Bootstrapping 2d loop models

$c = 13 - 6\beta^2 - 6\beta^{-2}$

	Not.	param.	$(P, \bar{P})$
Degenerate	$V^d_{\langle r,s\rangle}$	$r,s \in \mathbb{N}^*$	$(P_{(r,1)}, P_{(r,s)})$
Diagonal	V_P	$P \in \mathbb{C}$	(P, P)
Non-diag.	$V_{(r,s)}$	$r,s \in \mathbb{C}$	$(P_{(r,s)}, P_{(r,-s)})$

$P_{(r,s)} = \frac{1}{2}(\beta r - \beta^{-1} s)$

$\Delta = \frac{c-1}{24} + P^2$

Spin: $S = \bar{\Delta} - \Delta$

$S_{(r,s)} = rs$

OPEs:

$$V^d_{\langle 2,1\rangle} V_P \sim \sum_{\pm} V_{P \pm \beta/2}$$

$$V^d_{\langle 1,2\rangle} V_P \sim \sum_{\pm} V_{P \pm \frac{1}{2\beta}}$$

$$V^d_{\langle 2,1\rangle} V_{(r,s)} \sim \sum_{\pm} V_{(r\pm 1, s)} \quad \underline{\text{if}} \quad s \in \tfrac{1}{2}\mathbb{Z}$$

$$V^d_{\langle 1,2\rangle} V_{(r,s)} \sim \sum_{\pm} V_{(r, s\pm 1)} \quad \underline{\text{if}} \quad r \in \tfrac{1}{2}\mathbb{Z}$$

$$\left\langle V^{d}_{(1,2)} \prod_{i=1}^{N} V_{(r_i,s_i)} \right\rangle$$

$V^{d}_{(1,2)}$ around $V_{(r,s)}$

$$V^{d}_{(1,2)} V_{(r,s)} \sim \sum_{\pm} V_{(r,s\pm 1)}$$

$z_1 \quad z_2$

$$\boxed{\begin{gathered}\sum_{i=1}^{N} r_i \in \mathbb{Z} \\ r_i \in \tfrac{1}{2}\mathbb{Z}\end{gathered}}$$

$$e^{2\pi i \left(S_{(r,s\pm1)} - S_{(r,s)}\right)}$$

$$= e^{\pm 2\pi i r}$$

- $\beta^2 \notin \mathbb{R}$
- $V^d_{\langle 1,2\rangle}$

$$\lim_{\substack{P \to P_{(r,s)} \\ r \in \frac{1}{2}\mathbb{Z}}} V_P = V_{P_{(r,s)}} \neq V^d_{\langle r,s\rangle}$$

$$\tilde{S}^{loop} = \left\{ V^d_{\langle 1,s\rangle} \right\}_{s \in \mathbb{N}^*} \cup \left\{ V_{(r,s)} \right\}_{\substack{r \in \frac{1}{2}\mathbb{N}^* \\ s \in \frac{1}{2r}\mathbb{Z}}} \cup \left\{ V_P \right\}_{P \in \mathbb{C}}$$

$$V_{(r,s)} = V_{(-r,-s)} \qquad V_P = V_{-P}$$

$$V_{(0,s)} = V_{\frac{1}{2}\beta^{-1}s} \iff V_P \sim V_{(0,2\beta P)}$$

$$V^d_{\langle 1,2\rangle} V_{(r,s)} \sim \sum_{\pm} V_{(r,s\pm 1)}$$

$r = \frac{1}{2}$ $\quad V_{(\frac{1}{2},0)}, V_{(\frac{1}{2},\pm 2)}, V_{(\frac{1}{2},\pm 4)} \ldots$ (Integer spin)

$r = 1$ $\quad V_{(1,0)}, V_{(1,\pm 1)}, V_{(1,\pm 2)} \ldots$

$r = \frac{3}{2}$ $\quad V_{(\frac{3}{2},0)}, V_{(\frac{3}{2},\pm\frac{2}{3})}, V_{(\frac{3}{2},\pm\frac{4}{3})}, V_{(\frac{3}{2},\pm 2)} \ldots$

