise in a gravitational CFT [18]. It would be interesting to see if plugging the CFT eikonal result of [18] into the CFT dispersion relations [29] correctly reproduced the graviton pole.

4 Lower bound on the scattering amplitude at fixed angles in gravity

Recall that $S_{BH}(s) \sim (\sqrt{s})^{1+\frac{1}{D-3}} > \sqrt{s}$ which means that it violates the Cerulus-Martin bound [30]. A simple argument, see [31], then shows that *if* there is maximal analyticity in gravity *then* there is no Mandelstam representation (equivalently, the amplitude is not polynomially bounded for unphysical t).

The old argument of Cerulus and Martin [30] relates polynomial boundedness of the amplitude for *unphysical values* of t to certain properties of the amplitudes in the Regge limit and high energy scattering at fixed angles.[9] While the original argument was done for gapped theories, it is a straightforward exercise to relax this assumption.

Consider the scattering amplitude as a function of fixed angle $z \equiv \cos\theta$, namely $\mathcal{T}(s,z) \equiv T\Big(s, t = -\frac{s}{2}(1-z)\Big)$. We fix s to be real and positive,[10] and we consider $\mathcal{T}(s,z)$ in the complex z-plane. *Maximal analyticity* implies that $\mathcal{T}(s,z)$ is analytic, modulo the two cuts $z \in (-\infty,-1] \cup [1,\infty)$ which correspond to scattering in the u- and t-channel correspondingly.

We take three real z's such that $0 < z_1 < z_2 < z_3 < 1$ and we map the z-plane to the τ-plane. We first transform

$$w(z) = \frac{1}{z}(1 - \sqrt{1-z^2}), \tag{4.1}$$

which maps the cut z-plane inside the unit circle in the w-plane. We then consider the following map

$$\tau(w) = \frac{1}{w(z_1)}(w + \sqrt{w^2 - w(z_1)^2}). \tag{4.2}$$

This mapping maps the region $-z_1 \leq z \leq z_1$ in the z-plane to the unit circle $|\tau| = 1$.

Consider now another pair of circles in the τ-plane of radii $r_2 = \tau(w(z_2))$ and $r_3 = \tau(w(z_3))$. In the original z-plane the circles map in the oval shape region, see Figure 5.

By assumption, maximal analyticity implies that the scattering amplitude is analytic in the annulus $1 \leq |\tau| \leq r_3$. Let us introduce the maximal value of the amplitude on a given circle

$$M_r \equiv \max_{|\tau|=r} |\mathcal{T}(s,z)|. \tag{4.3}$$

[9] Recently, this argument was generalized to rely on the axiomatic QFT analyticity only [32]. The resulting Epstein-Martin bound however is very (ridiculously) weak.

[10] We approach the real axis from above as usual.

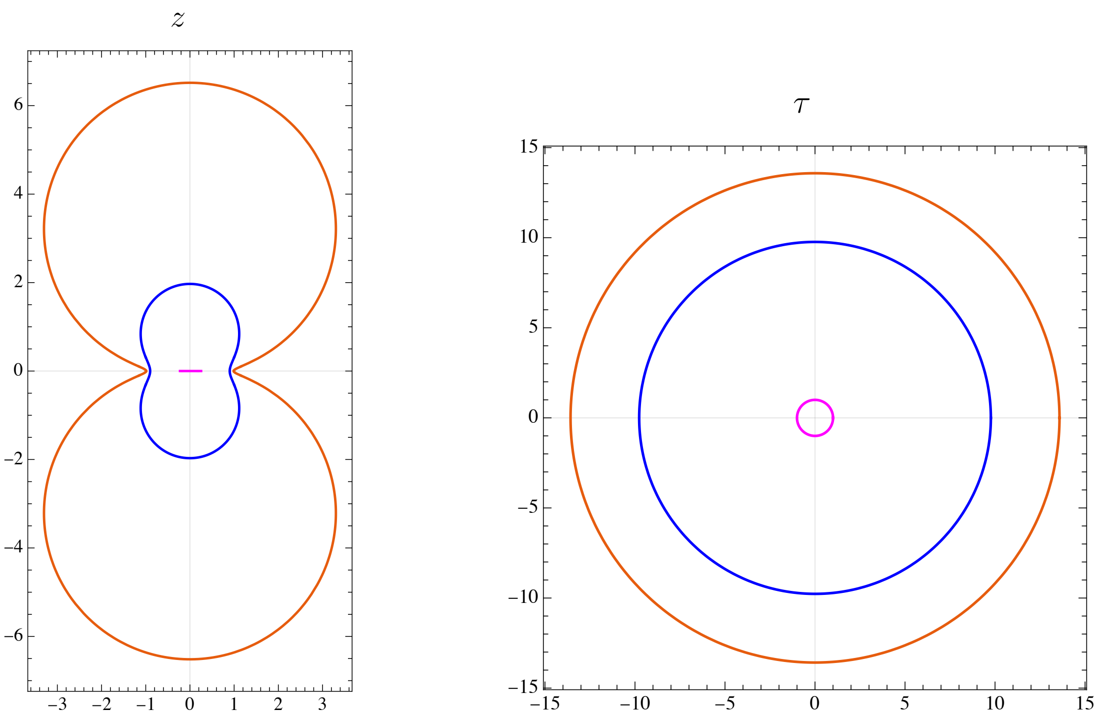

Figure 5: Here we depict regions in the z-plane which map to the concentric circles in the τ-plane. Here we consider $z_1 = \frac{1}{4}$ so that the region (magenta) $-\frac{1}{4} \leq z \leq \frac{1}{4}$ is mapped to the unit circle in the τ-plane. The blue region corresponds to $z_2 = 0.9$ and the red region comes from $z_3 = 0.99$ in the argument.

We then apply the Hadamard three-circle theorem, see e.g. chapter 23 in [33], that states that for $1 < r_2 < r_3$ we have the following inequality

$$M_{r_2} \leq M_1^{1-\frac{\log r_2}{\log r_3}} M_{r_3}^{\frac{\log r_2}{\log r_3}}. \tag{4.4}$$

This constraint becomes particularly interesting if we choose $z_{2,3} = 1 + \frac{2t_{2,3}}{s}$, where $t_2 < t_3 < 0$ are fixed momenta and we take $s \to \infty$.

