Bootstrapping 2d loop models

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Upcoming review article: github.com/ribault/CFT-Review/releases/latest

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$$Z = \sum_{\text{config}} n^{\#\text{loops}} \qquad n \in \mathbb{C}$$

$O(n)$ model $\quad (\forall d) \qquad v \in \mathbb{R}^n$

Potts model $\quad (2d) \qquad \{1 \ldots Q\} \qquad n = \sqrt{Q}$

Critical limits $\to$ CFT

1980s.

Methods:

Analytic methods: solve minimal models, Liouville theory

2 independent deg. fields.

Loop: 1 deg. field.

Results:

{Combinatorial maps}

↳ basis of sol^{n} of crossing sym.

2. Chiral 2d CFT

$(L_n)_{n\in\mathbb{Z}}, \ \mathbb{1}$

$[\mathbb{1}, L_n] = 0 \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$

$L_0 = \text{dilatation}$

Primary state: $\begin{cases} L_0 V = \Delta V \\ L_{n>0} V = 0 \end{cases}$ | $L_0 L_n V = (\Delta - n) L_n V$

$\mathcal{L} = \left\{ \prod_{i=1}^{k} L_{-n_i} \right\}_{0<n_1\leq n_2 \ldots \leq n_k} = \{1, L_{-1}, L_{-1}^2, L_{-2}, \ldots\}$

$$\mathcal{M}_\Delta = \mathrm{Span}(L V_\Delta)_{L \in \mathcal{L}}$$

$\hookrightarrow$ Verma module.

$\mathcal{M}_\Delta$ irred $\iff \nexists$ singular vector.

Singular vector: primary state $V \neq V_\Delta$

$$L_1 (L_{-1} V_\Delta) = 2\Delta V_\Delta$$

$$[L_1, L_{-1}] = 2 L_0$$

$$L_1 V_\Delta = 0$$

$L_{-1} V_\Delta$ is a sing. vector $\iff \Delta = 0$

$\mathcal{M}_0$ is reducible $\quad \mathcal{M}_1 \subset \mathcal{M}_0$

$$c = 1 - 6(\beta - \beta^{-1})^2$$

$$\Delta = \frac{c-1}{24} + P^2$$

$$\left(L_{-1}^{rs} + \dots\right) V_{\Delta_{(r,s)}}$$

M_Δ has a N.V. at level rs if $P = P_{(r,s)}$

$$\boxed{P_{(r,s)} = \frac{1}{2}\left(\beta r - \beta^{-1} s\right)}$$

$M_{\Delta_{(r,s)}}$ is reducible

$\Delta_{(r_1,s_1)} = \Delta_{(r_2,s_2)}$

$\Rightarrow c = \dots$

$\dfrac{M_{\Delta(r,s)}}{M_{\Delta(r,s)+rs}}$ is a irreducible coset $=$ deg. representation.

$$V_{\Delta_{(r,s)}} \qquad L_{(r,s)} \underbrace{V_{\Delta_{(r,s)}}}_{V^d_{\langle r,s\rangle}} = 0.$$

$$\begin{cases} L_{n>0} V_{\Delta_{(r,s)}} = 0 \\ L_0 V_{\Delta_{(r,s)}} = \Delta_{(r,s)} V_{\Delta_{(r,s)}} \end{cases}$$

$$\Delta = 0 \qquad L_{-1} V_0 = 0$$

$$L_{-1} V^d_{\langle 1,1\rangle} = 0$$

$$\left(L_{-1}^2 + \beta^2 L_{-2}\right) V^d_{\langle 2,1\rangle} = 0$$

$$V(z) \qquad z \in \overline{\mathbb{C}}$$

$$\boxed{\frac{\partial}{\partial z} V(z) = L^{(z)}_{-1} V(z)}$$

$$L_n V(z) \overset{\text{not.}}{=} L^{(z)}_n V(z)$$

$$T(y) = \sum \frac{L^{(z)}_n}{(y-z)^{n+2}}$$

energy-momentum tensor

$$T(y) V_\Delta(z) = \sum_{n \in \mathbb{Z}} \frac{L_n V_\Delta(z)}{(y-z)^{n+2}}$$