$\vdots$

$$S^{O(n)} = \left\{V^d_{(1,s)}\right\}_{s\in 2\mathbb{N}+1} \cup \left\{V_{(r,s)}\right\}_{\substack{r\in\frac{1}{2}\mathbb{N}^* \\ s\in\frac{1}{r}\mathbb{Z}}}$$

$\left(n = -2\cos(\pi\beta^2)\right)$ $\quad V^d_{(1,2)} \quad V_{(\frac{1}{2},0)} \sim V_{(\frac{1}{2},\pm 1)}$

$c = 13 - 6\beta^2 - 6\beta^{-2}$

$$\Delta_{(r,s)} + \bar{\Delta}_{(r,-s)} = \frac{c-1}{12} + \frac{1}{2}\left(r^2\beta^2 + s^2\beta^{-2}\right)$$

$\mathrm{Re}(\Delta + \bar{\Delta})$ bounded from below

$$\mathrm{Re}\,\beta^2 > 0 \iff \boxed{\mathrm{Re}\, c < 13}$$

$\cancel{AdS_3/CFT_2}$

$dS_3 \sim c = 13 + i\mathbb{R}$ [V. Godet 2024]

$\mathbb{R} \times \Sigma$

4. Bootstrap

$$\langle \underbrace{V_1(z) V_2(0)} V_3(\infty) V_4(1) \rangle = \sum_{h \in S} C_{12}^{h} C_{h34} \, g^{(s)}_{\Delta_h, \bar{\Delta}_h}(z)$$

$$g^{(s)}_{\Delta,\bar{\Delta}}(z) = \left| \mathcal{F}^{(s)}_{\Delta}(z) \right|^2$$

↳ conf. block ↳ Vir. block

$$\mathcal{F}^{(s)}_{\Delta} = \sum_{L \in \mathcal{L}} \ldots$$

$$\mathcal{L} = \{1, L_{-1}, L_{-2}, L_{-1}^2, \ldots\}$$

4pt struct cst

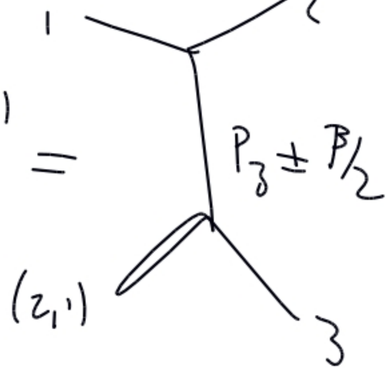
$$\langle\ \rangle = \sum_{k\in S^{(s)}} D_k^{(s)}\, g^{(s)}_{\Delta_k,\bar{\Delta}_k}(z) = \sum_{k\in S^{(t)}} D_k^{(t)}\, g_k^{(t)} = \sum_{k\in S^{(u)}} \cdots$$

$$D_k^{(s)} = C_{12}^{k}\, C_{k34}$$

$$G(z) = \left\langle V^{d}_{(2,1)}(z)\, V_1(0)\, V_2(\infty)\, V_3(1) \right\rangle$$

$$\mathcal{F}^{(s)}_{\pm} =$$

$$\mathcal{F}^{(t)}_{\pm} =$$



BPZ eqn (2nd order)

$$\left(L_{-2} - \beta^2 L_{-1}^2\right) V^{d}_{(2,1)} = 0$$

2 bases $\mathcal{F}^{(s)}_{\pm}$, $\mathcal{F}^{(t)}_{\pm}$

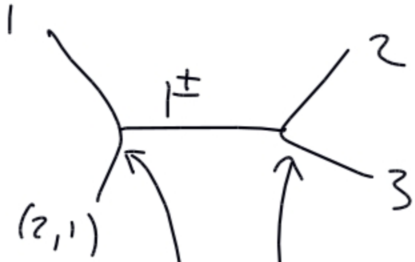
$$\mathcal{F}^{(s)}_{\varepsilon_1}(z) = \sum_{\varepsilon_3 = \pm} F_{\varepsilon_1, \varepsilon_3} \mathcal{F}^{(t)}_{\varepsilon_3}(z)$$