Let us introduce, following Cerulus and Martin, the upper bound on the fixed angle scattering amplitude at high energies

$$|\mathcal{T}(s,z)| \leq e^{-\phi(s)}, \quad -a \leq z \leq a < 1. \tag{4.5}$$

We can then rewrite (4.4) as follows

$$|T(s,t_2)| \leq e^{-C(a)\phi(s)\frac{\sqrt{-t_2}-\sqrt{-t_3}}{\sqrt{s}}} \max_{|\tau|=r(s,t_3)} |\mathcal{T}(s,z)|, \quad s \to \infty, \tag{4.6}$$

where $C(a) > 0$ and we used that $M_{r_2} \geq |T(s,t_2)|$. This equation bounds the Regge limit of the amplitude, the LHS, in terms of the fixed angle scattering, and the value of the amplitude in certain sub-domain of the z-plane, the RHS.

The validity of the Mandelstam representation implies that there should exist an integer N, such that

$$\max_{|\tau|=r(s,t_3)} |\mathcal{T}(s,z)| \leq |s|^N, \quad |s| \to \infty. \tag{4.7}$$

Assuming that $|T(s,t_2)|$ is polynomially bounded from below (it cannot decay too fast), we then get the Cerulus-Martin bound on the fixed angle scattering for amplitudes that admit Mandelstam representation

$$\phi(s) \leq c_0 \sqrt{s} \log s, \quad s \to \infty. \tag{4.8}$$

In other words, maximal analyticity implies the following schematic Cerulus-Martin relation

$$\text{Regge} \leq \text{Fixed angle} \times \text{Mandelstam}. \tag{4.9}$$

As we reviewed, the Regge limit for gravitational amplitudes is controlled by the large impact parameter scattering and its absolute value behaves polynomially in s. The fixed angle scattering, on the other hand, is believed to be entropically suppressed

$$\phi_{\text{QG}} \sim (\sqrt{s})^{1+\frac{1}{D-3}}, \tag{4.10}$$

which clearly violates (4.8).

Our conclusion is that the *expected* properties of the scattering amplitude in gravitational theories for physical t (polynomial behavior in the Regge limit and exponentially faster than $\sqrt{s}$ decay for fixed angle scattering) are *not* compatible with polynomial boundedness needed for the Mandelstam representation.

5 Stringy features

The only available UV completion of gravity in flat space is string theory. It comes with a new scale $M_s = \frac{1}{\ell_s}$ which we assume to be $M_s \ll M_{Pl}$ (or $g_s \ll 1$). It is also characterized by the string coupling g_s such that

$$16\pi G_N = 2^{-\frac{D-10}{2}} (2\pi)^{D-3} \frac{g_s^2}{M_s^{D-2}} \ . \tag{5.1}$$

In other words, $g_s^2 \sim \left(\frac{M_s}{M_{Pl}}\right)^{d-2}$, or in other words $M_{Pl} = \frac{M_s}{g_s^{2/(D-2)}}$. Stringy effects were discussed in [5, 34, 35].

An interesting scale to consider is the so-called *correspondence point* s_* defined by the condition $R_S(s_*) = \ell_s$,[11] which gives

$$\sqrt{s_*} \simeq \frac{M_s}{g_s^2} \ . \tag{5.2}$$

For a recent discussion of this regime see e.g. [36].

For scattering at energies $s < s_*$ we do not expect that black holes play any role, in other words gravity is *stringy*. A proposal is that in this region the expansion $\frac{R_S}{b}$ becomes $\frac{R_S}{\ell_s}$ and therefore we can explore stringy effects while neglecting radiative corrections.

Another important scale is

$$\sqrt{s_{str}} \simeq \frac{M_s}{g_s}, \tag{5.3}$$

[11] There could be other interesting scales, e.g. $R_S(s_*) = \ell_{KK}$.

at this energies stringy corrections to the tree-level phase shift acquire sizable imaginary part.

There are two main effects to consider: tidal excitation of the string due to the leading Regge trajectory exchange in the t-channel; string production in the s-channel. The key point about the first one is that it becomes important for $b \gg \ell_s$. On the other hand, the second effect only kicks in for $b \leq \ell_s \log \alpha' s$.

5.1 Correction to the phase shift

In the formula above we used the phase shift computed in general relativity. It receives stringy corrections as well. To compute them let us consider four dilaton amplitude in string theory

$$T(s,t) = 8\pi G_N \Big(\frac{su}{t} + \frac{tu}{s} + \frac{st}{u}\Big) \frac{\Gamma(1-s/2)\Gamma(1-t/2)\Gamma(1-u/2)}{\Gamma(1+s/2)\Gamma(1+t/2)\Gamma(1+u/2)}. \tag{5.4}$$

In the Regge limit $s \to \infty(1+i0)$ we get

$$T(s,t) = T^{Regge}(s,t)\Big(1 + \frac{t(t+2)}{2s} + \frac{t\left(3t^3+8t^2+48t-8\right)}{24s^2} + ...\Big) , \tag{5.5}$$

where

$$T^{Regge}(s,t) = -8\pi G_N \frac{s^{2+t}}{t} \frac{2^{-t}\Gamma(1-t/2)}{\Gamma(1+t/2)} e^{-\frac{1}{2}i\pi t} \tag{5.6}$$

$$= -8\pi G_N \frac{s^2}{t} e^{tY} \left(1 + \gamma_{\mathrm{E}} t + ...\right) \quad , \quad Y = \alpha' \log(-i\alpha' s/2) . \tag{5.7}$$

We can encode various correction by noticing that we can do the following substitution $t \to \partial_Y$.

