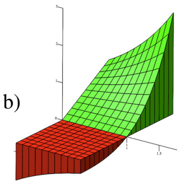
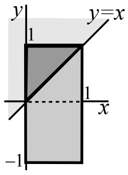
1. a) $\int_0^1\int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_{-1}^1\int_0^1 x^2 dx\,dy + \int_0^1\int_{-1}^1 y^2 dy\,dx = 2\int_0^1 x^2dx + 2\int_0^1 y^2 dy = \frac{4}{3}$.

b) $\int_{-1}^1\int_0^1 y\,e^{xy}dx\,dy = \int_{-1}^1 \big[e^{xy}\big]_0^1 dy = \int_{-1}^1 \big[e^y-1\big]\,dy = e-e^{-1}-2$ (≈ 0.35).

b) 

c) Como el integrando tiene dos expresiones en R, debemos hallar dos integrales:

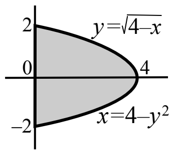
$\int_0^1\int_x^1 (y-x)\,dy\,dx + \int_0^1\int_{-1}^x (x-y)\,dy\,dx = \int_0^1 \big[\frac{(x-1)^2}{2}+\frac{(x+1)^2}{2}\big]\,dx = \int_0^1 (x^2+1)dx = \frac{4}{3}$.



d) $\int_{-1}^1\int_0^1 (xy)^2\cos x^3\,dx\,dy = \int_{-1}^1 y^2\big[\frac{1}{3}\,\text{sen}\,x^3\big]_0^1 dy = \frac{2}{3}\,\text{sen}\,1\int_0^1 y^2dy = \frac{2}{9}\,\text{sen}\,1$.

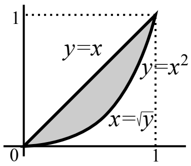
2. a) $\int_1^2\int_1^2 \log(xy)\,dx\,dy = \int_1^2 \log x\,dx + \int_1^2 \log y\,dy = 2[2\log 2-1]$, pues $\int_1^2 \log s\,ds = s\log s\big]_1^2 - \int_1^2 1\,ds = 2\log 2-1$.

b) $\int_{-2}^2\int_0^{4-y^2} x^3y\,dx\,dy = \frac{1}{4}\int_{-2}^2 \underbrace{y\,(4-y^2)^4}_{\text{impar}}\,dy = 0$. O bien $\int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} x^3y\,dy\,dx = \int_0^4 0\,dx = 0$.



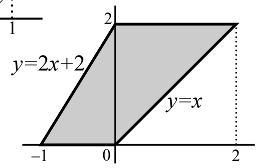
c) $\int_0^1\int_{x^2}^x xy\,dy\,dx = \int_0^1 x\big[\frac{y^2}{2}\big]_{x^2}^x dx = \frac{1}{2}\int_0^1 [x^3-x^5]\,dx = \frac{1}{2}\big[\frac{1}{4}-\frac{1}{6}\big] = \frac{1}{24}$.

O bien $\int_0^1\int_y^{\sqrt{y}} xy\,dx\,dy = \int_0^1 y\big[\frac{x^2}{2}\big]_y^{\sqrt{y}} dy = \frac{1}{2}\int_0^1 [y^2-y^3]\,dy = \frac{1}{2}\big[\frac{1}{3}-\frac{1}{4}\big] = \frac{1}{24}$.



d) $\int_0^2\int_{y/2-1}^y e^{x-y}\,dx\,dy = \int_0^2 \big[1-e^{-1-y/2}\big]dy = 2\big[1-\frac{1}{e}+\frac{1}{e^2}\big]$, o más largo:

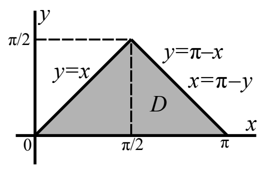
$\int_{-1}^0\int_0^{2x+2} e^{x-y}\,dy\,dx + \int_0^2\int_x^2 e^{x-y}\,dy\,dx = \int_{-1}^0 \big[e^x-e^{-x-2}\big]dx + \int_0^2 \big[1-e^{x-2}\big]dx$.



e) Mejor: $\int_0^{\pi/2}\int_y^{\pi-y} \text{sen}\,x\,dx\,dy = \int_0^{\pi/2}\big[\cos y - \underbrace{\cos(\pi-y)}_{=-\cos y}\big]\,dy = \big[2\,\text{sen}\,y\big]_0^{\pi/2} = 2$.