$$F_{\varepsilon_1 \varepsilon_3} = \frac{\Gamma(1 + 2\beta\varepsilon_1 P_1)\Gamma(-2\beta\varepsilon_3 P_3)}{\prod_{\pm} \Gamma\left(\frac{1}{2} + \beta\varepsilon_1 P_1 \pm \beta P_2 - \beta\varepsilon_3 P_3\right)}$$

$$G(z) = \sum_{\pm} d_\pm^{(x)} \left| \mathcal{F}_\pm^{(x)}(z) \right|^2 \qquad x \in \{s, t, u\}$$

$$\frac{d_-^{(s)}}{d_+^{(s)}} = - \frac{\overline{F_{++}}\ \overline{F_{+-}}}{F_{++}\ F_{+-}}$$

$$V_1 \to \lambda_1 V_1$$



$$d_\pm^{(s)} = C_1^{\pm}\, C_{1^\pm 23}$$

$$V^{d}_{(2,1)} V_1 \sim \sum_\pm C_1^{\pm} V_{1^\pm}$$

$$\frac{C_{\Gamma 23}}{C_{\Gamma^{\dagger} 23}} = (-1)^{2s_2}\, \beta^{4\beta^2 r_1} \frac{\prod_{\pm\pm} \Gamma\left(\frac{1}{2} - \beta\bar{P}_1 \pm \beta\bar{P}_2 \pm \beta\bar{P}_3\right)}{\prod_{\pm\pm} \Gamma\left(\frac{1}{2} + \beta P_1 \pm \beta P_2 \pm \beta P_3\right)}$$

$V^d_{(2,1)}$

$V^d_{(1,2)}$

$V^d_{(1,3)}$

$$\frac{C_{\Gamma 23}}{C_{1\dagger 23}} = (-1)^{2r_2}\, \beta^{-4\beta^{-2} s_1} \frac{\prod_{\pm\pm} \Gamma\left(\frac{1}{2} - \beta^{-1}\bar{P}_1 \pm \beta^{-1}\bar{P}_2 \pm \beta^{-1}\bar{P}_3\right)}{\prod_{\pm\pm} \Gamma\left(\frac{1}{2} - \beta^{-1} P_1 \pm \beta^{-1} P_2 \pm \beta^{-1} P_3\right)}$$

Minimal models.

$$\frac{C_{(r_1+1,s_1)(r_2 s_2)(r_3 s_3)}}{C_{(r_1-1,s_1)(r_2 s_2)(r_3 s_3)}} = \text{known} \Rightarrow \text{all 3pt } C_{(r_1,s_1)(r_2 s_2)(r_3 s_3)}$$

$$\left.\begin{array}{l} r_1 \to r_1+2 \\ s_1 \to s_1+2 \end{array}\right\} \text{known}$$

$$\beta^2 \in \mathbb{R}_{<0} - \mathbb{Q}$$

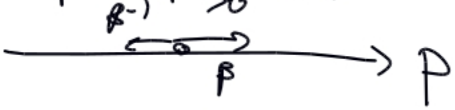
Liouville theory:

$$P \in \mathbb{R}$$

$$\beta^2 \in \mathbb{R}_{>0} - \mathbb{Q}$$

$$\frac{C_{P_1-\beta/2,\, P_2,\, P_3}}{C_{P_1+\beta/2,\, P_2,\, P_3}} = \text{known}$$

$$\left\{\begin{array}{l} P \to P+\beta \\ P \to P+\beta^{-1} \end{array}\right.$$