We next transform to the impact parameter space. To do it is useful to rewrite the basic amplitude as follows

$$8\pi G_N \frac{s^2}{(-t)} e^{tY} = 8\pi G_N s^2 Y \int_0^1 \frac{d\rho}{\rho^2} e^{\frac{tY}{\rho}}. \tag{5.8}$$

After this the transform becomes a simple Gaussian integral and we get

$$T_{string-tree}(s,b) \equiv \frac{1}{2s} \int d^{D-2}\vec{q} e^{i\vec{q}\vec{b}} \left(8\pi G_N \frac{s^2}{\vec{q}^2} e^{-\vec{q}^2 Y}\right) = 4\pi G_N \frac{(4\pi)^{1-\frac{d}{2}} s}{Y^{\frac{d-4}{2}}} \int_0^1 d\rho \rho^{d/2-3} e^{-\frac{b^2 \rho}{4Y}} \tag{5.9}$$

$$= \left(\Gamma\left(\frac{D-4}{2}\right) - \Gamma\left(\frac{D-4}{2}, \frac{b^2}{4Y}\right)\right) \frac{1}{\pi^{(D-4)/2}} \frac{G_N s}{b^{D-4}}. \tag{5.10}$$

We can now consider the large $b \gg Y \sim \log \alpha' s$ expansion of this formula we get

$$T_{string-tree}(s,b) \simeq T_{tree}(s,b) - 2^{6-D} \pi^{2-\frac{D}{2}} \frac{G_N s}{b^{D-4}} e^{-\frac{b^2}{4Y}} \left(\frac{b^2}{Y}\right)^{\frac{D}{2}-3} . \tag{5.11}$$

The phase shift therefore develops an imaginary part

$$\mathrm{Im} 2\delta_{string-tree}(s,b) \simeq 2\pi^2 e^{-\frac{b^2}{4Y}} \frac{G_N s}{(4\pi Y)^{\frac{D-2}{2}}}, \tag{5.12}$$

and we observe the transverse spreading of the string $b_s^2 \sim \alpha' \log \alpha' s$ at high energies. At small impact parameters $b \ll Y$ instead we have

$$2\delta_{string-tree}(s,b) \simeq \frac{2G_N s}{(4\pi Y)^{\frac{D-4}{2}}}\Big(\frac{1}{D-4} - \frac{1}{D-2}\frac{b^2}{4Y} + ...\Big). \tag{5.13}$$

This imaginary part is related to production of strings in the s-channel.

5.2 Propagation through the shock wave

First let us consider string propagation through a shock wave. The problem was first solved in [37] (and many papers followed). Recall that the string mode operators obey

$$[\alpha_n^i, \alpha_m^j] = n\delta_{n+m,0}\delta^{ij}, \tag{5.14}$$

where negative modes $n < 0$ create string excitations, while positive modes $n > 0$ annihilate the vacuum

$$\alpha_{n>0}|0\rangle = 0. \tag{5.15}$$

We choose the conformal gauge for the worldsheet metric $h^{\alpha\beta} = \eta^{\alpha\beta}$, and fix the light-cone gauge

$$u(\sigma,\tau) = P^u \tau. \tag{5.16}$$

The closed string mode expansion takes the form

$$X^i(\sigma,\tau) = x^i + p^i\tau + \frac{i}{2}\sqrt{\alpha'}\sum_{n\neq 0}\left[\tilde{\alpha}_n^i e^{-2in\tau} - \alpha_n^{i\dagger}e^{2in\tau}\right]e^{-2in\sigma}, \quad (i = 2,\dots,D-1), \tag{5.17}$$

where $\alpha_n^{i\dagger} = \alpha_{-n}^i$.

Before and after a shock, the string propagates freely. If the shock geometry has the metric

$$ds^2 = -dudv + \delta(u)f(\vec{x})du^2 + d\vec{x}^2\,, \tag{5.18}$$

the transition through the shock is described by the S-matrix [37],

$$S_{\text{shock}} = e^{\frac{i}{2}P^u \int_0^\pi f(\vec{X}(\sigma,0))\frac{d\sigma}{\pi}}. \tag{5.19}$$

As an example, consider a shock created by a fast-moving particle at position $\vec{x}_a$,

$$f_a(\vec{X}(\sigma,0)) = \frac{\Gamma(\frac{D-4}{2})}{\pi^{\frac{D-2}{2}}}\frac{2G_N p^v}{((\vec{X}(\sigma,0)-\vec{x}_a)^2)^{\frac{D-4}{2}}} \tag{5.20}$$

In writing f_a above, there is an ambiguity in the ordering of operators $X^i(\sigma,0)$. However, this ambiguity is proportional to $\vec{q}_1^{\,2}$ and is localized at zero impact parameter upon doing the Fourier transform. It is therefore irrelevant for our purposes.

Note that the the operator S_{shock} is diagonal in the position basis $X^i(\sigma,0)$ for the transverse oscillators. Thus, it instantaneously changes the momenta of the oscillators without affecting their positions. Overall, the effect of the shock on the string is the same as in the geodesic calculation: the center of mass of the string moves in the v direction (Shapiro time delay), and the transverse modes receive an instantaneous kick that depends on the profile $f_a(\vec{X})$. Essentially, each part of the string individually follows a geodesic through the shock.

Let us then compute the leading order stringy correction to the propagation through the shock at large impact parameters. We can do it by expanding f_a in powers of $\vec{X}(\sigma, 0)$ with the leading contribution coming the quadratic term

$$\hat{\delta} = \delta(s,b) + \frac{1}{2}\frac{\partial^2 \delta}{\partial b^i \partial b^j}(\hat{X}^u \hat{X}^u + \hat{X}^d \hat{X}^d) + \dots \ . \tag{5.21}$$

The quartic term is eventually suppressed by $\frac{\ell_s^2 \log s}{b^2} \ll 1$ for $b > \ell_s \sqrt{\log s}$ and by $1/Y$ in the opposite regime $b \ll \ell_s \sqrt{\log s}$. We can write

$$\frac{\partial^2 \delta}{\partial b^i \partial b^j} = \Delta_\perp (\delta_{ij} - \frac{b_i b_j}{b^2}) + \Delta_{||} \frac{b_i b_j}{b^2} \ . \tag{5.22}$$

which for $b \gg \ell_s$ becomes

$$\langle 0 | S_{\text{shock}} | 0 \rangle = \Big(\frac{\pi \Delta}{\sinh \pi \Delta} \Big)^{D-3} \frac{\pi (D-3)\Delta}{\sinh \pi (D-3)\Delta} \sim e^{-2\pi (D-3)\Delta}, \tag{5.23}$$

and we have $\Delta_{||} = (D-3)\Delta_\perp \equiv (D-3)\Delta$ with

$$\Delta = \pi^{2-\frac{D}{2}} \Gamma \left(\frac{D}{2} - 1 \right) \frac{G_N s \alpha'}{b^{D-2}}, \tag{5.24}$$

where $\alpha' = \ell_s^2$. This is the promised tidal effect on a string propagating through the shock wave.