Peor: $\int_0^{\pi/2}\int_0^x \text{sen}\,x\,dy\,dx + \int_{\pi/2}^{\pi}\int_0^{\pi-x} \text{sen}\,x\,dy\,dx = \int_0^{\pi/2} x\,\text{sen}\,x\,dx + \int_{\pi/2}^{\pi}(\pi-x)\,\text{sen}\,x\,dx$

$= -\big[x\cos x\big]_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx - \big[(\pi-x)\cos x\big]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\cos x\,dx = 2$.

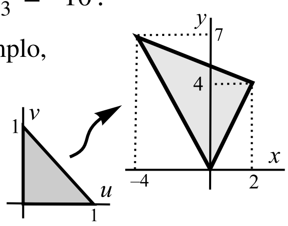


f) $\int_{-4}^0\int_{-7x/4}^{5-x/2} x\,dy\,dx + \int_0^2\int_{2x}^{5-x/2} x\,dy\,dx = 5\int_{-4}^0\big[x+\frac{x^2}{4}\big] + 5\int_0^2\big[x-\frac{x^2}{2}\big] = -\frac{40}{3}+\frac{10}{3} = -10$.

Con un cambio lineal podríamos llevar el triángulo a uno más sencillo; por ejemplo, el que lleva $(1,0)$ a $(2,4)$ y $(0,1)$ a $(-4,7)$, es decir, el dado por la matriz:

$\mathbf{A} = \begin{pmatrix}2 & -4\\ 4 & 7\end{pmatrix}$, o sea, $\begin{pmatrix}x\\ y\end{pmatrix} = \mathbf{A}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}2u-4v\\ 4u+7v\end{pmatrix}$; $\big|\frac{\partial(x,y)}{\partial(u,v)}\big| = \begin{vmatrix}2 & -4\\ 4 & 7\end{vmatrix} = 30$;

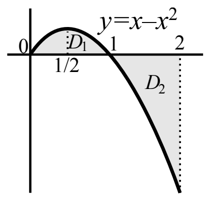
$\int_0^1\int_0^{1-u} 60(u-2v)\,dv\,du = 60\int_0^1\big[u(1-u)-(1-u)^2\big]du = -10$.



g) El recinto aparece dividido en dos regiones: $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$.

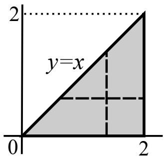
$\int_0^1\int_0^{x-x^2}(x^2+6xy^2)dy\,dx + \int_1^2\int_{x-x^2}^0 (x^2+6xy^2)dy\,dx = \int_0^1\big[x^2(x-x^2)+x(x-x^2)^3\big]dx - \int_1^2 [\cdot\cdot]\,dx$

$= \int_0^1 (x^3+x^4-6x^5+6x^6-2x^7)\,dx - \int_1^2(\cdot\cdot)\,dx = \frac{2}{35}+\frac{278}{35} = 8$.



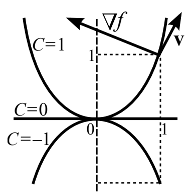
3. $\int_0^2\int_y^2 e^{x^2}dx\,dy = \int_0^2\int_0^x e^{x^2}dy\,dx = \int_0^2 x\,e^{x^2}dx = \frac{1}{2}\,e^{x^2}\big]_0^2 = \frac{1}{2}\big[e^4-1\big]$.

(La integral inicial no tiene primitiva elemental).



4. a) $\boxed{f(x,y) = \frac{y}{x^2}} = 0,1,-1 \;\to\; y=0,\ y=x^2,\ y=-x^2$ (parábolas).