Barnes double Gamma fct

$\rightarrow$ poles $x \in -\beta\mathbb{N} - \beta^{-1}\mathbb{N}$

$$\frac{\Gamma_\beta(x+\beta)}{\Gamma_\beta(x)} = \frac{\sqrt{2\pi}\,\beta^{\beta x - \frac{1}{2}}}{\Gamma(\beta x)} \qquad \frac{\Gamma_\beta(x+\beta^{-1})}{\Gamma_\beta(x)} = \frac{\sqrt{2\pi}\,\beta^{\frac{1}{2} - \beta^{-1}x}}{\Gamma(\beta^{-1}x)}$$

$\beta^2 \in \mathbb{R}_{>0}$ $(c \leq 1)$

$$C_{P_1P_2P_3} = \prod_{\pm\pm\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} \pm P_1 \pm P_2 \pm P_3\right) \qquad \left(C_{\sigma\sigma\varepsilon} = \frac{1}{2} \quad \beta^2 = \frac{3}{4}\right)$$

MM: $P_i \to P_{(r_i,s_i)}$

$\beta^2 = p/q$

$$C_{P_1P_2P_3} \to \lambda_{P_1}\lambda_{P_2}\lambda_{P_3} C_{P_1P_2P_3} \Rightarrow C_{\mathbb{1}jj} = 1$$

DOZZ - Wormille

$$\lim_{P \to P_{(r,s)}} V_P \sim V^d_{(r,s)}$$

$$C^{DOZZ} = \frac{\prod_{\pm\pm\pm} \Gamma_{i\beta}\left(\frac{i}{2}(\beta-\beta^{-1}) \pm iP_1 \pm iP_2 \pm iP_3\right)}{\prod_{k=1}^{3} \Gamma_{i\beta}(2iP_k)\,\Gamma_{i\beta}\left(i(\beta-\beta^{-1}) - 2iP_k\right)}$$

Loop models.

$V^d_{(1,2)} \rightsquigarrow$ $V_{(r,s)}$ $\quad s \in \frac{1}{2r}\mathbb{Z}$ $\quad V_P$

$s \to s+2$ $\quad P \to P+\beta^{-1}$

$\cancel{r \to r+2}$ $\quad \cancel{P \to P+\beta}$

$$C^{\text{ref}}_{(r_1,s_1)(r_2 s_2)(r_3,s_3)} = \prod_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} + \frac{\beta}{2}\left|\sum \varepsilon_i r_i\right| + \frac{\beta^{-1}}{2}\sum \varepsilon_i s_i\right)$$

$r \in \frac{1}{2}\mathbb{Z}$

$s \in \frac{1}{2r}\mathbb{Z}$

- Solves shift eqn up to signs.
- permutations of 3 fields.
- Reduces to Liouville 3pt fct $(r_i, s_i) = (0, 2\beta P_i)$

Numerics $\to$ $C^{\text{num}} \to \dfrac{C^{\text{num}}}{C^{\text{ref}}}$

Numerical bootstrap. $\boxed{-1 < s \leq 1}$

$$\sum_{k\in S^{(s)}} D_k^{(s)} g^{(s)}_{\Delta_k, \bar\Delta_k} = \dots$$

Unknowns $\{D_k^{(s)}\}$

$D^{(z)}_{(r,s)}$

$V^{\uparrow}_{(r,s)}$

$s \to 2s+2$

$V_{(\frac{1}{2},0)}$

$V_{(3,0)}\ V_{(1,\cdot)}$

$V_{(\frac{3}{2},0)}\ V_{(\frac{3}{2},\pm\frac{2}{3})}$

$\dots$

Interchiral blocks

$$\tilde{g}_{(r,s)} = \sum_{k\in\mathbb{Z}} \frac{D^{(s)}_{(r,s+2k)}}{D_{(r,s)}} \mathcal{F}_{P_{(r,s+2k)}} \bar{\mathcal{F}}_{P_{(r,-s-2k)}}$$

$TVV \to$ conf. blocks

$V^d_{(1,2)}VV \to$ interchiral blocks.