In the treatment above we treated the background and the probe asymetrically. A more symmetric formula takes the form

$$S_{ACV} = e^{\int_0^\pi \frac{d\sigma_u d\sigma_d}{\pi^2} : \hat{\delta}_{tree}\left(s, X_u^i(\sigma_u) - X_d^i(\sigma_u)\right):}, \tag{5.25}$$

which is indeed what comes out from doing the stringy eikonal. This is a manifestly unitary S-matrix that acts on the two-string Hilbert state. Evaluating the S-matrix in the oscillator vacuum we then get [5]

$$\langle 0 | S_{\text{shock}} | 0 \rangle = e^{2i\delta} \left[\Gamma(1 - i\Delta_\perp) \right]^{2(D-3)} \Gamma^2(1 - i\Delta_{||}) \tag{5.26}$$

For small impact parameters we get instead

$$S(s,b) \sim e^{-\frac{G_N s}{(\ell_s \sqrt{Y})^{D-2}}}, \tag{5.27}$$

at the correspondence point this matches $\sim e^{-S_{BH}}$.

5.3 Partial waves

It is interesting to contrast the high-energy behavior of partial waves and the fixed-angle scattering in this case. Consider next the partial wave integral $S_J = 1 + i a_J$

$$a_J(s) = \sqrt{s} \int_0^\infty db J_{D-3+2J}(b\sqrt{s}) T(s,b). \tag{5.28}$$

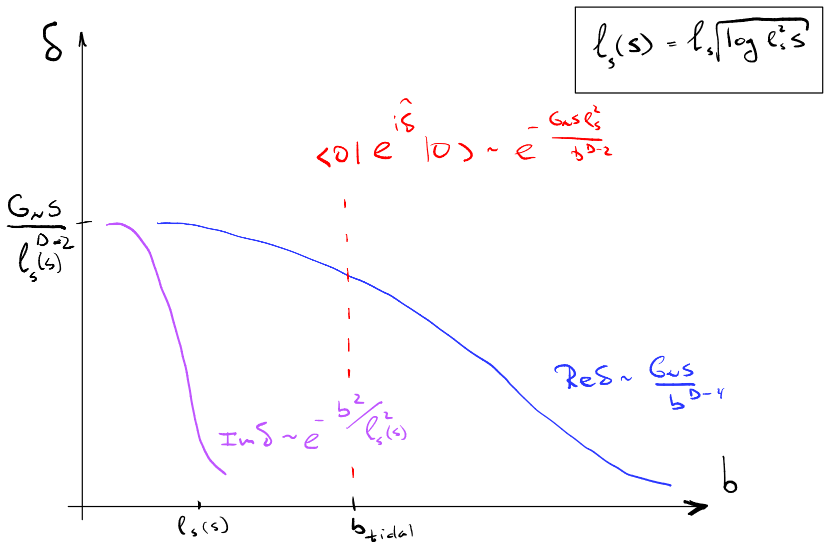

Figure 6: Stringy features of the phase shift.

This integral has a saddle point which is exponentially suppressed and which is related to the exponential suppression of the amplitude at fixed angles. Plugging the small b expansion (5.13) into the integral (5.28) we get

$$a_J(s) = \frac{(4\pi)^{1-\frac{D}{2}} sY^{2-\frac{D}{2}}}{\frac{D}{2}-2}\Big(1 - \frac{(D-4)(D+2J-4)(D+2J-2)}{4(D-2)sY} + ...\Big). \tag{5.29}$$

We see that the small b expansion becomes the large s expansion of the partial waves. This reproduces the result of Muzinich and Soldate [17]. There are also non-perturbative correction to this related to the usual saddle. From the naive saddle point analysis another saddle at fixed distance emerges when considering extrema of $e^{\alpha \log b} e^{ib\sqrt{s}} e^{-c_0 b^2}$.

Note that in this model

$$T(s,\theta) \overset{s\to\infty}{\simeq} e^{-\# s}, \tag{5.30}$$

whereas

$$T(s,b) \overset{s\to\infty}{\simeq} \frac{s}{Y^{\frac{d-4}{2}}}. \tag{5.31}$$

The impact parameter integral can be written as

$$\int_0^\infty dbb^{d-3}(bq)^{\frac{4-d}{2}} J_{\frac{d-4}{2}}(bq) T(s,b) \sim e^{-\# s}. \tag{5.32}$$

This time we don't get the contribution from the small b expansion thanks to the following identity

$$\int_0^\infty dbb^{d-3}(bq)^{\frac{4-d}{2}} J_{\frac{d-4}{2}}(bq) b^{2n} = 0, \quad n \in \mathbb{N}. \tag{5.33}$$

Note that

$$\int_0^\infty dbb^{d-3}(bq)^{\frac{4-d}{2}} J_{\frac{d-4}{2}}(bq) b^{2n+1} \neq 0, \quad n \in \mathbb{N}. \tag{5.34}$$

Therefore the fact that we have exponential suppression in the fixed angle regime is tight to the fact that in the small impact parameter expansion of the stringy amplitude the terms b^{2n+1} are absent.

5.4 The ACV S-matrix: fixed angles

We next combine the two last subsections and put the full tree-level phase shift into the exponent [34]. First, it is interesting to see how the ACV saddle (2.26) gets corrected. Recall that the saddle point is located at

$$q = -\frac{\partial 2\delta}{\partial b}|_{b=b_*}. \tag{5.35}$$

The scattering angle is $\theta \simeq \frac{2q}{\sqrt{s}}$. There are two real saddles for $\theta \leq \theta_M$

$$\theta_M = \left(\frac{R_S}{\ell_s\sqrt{\log\alpha' s}}\right)^{D-3} . \tag{5.36}$$

Consider first $1 \gg \theta_M \gg \theta$. The first saddle is the old one with $b \gg \ell_s\sqrt{\log\alpha' s}$, see (2.26). The new saddle is for $b \ll \ell_s\sqrt{\log\alpha' s}$ and it takes the form

$$\theta = \frac{2G_N\sqrt{s}}{(4\pi Y)^{\frac{D-4}{2}}}\frac{1}{D-2}\frac{b}{Y}, \tag{5.37}$$

which becomes

$$b \sim \sqrt{Y}\frac{\theta}{\theta_M} . \tag{5.38}$$

The real saddle with $b > \ell_s\sqrt{\log\alpha' s}$ dominates the amplitude. Evaluating the tidal correction on this saddle we get

$$|T(s,\theta)| \sim e^{-c_0\frac{1}{G_N^{1/(D-3)}m_*^2}(\sqrt{s})^{1-\frac{1}{D-3}}\theta^{\frac{D-2}{D-3}}} . \tag{5.39}$$

This is expected to be valid for not too high energies $\frac{\sqrt{s}}{M_P}\theta^{(3D-10)/(D-2)} \ll 1$ when gravity wave production becomes important and the amplitude becomes even more damped.