$$= \frac{\Delta}{(y-z)^2} V_\Delta(z) + \frac{\partial V_\Delta(z)}{y-z}$$

$$+ \mathcal{O}(1)$$

$$\frac{\partial}{\partial z} T(y) = 0$$

$$V_1(z_1)V_2(z_2) = \sum_k C_{12}^k(z_1,z_2)V_k(z_2)$$

$\llcorner$ basis of fields

$$\frac{\partial}{\partial z_1} z_{12}^{\Delta_k-\Delta_2} = 0 \iff \Delta_k = \Delta_2$$

$$V_1(z_1)V_2(z_2) = V_2(z_2)V_1(z_1)$$

$|L| =$ level

$T\, V_1 V_2$ associativity

$|L_{-1}^2| = 2$

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) = \sum_k C_{12}^k z_{12}^{\Delta_k-\Delta_1-\Delta_2}\left(V_k(z_2) + \sum_{L\in\mathcal{L}-\{1\}} z_{12}^{|L|} \underbrace{f_{\Delta_1\Delta_2}^{\Delta_k,L}}_{\text{known}} L V_k(z_2)\right)$$

$\llcorner$ primaries

$\to$ OPE struct cst

$\Downarrow$

$V^d_{(1,1)}$

$$L_{-1}V^d_{(1,1)} = 0 = \frac{\partial}{\partial z_1}V^d_{(1,1)}(z_1) =$$

$$V^d_{(1,1)}V_\Delta \sim V_\Delta$$

$$V^d_{\langle 1,1\rangle} V_p = \sum_{\pm} V_{p \pm \beta/2}$$

$$(L_{-1}^2 + \beta^2 L_{-2}) V^d_{\langle 2,1\rangle}$$

$$V^d_{\langle 1,2\rangle} V_p = \sum_{\pm} V_{p \pm \frac{1}{2\beta}}$$

OPE associativity

- OPE of deg. fields $\rightarrow$ deg. fields.
- $V^d_{(2,1)} V^{\cancel{d}}_{(2,1)} = V^d_{(1,1)} + V^d_{(3,1)}$

$$P_{(1,1)}\ ,\ P_{(3,1)}$$

$$V^d_{(2,1)} V^d_{(r,s)} = \sum_{\pm} V^d_{(r\pm 1,s)} \qquad P_{(r,s)} = \frac{1}{2}\left(\beta r - \beta^{-1} s\right)$$

$$V^d_{(1,2)} V^d_{(r,s)} = \sum_{\pm} V^d_{(r,s\pm 1)}$$

$$V^d_{(r_1,s_1)} V^d_{(r_2,s_2)} = \sum_{r \overset{2}{=} |r_1-r_2|+1}^{r_1+r_2-1} \; \sum_{s \overset{2}{=} |s_1-s_2|+1}^{s_1+s_2-1} V^d_{(r,s)}$$

$$\sigma = V^d_{(2,1)} = V^d_{(2,2)}$$

$$\Delta_{(2,1)} = \Delta_{(2,2)}$$

$$\Rightarrow \quad P_{(2,1)} + P_{(2,2)} = 0 = \frac{1}{2}\left(4\beta - 3\beta^{-1}\right) \Rightarrow \beta^2 = \frac{3}{4} \Rightarrow c = \frac{1}{2}$$

$$\sigma\sigma = V^d_{(3,1)} + V^d_{(1,1)} = \varepsilon + 1 \qquad \varepsilon = V^d_{(3,1)} = V^d_{(1,2)} \qquad c = 1 - 6\left(\beta - \beta^{-1}\right)^2$$

$$\varepsilon\varepsilon = 1 \qquad \varepsilon\sigma = \sigma$$

$$S^{MM}_{p,q} = \left\{ V^d_{(r,s)} = V^d_{(p-r,q-s)} \right\}_{(r,s)\in(0,p)\times(0,q)\cap\mathbb{N}^*\times\mathbb{N}^*}$$

(Kac table)

$\beta^2 = \frac{p}{q}$

$c \in (-\infty, 1)$

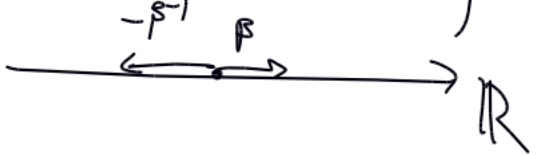
$$\beta^2 \to \underline{\beta_0^2 \in \mathbb{R}_{>0}}$$

fixed r,s $\Rightarrow$ $S^{GMM} = \left\{ V^d_{(r,s)} \right\}_{(r,s)\in\mathbb{N}^*\times\mathbb{N}^*}$