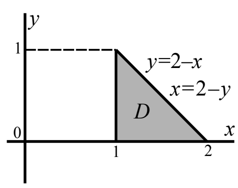
$\nabla f = \big(\frac{-2y}{x^3}, \frac{1}{x^2}\big)\Big|_{(1,1)} = (-2,1)$. $\Delta f = \frac{6y}{x^4}$. $D_{\mathbf{v}}f(1,1) = (-2,1)\cdot\big(\frac{3}{5},\frac{4}{5}\big) = -\frac{2}{5}$.



b) $\iint_D f = \int_1^2\int_0^{2-x}\frac{y}{x^2}\,dy\,dx = \frac{1}{2}\int_1^2\frac{4-4x+x^2}{x^2}\,dx = \big[-\frac{2}{x}-2\ln|x|\big]_1^2 + \frac{1}{2} = \frac{3}{2}-2\ln 2$.

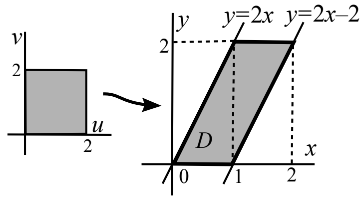
Algo más corto que:

$\int_0^1\int_1^{2-y}\frac{y}{x^2}\,dx\,dy = \int_0^1 y\,\big[1-\frac{1}{2-y}\big]\,dy = \int_0^1\big[y+1-\frac{2}{2-y}\big]\,dy = \frac{1}{2}+1+2\ln 2$.



5. a) i) Para no hacer 2 integrales, es mejor integrar primero respecto a x:

$$\int_0^2\int_{y/2}^{y/2+1}(2x-y)^3\,dx\,dy=\int_0^2\tfrac{1}{8}\big[(2x-y)^4\big]_{y/2}^{y/2+1}dy=\int_0^2 2\,dy=\boxed{4}\,.$$



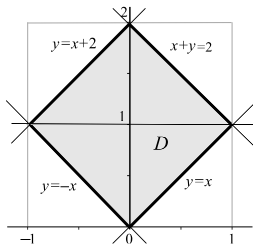
Más largo: $\int_0^1\int_0^{2x}(2x-y)^3dy\,dx+\int_1^2\int_{2x-2}^{2}(2x-y)^3dy\,dx$

$$=\int_0^1 4x^4dx+\int_1^2\big[4-4(x-1)^4\big]\,dx=\tfrac{4}{5}+4-\tfrac{4}{5}=4\,.$$

ii) Con el cambio: $\begin{matrix}u=2x-y\\ v=y\end{matrix}$, $\begin{matrix}x=\frac{u+v}{2}\\ y=v\end{matrix}$, $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/2&1/2\\0&1\end{vmatrix}=\frac{1}{2}$. $\frac{1}{2}\int_0^2\int_0^2 u^3du\,dv=1\cdot\big[\frac{1}{4}u^4\big]_0^2=\boxed{4}$.

b) $\displaystyle\int_{-1}^{0}\int_{-x}^{x+2}e^{y-x}\,dy\,dx+\int_0^1\int_x^{2-x}e^{y-x}\,dy\,dx=\int_{-1}^{0}\big[e^2-e^{-2x}\big]\,dx+\int_0^1\big[e^{2-2x}-1\big]dx$

$$=e^2-1+\tfrac{1}{2}\big[e^{-2x}\big]_{-1}^{0}-\tfrac{1}{2}\big[e^{2-2x}\big]_0^1=\boxed{e^2-1}\,.$$

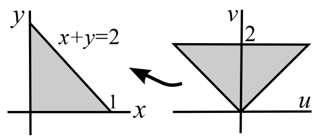


$$\int_0^1\int_{-y}^{y}e^{y-x}\,dx\,dy+\int_1^2\int_{y-2}^{2-y}e^{y-x}\,dx\,dy=\int_0^1\big[e^{2y}-1\big]\,dy+\int_1^2\big[e^2-e^{2y-2}\big]\,dy=\cdots=\boxed{e^2-1}\,.$$

Pide hacer $\begin{cases}u=y-x\\ v=y+x\end{cases}$. Despejando $\begin{cases}x=\frac{1}{2}(v-u)\\ y=\frac{1}{2}(u+v)\end{cases}$, $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-1/2&1/2\\1/2&1/2\end{vmatrix}=-\frac{1}{2}$, $|J|=\frac{1}{2}$.