Consider next angles $1 \gg \theta \gg \theta_M$. This regime was analyzed using a complex saddle $b_* \sim -i\ell_s\sqrt{\log\alpha' s}$ in [] with the result

$$|T_{ACV}(s,\theta)| \sim e^{-\sqrt{s}\ell_s\theta\sqrt{\log\frac{\theta}{\theta_M}\log\alpha' s}}. \tag{5.40}$$

The fact that as we change θ the saddle never probes $|b| < \ell_s\sqrt{\log\alpha' s}$ was called *generalized uncertainty principle*

$$\Delta X \geq \frac{1}{\Delta P} + \alpha'\Delta P. \tag{5.41}$$

The result above can be compared to the Borel-resummed saddle point analysis of Gross-Mende-Ooguri which at small angles gives

$$|T_{GMO}(s,\theta)| \sim e^{-\sqrt{s}\ell_s\theta\sqrt{\log\frac{4}{\theta^2}\log g_s^{-2}}}. \tag{5.42}$$

As angle grows the amplitude becomes more damped.

Further effects include production of strings in the s-channel which are encoded by the imaginary part of the exponentiated phase shift, see [38]. In this case one can introduce a unitary operator

$$S = e^{i\hat{I}}, \tag{5.43}$$

where $\hat{I}$ acts on the extended Hilbert space which includes gravi-Reggeons (excited closed string states)

$$\hat{I} = \hat{\delta} + \hat{\delta}^\dagger + \sqrt{-2i(\hat{\delta} - \hat{\delta}^\dagger)}(C + C^\dagger) \ , \tag{5.44}$$

where

$$[C, C^\dagger] = 1 \ , \tag{5.45}$$

and all other commutators are zero. This can be rewritten as

$$S = e^{2i\hat{\delta}} e^{i\sqrt{-2i(\hat{\delta}-\hat{\delta}^\dagger)}C^\dagger} e^{i\sqrt{-2i(\hat{\delta}-\hat{\delta}^\dagger)}C}. \tag{5.46}$$

The exclusive amplitude corresponds to evaluating this in the vacuum for C which of course reproduces the expected $S = e^{2i\hat{\delta}}$.

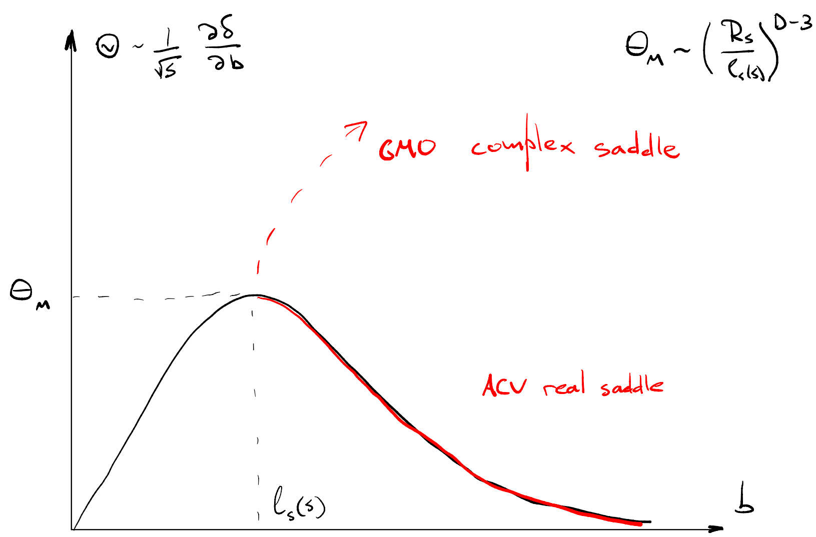

Figure 7: Scattering angle as a function of the dominant impact parameter. For $\theta > \theta_M$ the dominant saddle becomes complex.

5.5 Graviton pole

Consider the string-like amplitude in the Regge limit

$$T_s(s, -q^2) = \frac{G_N}{\alpha'} (s\alpha')^{2-\alpha' q^2}. \tag{5.47}$$

It leads to the $-\frac{8\pi G_N s^2}{t}$ pole if we plug it in the twice-subtracted dispersion relation.

For a given t, the relevant energy scale in the sum-rule is

$$\log(s_* \alpha')\alpha'|t| = 1\,, \quad s_* = \frac{1}{\alpha'} \exp\left(\frac{1}{\alpha'|t|}\right). \tag{5.48}$$

We expect the tree-level approximation to be valid as long as $G_N s_* |t|^{\frac{D-4}{2}} \ll 1$. This translates into the following bound

$$\frac{1}{\log\left(\frac{\alpha'^{\frac{D-2}{2}}}{G_N}\right)} \ll \alpha' t \ll 1 \tag{5.49}$$

Recall that $g_s^2 = \frac{G_N}{\ell_s^{D-2}}$ and $\ell_s = \sqrt{\alpha'}$ we get

$$\frac{1}{\log\left(\frac{1}{g_s^2}\right)} \ll \alpha' t \ll 1. \tag{5.50}$$

Therefore we see that the graviton pole has a universal origin (due to eikonal) in the limit $t \to 0$. In string theory, however, when $g_s \ll 1$ the tree-level amplitude reproduces the pole behavior at intermediate t's. If we take $g_s \to 0$ limit of twice-subtracted dispersion relations, we get that the pole is reproduced by the stringy amplitude.

5.6 Local growth

Twice-subtracted dispersion relation implies a bound on the local growth of the amplitude [21].

Let us consider the four-dilaton scattering in type II superstring theory

$$T^{VS}(s,t) = 8\pi G_N \Big(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\Big) \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}, \quad s+t+u=0, \tag{5.51}$$

where we set $\alpha' = 4$ so that the gap in the spectrum is 1.