$$S_\Lambda = \{ k \in S \mid \mathrm{Re}(\Delta_k + \overline{\Delta}_k) < \Lambda \}$$

$\hookrightarrow \sim \beta^2 r^2 + \beta^{-2} s^2$

truncate $\{ V_{(r,s)} \} \to$ finite $\quad \mathrm{Re}\,\beta^2 > 0$

finite nb of unknowns $\{ D_k^{(z_0)} \}$ $\quad Z = \{ z_i \}$

crossing sym $\forall z \in \overline{\mathbb{C}}$

$z = z_0$

$(0,1,\infty) \to (1,0,\infty)$

$\frac{az+b}{cz+d}$ $\quad z \mapsto 1-z \Rightarrow$ fixed $z_0 = \frac{1}{2}$

$(0,1,\infty) \to (1,\infty,0)$

$z_0 = e^{\pm i\frac{\pi}{3}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\left\{ D_k^{(x),\Lambda,Z} \right\}_{x,k} = 1 \text{ approx sol}^{\underline{n}} .$$

$$\lim_{\Lambda \to \infty} D_k^{(x),\Lambda,Z} \overset{?}{=} D_k^{(x)}$$

$$Z = \{ z_j \}$$

$$\varepsilon_k^{(x),\Lambda} = \left| 1 - \frac{D_k^{(x),\Lambda,Z_1}}{D_k^{(x),\Lambda,Z_2}} \right|$$

$$\varepsilon \sim O(10^{-2})$$

$$\log \left(\varepsilon_k^{(x),\Lambda} D_k^{(x)} g^{(x)}_{D_k \bar{D} W} \right) \sim -\Lambda$$

$$\Big\langle V_{(\frac{3}{2},\frac{2}{3})}\, \overset{c\leq 1}{V_{(1,0)}}\, \overset{c\geq 25}{V_{(1,0)}}\, V_{(\frac{1}{2},0)} \Big\rangle \quad \text{in } O(n)$$

$$S^{(s)} = S^{(u)} = \{V_{(r,s)}\}_{\substack{r\in \frac{1}{2}\mathbb{N} \\ s\in\frac{1}{2}\mathbb{Z}}}$$

$$S^{(t)} = \{V_{(r,s)}\}_{\substack{r\in \mathbb{N}^* \\ s\in\frac{1}{2}\mathbb{Z}}}$$

$\Uparrow$ unique sol^n

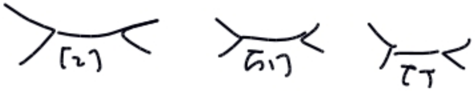
$$D^{(s)}_{(\frac{1}{2},0)} = 1, \quad D^{(t)}_{(1,0)} = D^{(t)}_{(1,1)} = D^{(u)}_{(\frac{1}{2},0)} = 0$$

$V_{(\frac{1}{2},0)}$ $[1]$ $\quad V_{(1,0)}$ $[2]$ $\quad V_{(1,1)}$ $[11]$

$V_{(\frac{3}{2},0)}$ $[3]+[111]$ $\quad V_{(\frac{3}{2},\frac{1}{2})}$ $[21]$

$$\langle V_{(\frac{1}{2},0)}^{4} \rangle$$

$$\dim \mathrm{Inv}\left([1]^{\otimes 4} \to \mathrm{id}\right) = 3$$



$$V_{(\frac{1}{2},0)}^{[1]}$$

$$[1]\otimes[1] = [2]+[11]+[\,]$$

$\to 3$ sol^n ?

$\lambda \in \{1,2,3\}$

$$\sum_{\lambda} \langle\ \ \rangle T^{\lambda}(s)$$

$$= \sum_{\lambda'} \langle \cdots \rangle T^{\lambda'}(t).$$

$$\langle V^{i_1} V^{i_2} V^{i_3} V^{i_4} \rangle T^{\lambda}_{i_1 i_2 i_3 i_4}$$