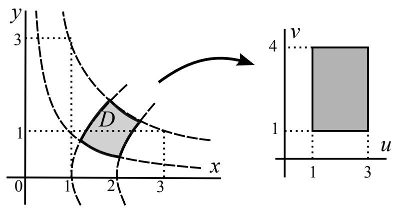
Los lados pasan a ser $u,v=0,2$. La integral se convierte en: $\frac{1}{2}\int_0^2\int_0^2 e^u\,du\,dv=\frac{1}{2}\int_0^2\big[e^2-1\big]\,dv=\boxed{e^2-1}$.

6. $\begin{cases}u=y-x\\ v=y+x\end{cases}\Leftrightarrow\begin{cases}x=(v-u)/2\\ y=(u+v)/2\end{cases}$. $J=\begin{vmatrix}-1/2&1/2\\1/2&1/2\end{vmatrix}=-\frac{1}{2}$. $\begin{matrix}x=0\to u=v\\ y=0\to u=-v\\ x+y=2\to v=2\end{matrix}$



Luego, $\displaystyle\iint_D e^{(y-x)/(y+x)}\,dx\,dy=\frac{1}{2}\int_0^2\int_{-v}^{v}e^{u/v}du\,dv=\int_0^2 v(e-\tfrac{1}{e})\,dv=e-\tfrac{1}{e}$.

7. $u=xy$, $v=x^2-y^2\to\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}y&x\\2x&-2y\end{vmatrix}=-2(x^2+y^2)\to$



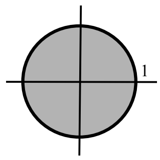
$$\iint_D(x^2+y^2)\,dx\,dy=\int_1^3\int_1^4\tfrac{1}{2}\,dv\,du=3\,.$$

↘ [Casualidad que coincida casi con el jacobiano].

[Despejar x,y en función de u,v es complicado].

8. a) $\displaystyle\int_{-1}^{1}x^3\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\,dx=2\int_{-1}^{1}\underbrace{x^3\sqrt{1-x^2}}_{\text{impar}}\,dx=0$. $\displaystyle\int_0^1 r^4\,dr\int_0^{2\pi}\cos^3\theta\,d\theta=\frac{1}{5}\int_0^{2\pi}\cos\theta(1-\mathrm{sen}^2\theta)\,d\theta=0$.

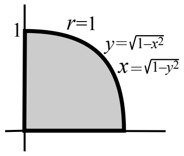
[integral de función impar en recinto simétrico]



b) $\displaystyle\int_0^1 r^5dr\int_0^{2\pi}\cos^4\theta\,d\theta=\frac{1}{24}\int_0^{2\pi}\big(1+2\cos2\theta+\tfrac{1}{2}[1+\cos4\theta]\big)\,d\theta=\frac{1}{24}\cdot2\pi\cdot\frac{3}{2}=\frac{\pi}{8}$.

$$\int_{-1}^{1}x^4\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\,dx=4\int_0^1 x^4\sqrt{1-x^2}\,dx=4\int_0^{\pi/2}\mathrm{sen}^4t\cos^2t\,dt=\cdots=\frac{\pi}{8}\,.$$

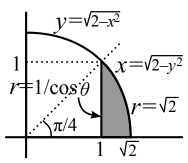
9. a) i) Cartesianas: $\displaystyle\iint_D f=\int_0^1\int_0^{\sqrt{1-x^2}}x^2y\,dy\,dx=\frac{1}{2}\int_0^1 x^2(1-x^2)\,dx=\frac{1}{2}\big[\tfrac{1}{3}-\tfrac{1}{5}\big]=\frac{1}{15}$.