To apply the local growth bound we first need to subtract the contribution of the residues at $s, u = 0$ which gives

$$\hat{T}^{VS}(s,t) = T^{VS}(s,t) + 8\pi G_N \frac{t(2s+t)^2}{s(s+t)}. \tag{5.52}$$

Note that the subtraction term behaves as s^0 at large s and is therefore highly sub-leading in the Regge limit. With this explicit example, we checked that indeed the local bound on scattering is satisfied for (5.52), see Figure 8,

$$y\partial_y \log \mathrm{Im}\left[\hat{T}^{VS}_{\psi_1}(x+iy)\right] \leq 1. \tag{5.53}$$

This example was built using the functional $\psi_1(q) = q(1-q^2)(1-q)^2$, valid in $d = 10$. We observe that the local growth of the Virasoro-Shapiro amplitude is consistent with our bound in its region of validity.

To better understand its implication, let us imagine again that the amplitude locally takes the form $T_{\text{tree}}(s,t) \sim f(t)(-is)^{j(t)}$, $f(t) \in \mathbb{R}$. Considering $y \gg s_0$ and taking $j(t) \in \mathbb{R}$, the bound (5.53) becomes

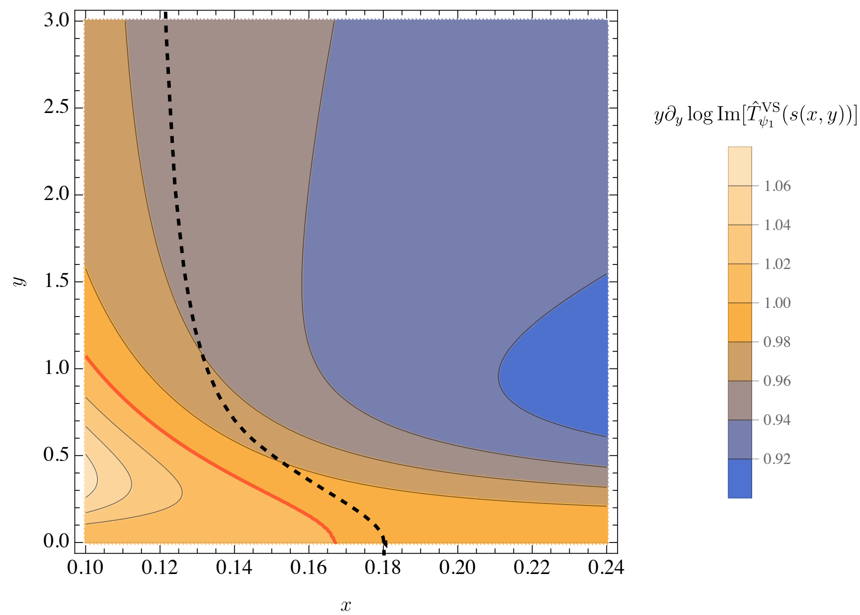

Figure 8: Comparison of the region where the bound on the local growth of the amplitude applies (black dashed line) using the functional $\psi_1(q) = q(1-q^2)(1-q)^2$ with the explicit example of the Virasoro-Shapiro amplitude. We observe that $y\partial_y \log \mathrm{Im}\left[\hat{T}^{VS}_{\psi_1}(x+iy)\right] \leq 1$ in the region enclosed by the red line which includes the region predicted by the local growth bound. Recall that the bound is saturated if the amplitude behaves as $T(s,t) \sim \lambda(t)s^2$.

$$|\langle j(t)\rangle_\psi| = \left| \frac{\int_{-M^2_{\rm gap}}^0 dt\ \psi(t) j(t) f(t) y^{j(t)-1}}{\int_{-M^2_{\rm gap}}^0 dt\ \psi(t) f(t) y^{j(t)-1}} \right| \leq 2, \tag{5.54}$$

where we used that $\mathrm{Im}(y - is_0)^{j(t)} \simeq -y^{j(t)-1}s_0$. The saturation of this bound is possibly also excluded (but it requires further checking) with the departure from 2 controlled by M_{UV} where the dispersive states enter.

6 Black holes

At high energies and small impact parameters a black hole is expected to be produced in high energy collisions of particles (together with radiation). With time it evaporates and therefore typical outcome of the experiment is a multi-particle state made out of many soft quanta (Hawking radiation).

For the two-to-two scattering it means that scattering is mostly inelastic and we expect that (possibly upon a bit of smearing in s?)

$$S_J(s) = 0, \quad s \to \infty, \quad J - \text{fixed}\ . \tag{6.1}$$

This agrees with the expected suppression due to many other effects discussed earlier that have nothing to do with black holes. As well as the picture of strong suppression due to black holes at small impact parameters (remember the string theory example). In particular we expect the condition above to hold for

$$J \leq J_{BD}(s), \tag{6.2}$$

which is the black disc model of scattering.

This black disc behavior holds for spins which are not too high. Recall that $J \sim \sqrt{s}b$ and when $b \sim s^{1/(D-3)}$ we expect instead

$$\lim_{s\to\infty} |S_J(s)| = 1, \quad J \geq (\sqrt{s})^{1+2/(D-3)}, \tag{6.3}$$

whereas the $b \sim R_S$ corresponds to $J \lesssim (\sqrt{s})^{1+/(D-3)}$. Therefore we see that partial waves transition from 'transparent' to 'opaque' behavior as we decrease spin.

6.1 Violations of symmetries

We expect that there is a lower bound on global symmetry violations due to black holes. Imagine four free scalars minimally coupled to gravity and consider $1, 2 \to 3, 4$. To all orders in G_N this amplitude is zero. We would like to understand if it is also zero non-perturbatively. This argument in a slightly different context (which we find actually much harder to justify) has been presented in [39–41].

Using unitarity we get

$$\text{Im} f_J^{1,2\to3,4}(s) = \sum_n \langle 1, 2|n\rangle\langle n|3, 4\rangle. \tag{6.4}$$

Given that perturbatively the amplitude is zero, we next model intermediate states as black holes BH_I to get

$$\text{Im} f_J^{1,2\to3,4}(s) = \sum_{I=1}^{e^{S_{BH}}} \langle 1, 2|\text{BH}_I\rangle\langle \text{BH}_I|3, 4\rangle, \tag{6.5}$$

and we impose

$$\sum_{I=1}^{e^{S_{BH}}} |\langle 1, 2|\text{BH}_I\rangle|^2 \simeq 1, \tag{6.6}$$

which is we say that *'black hole is formed with probability one'*. The solution to this ansatz is

$$\langle 1, 2|\text{BH}_I\rangle = e^{-S_{BH}/2} e^{i\phi_I^{1,2}}, \tag{6.7}$$

where $\phi_I^{1,2}$ is a pseudo-random phase. We know can estimate the symmetry violating amplitude (6.5) to be

$$\text{Im} f_J^{1,2\to3,4}(s) = e^{-S_{BH}(s)} \sum_{I=1}^{e^{S_{BH}(s)}} e^{i(\phi_I^{1,2}-\phi_I^{3,4})} \simeq e^{-S_{BH}(s)/2}. \tag{6.8}$$

We know write down twice-subtracted dispersion relations. The discontinuity of the amplitude is given by the formula above and we get for the symmetry violating Wilson coefficients

$$c_n \gtrsim e^{-S_{BH}(M_S)/2}, \tag{6.9}$$

where M_S is the mass of the lightest semi-classical black hole (given by the correspondence point in string theory). The formula above is not rigorous because the discontinuity is not positive-definite and there could be some extra cancelations. It would be interesting to test it in string theory.