$P_{(r,s)} \to P_0 \in \mathbb{R} \Rightarrow S^{Liou} = \left\{ V_{P_0} \right\}_{P_0\in\mathbb{R}}$

$c \in \mathbb{C}$

$$P_{(r,s)} = \frac{1}{2}\left(\beta r - \beta^{-1} s\right)$$



$$V_{P_1} V_{P_2} \sim \int_{\mathbb{R}} dP\, V_P$$

Conf. alg $= \mathrm{Vir} \times \overline{\mathrm{Vir}} \rightsquigarrow (L_n)_{n\in\mathbb{Z}}$

$\frac{\partial}{\partial z} V(z) = L_{-1} V(z) \hookrightarrow \frac{\partial}{\partial \bar{z}} V(z) = \bar{L}_{-1} V(z)$

$V_{\Delta, \bar{\Delta}}(z) \qquad S = \bar{\Delta} - \Delta = \text{conf. spin}.$

$\langle V_1(z_1) V_2(z_2) \rangle = B_{12} \left| \delta_{\Delta_1, \Delta_2} z_{12}^{-2\Delta_1} \right|^2$

$|f(\Delta, z)|^2 = f(\Delta, z) f(\bar{\Delta}, \bar{z})$

$$\langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle = C_{123} \left| z_{12}^{\Delta_3-\Delta_1-\Delta_2} z_{23}^{\Delta_1-\Delta_2-\Delta_3} z_{31}^{\Delta_2-\Delta_1-\Delta_3} \right|^2$$

Single-valued.

$$\Updownarrow$$

$$\begin{cases} S_i \in \frac{1}{2}\mathbb{Z} \\ S_1 + S_2 + S_3 \in \mathbb{Z} \end{cases}$$

$$z_{ij} \to e^{2\pi i} z_{ij}$$

$$z_{12}^{\delta} \bar{z}_{12}^{\bar{\delta}}$$

$$\left(\frac{z_{12}}{\bar{z}_{12}} \right)^{\delta - \bar{\delta}}$$

$$B_{ij} = B_i \delta_{ij} \qquad I = \text{identity}$$

$$C_{ijk} = C_{ij}^{k} B_k \qquad C_{Ijj} = B_j$$

Spectrums of loop models.

$c \in \mathbb{C}$

	Not.	Var.	$(P, \bar{P})$
Degenerate	$V^d_{\langle r,s\rangle}$	$(r,s)\in \mathbb{N}^*$	$\left(P_{(r,1)}, P_{(r,s)}\right)$
Diagonal	V_P	$P \in \mathbb{C}$	(P, P)
Non-diagonal	$V_{(r,s)}$	$r,s \in \mathbb{C}$	$\left(\Delta_{(r,s)}, \Delta_{(r,-s)}\right)$

$$\Delta_{(r,-s)} - \Delta_{(r,s)} = rs = S_{(r,s)}$$

- c is generic $\beta^2 \notin \mathbb{Q}$
- spins are half-integer $rs \in \frac{1}{2}\mathbb{Z}$

$$V^d_{(2,1)} V_p = \sum_{\pm} V_{p \pm \beta/2}$$

$$= \sum_{\pm\pm} V_{p\pm\beta/2,\, p\pm\beta/2}$$

$$\beta^2 \notin \mathbb{Q}$$

$$\boxed{V^d_{(2,1)} V_{(r,s)} = \sum_{\pm} V_{(r\pm 1,\, s)}} + \cancel{\sum_{\pm} V_{(r,\, s\pm\beta^2)}}$$

$$\cancel{rs,\ s,\ r\beta^2 \in \tfrac{1}{2}\mathbb{Z}}$$

$$S_{(r\pm1,s)} - S_{(r,s)} = \pm s$$

$$S = rs \qquad S_{(r,\, s\pm\beta^2)} - S_{(r,s)} = \pm r\beta^2$$

$$\boxed{s \in \tfrac{1}{2}\mathbb{Z}}$$

$$V^d_{\langle 2,1\rangle} V_{(r,s)} = \sum_{\pm} V_{(r\pm 1,s)}$$

$$V^d_{\langle 1,2\rangle} V_{(r,s)} = \sum_{\pm} V_{(r,s\pm 1)}$$

Liouville

$$\lim_{P\to P_{(r,s)}} V_P \overset{?}{=} V^d_{(r,s)}$$

yes if $c\in \mathbb{C}-(-\infty,1)$

no if $c\in(-\infty,1)$