Más largo: $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}}x^2y\,dx\,dy=\frac{1}{3}\int_0^1 y\,(1-y^2)^{3/2}\,dy=-\frac{1}{15}\,(1-y^2)^{5/2}\Big]_0^1=\frac{1}{15}$.

ii) En polares: $\displaystyle\iint_D f=\int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\ \mathrm{sen}\,\theta\,dr\,d\theta=\big[\tfrac{1}{5}r^5\big]_0^1\big[-\tfrac{1}{3}\cos^3\theta\big]_0^{\pi/2}=\frac{1}{5}\cdot\frac{1}{3}=\frac{1}{15}$.

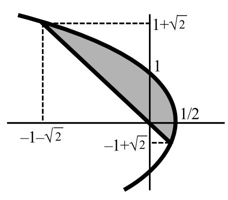
b) i) $\displaystyle\int_1^{\sqrt2}\int_0^{\sqrt{2-x^2}}x\,dy\,dx=\int_1^{\sqrt2}x\sqrt{2-x^2}\,dx=-\frac{1}{3}(2-x^2)^{3/2}\Big]_1^{\sqrt2}=\frac{1}{3}$, o bien



$$\int_0^1\int_1^{\sqrt{2-y^2}}x\,dx\,dy=\int_0^1\big[\tfrac{x^2}{2}\big]_1^{\sqrt{2-y^2}}dy=\int_0^1\frac{1-y^2}{2}\,dy=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}\,.$$

ii) $\displaystyle\int_0^{\pi/4}\int_{1/\cos\theta}^{\sqrt2}r^2\cos\theta\,dr\,d\theta=\frac{1}{3}\int_0^{\pi/4}\big[2\sqrt2\,\cos\theta-\tfrac{1}{\cos^2\theta}\big]d\theta=\frac{1}{3}[2\sqrt2\ \mathrm{sen}\,\theta-\tan\theta]_0^{\pi/4}=\frac{1}{3}[2-1]=\frac{1}{3}$.

c) $r+r\cos\theta=1$, $x^2+y^2=(1-x)^2$, $x=\frac{1-y^2}{2}$ que corta $y=-x$ si $y=-1\pm\sqrt2\ \to$

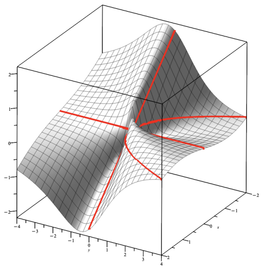


i) $A=\frac{1}{2}\int_{1-\sqrt2}^{1+\sqrt2}\big(1-y^2+2y\big)\,dy=\sqrt2+\frac{1}{2}\big[y^2-\frac{1}{3}y^3\big]_{1-\sqrt2}^{1+\sqrt2}=\frac{4}{3}\sqrt2$.

$\tan\frac{3\pi}{8}=1+\sqrt2$

ii) $\displaystyle\int_{-\pi/4}^{3\pi/4}\int_0^{1/(1+\cos\theta)}r\,dr\,d\theta=\frac{1}{2}\int_{-\pi/4}^{3\pi/4}\frac{d\theta}{(1+\cos\theta)^2}=\big[u=\tan\tfrac{\theta}{2}\ \cdots\big]=\frac{1}{4}\big[\tan\tfrac{\theta}{2}+\tfrac{1}{3}\tan^3\tfrac{\theta}{2}\big]_{-\pi/4}^{3\pi/4}\overset{\downarrow}{=}\frac{4}{3}\sqrt2$.

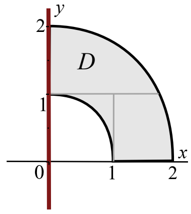
10. $\boxed{g(x,y)=\frac{xy-x^3}{x^2+y^2}}$ a) $g(x,0)=-x\ \forall x \Rightarrow g_x(0,0)=-1$. $g(0,y)=0\ \forall y \Rightarrow g_y(0,0)=0$.