Let's consider string scale to be 10^{16} GeV then at the correspondence point $S \sim (\frac{\ell_s}{\ell_P})^2 \sim 10^6$, therefore the effect is $\sim e^{-10^6}$ which seems to be completely irrelevant phenomenologically.

A more refined computation was done in [11]. One considers a two-point function of shell operators in AdS. More precisely, the observbale of interest is

$$T_{t,E,\Delta E} = \int dt' e^{it'E} e^{-\Delta E^2 (t-t')^2} \langle B(t) A(0) \rangle \tag{6.10}$$

which is a symmetry violating process that is zero to all orders in perturbation theory. The following regime is studied $t \gg \frac{1}{\Delta E} \gg \beta(E)$.

The idea is that there is a wormhole geometry that computes $|T_{t,E,\Delta E}|^2$ in 'a suitable averaged sense'. Finite ΔE and large t guarantee projection to the Hartling-Hawking state on the bow-tie geometry (all excitations decay controlled by QNMs). The result is that

$$\overline{|T_{t,E,\Delta E}|^2} \simeq e^{-S_{BH}(E)}, \tag{6.11}$$

where bar stands for some kind of further averaging over $(t, E, \Delta E)$.

6.2 Black hole ansatz

A more refined version of the black disc statement is called *the black hole ansatz* [42], which (up to smearing) takes the form

$$|S_J(s)| \simeq e^{-S_{BH}(s)/2}, \tag{6.12}$$

with some possible corrections due to spin. In our opinion it is very hard to test or justify it. In particular, to have the collapse argument under control we needed to have wavepackets of finite size and this increases the wavepacket tail effects. So this is something that needs to be explored using bootstrap and smearing.

Perhaps an interesting scenario is to explore 'maximal blackness' above the correspondence point in the collapse region?

One way to get this result is to try using

$$S_J(s) = \sqrt{s} \int_0^\infty J_{d-3+2J}(\tilde{b}\sqrt{s}) e^{2i\delta(s,\tilde{b})} d\tilde{b}\,, \tag{6.13}$$

and model $\delta(s, b)$ as we discussed in the previous sections.

6.3 Fixed angles and chaos

Another conjecture that has been often stated in various degrees of vagueness is [39–41]

$$T(s,\theta) \simeq e^{-\frac{1}{2}S_{BH}(s)f(\theta)}e^{i\phi(s,\theta)}, \tag{6.14}$$

where $\phi(s,\theta)$ is a pseudorandom phase and $f(\theta) \sim O(1)$. We think this is also hard to justify as opposed to violations of symmetries, where we are talking about leading effects.

In the context of quantum mechanics the randomness of phase is related to chaos [43]. It would be very interesting to explore the properties of the scattering amplitude phase in QFT and gravity as suggested in [44].

7 Conclusions

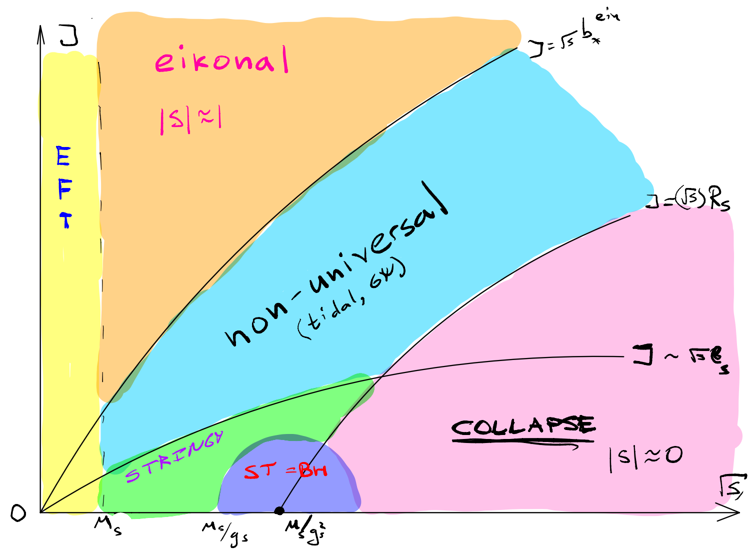

Figure 9: Map of gravitational scattering in the (s, J)-plane.

The basic conclusion is that gravity looks consistent with the basic principles of S-matrix theory, but black hole physics (or gravity waves for that matter) has not been really used so far.

A big challenge is to turn the statements in these notes into interesting, quantitative predictions about low-energy physics accessible to reasonable observers. To some extent the twice-subtracted dispersion relations has been the most useful in this regard so far.

The challenge is that many of the statements involve high energies which to the best of our knowledge decouple from low-energy dynamics (e.g. crossing does not introduce nontrivial UV/IR mixing). For example, the expected symmetry violations due to nonperturbative effects look negligibly small. It would be still interesting to use bootstrap methods and explicit constructions in string theory to put the basic picture outlined in this note under scrutiny and sharpen it further.

Perhaps the picture of collapse invites us to explore what is the size and intensity of black discs in the S-matrix bootstrap? Can we use the bootstrap methods to get some further quantitative insights into the gravitational scattering in the inelastic regime? What is the maximal inelasticity consistent with the basic principles of S-matrix theory?

We also discussed here only scattering of light strings. It is very interesting to understand physics of high-energy scattering of other objects in string theory, e.g. excited strings, $D0$ branes, string-brane systems.

Finally, understanding better the implications of chaos for the S-matrix bootstrap seems like an interesting open problem.

References

[1] S. W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, *Phys. Rev. D* **14** (1976) 2460–2473.

[2] S. B. Giddings and R. A. Porto, *The Gravitational S-matrix*, *Phys. Rev. D* **81** (2010) 025002, [0908.0004].

[3] G. F. Giudice, R. Rattazzi and J. D. Wells, *Quantum gravity and extra dimensions at high-energy colliders*, *Nucl. Phys. B* **544** (1999) 3–38, [hep-ph/9811291].

[4] G. 't Hooft, *Graviton Dominance in Ultrahigh-Energy Scattering*, *Phys. Lett. B* **198** (1987) 61–63.

[5] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, *Int. J. Mod. Phys. A* **3** (1988) 1615–1661.

[6] L. Rastelli, *Strings 2021, Review*, .

[7] J. Polchinski, *Chaos in the black hole S-matrix*, 1505.08108.

[8] J. Maldacena, *Strings 2024, Review*, .

[9] S. M. Chester, S. S. Pufu and X. Yin, *The M-Theory S-Matrix From ABJM: Beyond 11D Supergravity*, *JHEP* **08** (2018) 115, [1804.00949].

[10] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060, [hep-th/0601001].

[11] I. Bah, Y. Chen and J. Maldacena, *Estimating global charge violating amplitudes from wormholes*, *JHEP* **04** (2023) 061, [2212.08668].

[12] P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, *The gravitational eikonal: from particle, string and brane collisions to black-hole encounters*, 2306.16488.

[13] M. Correia and G. Isabella, *The Born regime of gravitational amplitudes*, 2406.13737.

[14] D. M. Eardley and S. B. Giddings, *Classical black hole production in high-energy collisions*, *Phys. Rev. D* **66** (2002) 044011, [gr-qc/0201034].

[15] S. B. Giddings and V. S. Rychkov, *Black holes from colliding wavepackets*, *Phys. Rev. D* **70** (2004) 104026, [hep-th/0409131].

[16] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020, [1407.5597].

[17] I. J. Muzinich and M. Soldate, *High-Energy Unitarity of Gravitation and Strings*, *Phys. Rev. D* **37** (1988) 359.

[18] L. Cornalba, M. S. Costa and J. Penedones, *Eikonal approximation in AdS/CFT: Resumming the gravitational loop expansion*, *JHEP* **09** (2007) 037, [0707.0120].

[19] V. S. Rychkov, *Tests of classical gravity description for microscopic black hole production*, hep-ph/0405104.

[20] M. Correia, A. Sever and A. Zhiboedov, *An analytical toolkit for the S-matrix bootstrap*, JHEP **03** (2021) 013, [2006.08221].

[21] K. Häring and A. Zhiboedov, *Gravitational Regge bounds*, SciPost Phys. **16** (2024) 034, [2202.08280].

[22] P. Kravchuk, J. Qiao and S. Rychkov, *Distributions in CFT. Part I. Cross-ratio space*, JHEP **05** (2020) 137, [2001.08778].

[23] S. Caron-Huot, D. Mazac, L. Rastelli and D. Simmons-Duffin, *Sharp boundaries for the swampland*, JHEP **07** (2021) 110, [2102.08951].

[24] A. Martin, *Improvement of the greenberg-low bound*, *Il Nuovo Cimento A (1965-1970)* **42** (1966) 901–907.

[25] J. Qiao and S. Rychkov, *Cut-touching linear functionals in the conformal bootstrap*, JHEP **06** (2017) 076, [1705.01357].

[26] J. Isabella, P. Tourkine and A. Zhiboedov, *Partial waves at high energy (unpublished)*, .

[27] K. Haring, *A tauberian theorem for the graviton pole*, .

[28] S. Caron-Huot and J. Tokuda, *String loops and gravitational positivity bounds: imprint of light particles at high energies*, 2406.07606.

[29] D. Carmi and S. Caron-Huot, *A Conformal Dispersion Relation: Correlations from Absorption*, JHEP **09** (2020) 009, [1910.12123].

[30] F. A. Cerulus and A. Martin, *A lower bound for large-angle elastic scattering at high energies*, Phys. Lett. **8** (1964) 80–82.

[31] P. Tourkine and A. Zhiboedov, *Scattering amplitudes from dispersive iterations of unitarity*, JHEP **11** (2023) 005, [2303.08839].

[32] H. Epstein and A. Martin, *Rigorous lower bound on the scattering amplitude at large angles*, Phys. Rev. D **99** (2019) 114025, [1903.00953].

[33] D. C. Ullrich, *Complex made simple*, vol. 97. American Mathematical Soc., 2008.

[34] D. Amati, M. Ciafaloni and G. Veneziano, *Can Space-Time Be Probed Below the String Size?*, Phys. Lett. B **216** (1989) 41–47.

[35] S. B. Giddings, D. J. Gross and A. Maharana, *Gravitational effects in ultrahigh-energy string scattering*, Phys. Rev. D **77** (2008) 046001, [0705.1816].

[36] A. Bedroya, *High energy scattering and string/black hole transition*, 2211.17162.

[37] D. Amati and C. Klimcik, *Strings in a Shock Wave Background and Generation of Curved Geometry from Flat Space String Theory*, Phys. Lett. B **210** (1988) 92–96.

[38] G. Veneziano, *String-theoretic unitary S-matrix at the threshold of black-hole production*, JHEP **11** (2004) 001, [hep-th/0410166].

[39] T. Banks and W. Fischler, *A Model for high-energy scattering in quantum gravity*, hep-th/9906038.

[40] N. Arkani-Hamed, S. Dubovsky, A. Nicolis, E. Trincherini and G. Villadoro, *A Measure of de Sitter entropy and eternal inflation*, *JHEP* **05** (2007) 055, [0704.1814].

[41] N. Arkani-Hamed, M. Pate, A.-M. Raclariu and A. Strominger, *Celestial amplitudes from UV to IR*, *JHEP* **08** (2021) 062, [2012.04208].

[42] S. B. Giddings and M. Srednicki, *High-energy gravitational scattering and black hole resonances*, *Phys. Rev. D* **77** (2008) 085025, [0711.5012].

[43] D. J. Gross and V. Rosenhaus, *Chaotic scattering of highly excited strings*, *JHEP* **05** (2021) 048, [2103.15301].

[44] V. Rosenhaus, *Chaos in the Quantum Field Theory S-Matrix*, *Phys. Rev. Lett.* **127** (2021) 021601, [2003.07381].