Por rectas: $g(x,mx)=\frac{m+x}{1+m^2}\xrightarrow{x\to 0}\frac{m}{1+m^2}$ distinto para cada $m \Rightarrow$

g es **discontinua** en el origen $\Rightarrow$ g **no es diferenciable** en $(0,0)$.

b) $g(r,\theta)=\cos\theta\,\mathrm{sen}\,\theta-r\cos^3\theta$ ($\to cs$ distinto según θ, otra prueba de la discontinuidad).



$$\int_0^{\pi/2}\!\!\int_1^2\big(rcs-r^2c^3\big)\,d\theta=\int_0^{\pi/2}\big[\tfrac{3}{2}cs-\tfrac{7}{3}(c-cs^2)\big]\,d\theta$$
$$=\big[\tfrac{3}{4}s^2-\tfrac{7}{3}s+\tfrac{7}{9}s^3\big]_0^{\pi/2}=\tfrac{3}{4}-\tfrac{7}{3}+\tfrac{7}{9}=\boxed{-\tfrac{29}{36}}\,.$$

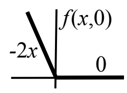
[En cartesianas, mucho peor, es mejor $dx\,dy$ y habría que hacer dos integrales, alguna larga con partes:

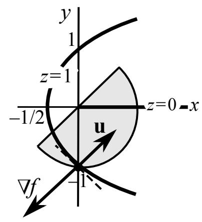
$$g(x,y)=\frac{xy}{x^2+y^2}-x+\frac{xy^2}{x^2+y^2}\ \to\ \int g(x,y)\,dx=\tfrac{1}{2}\big[(y+y^2)\ln(x^2+y^2)-x^2\big]\equiv G(x,y)\,,$$

$$\int_0^1[G]_{\sqrt{1-y^2}}^{\sqrt{4-y^2}}dy+\int_1^2[G]_0^{\sqrt{4-y^2}}dy=\int_0^1\big[\ln 2(y+y^2)-\tfrac{3}{2}\big]\,dy+\int_1^2\big[(\ln 2-\ln y)(y+y^2)+\tfrac{1}{2}y^2-2\big]\,dy=\cdots\ ].$$

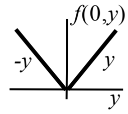
11. $\boxed{f(x,y)=\sqrt{x^2+y^2}-x}$ a) $f(x,0)=|x|-x$, $f(0,y)=|y|$ no derivables

$\Rightarrow \nexists$ **parciales** en $(0,0)$ $\Rightarrow$ f **no es diferenciable**.





[Que f es continua en $\mathbf{R}^2$ es obvio, pues lo es la raíz para valores positivos].



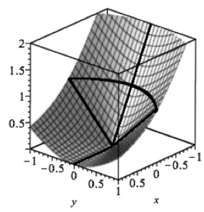
b) $f=1 \Rightarrow x^2+y^2=x^2+2x+1$, $x=\frac{1}{2}[y^2-1]$ [parábola con $x'(-1)=-1$].

$\nabla f=\Big(\frac{x}{\sqrt{x^2+y^2}}-1\,,\,\frac{y}{\sqrt{x^2+y^2}}\Big)$, $\nabla f(0,-1)=(-1,-1) \Rightarrow \bar{u}=\big(\frac{1}{\sqrt{2}}\,,\,\frac{1}{\sqrt{2}}\big)$,

pues la derivada direccional el mínima en sentido opuesto al gradiente.

[En polares: $\nabla f=f_r\,\mathbf{e_r}+\frac{1}{r}f_\theta\,\mathbf{e}_\theta=(\cos\theta-1,\,\mathrm{sen}\,\theta)$].

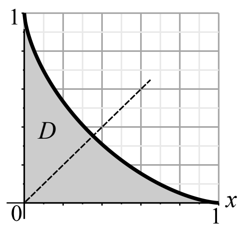
[También se deduce del hecho de que el gradiente es perpendicular a la curva de nivel en el punto y de que apunta hacia donde crece el campo].



dibujo de f con Maple

c) $\displaystyle\int_{-3\pi/4}^{\pi/4}\!\!\int_0^1 r^2(1-\cos\theta)\,dr\,d\theta=\tfrac{1}{3}\big(\pi-[\,\mathrm{sen}\,\theta]_{-3\pi/4}^{\pi/4}\big)=\boxed{\tfrac{\pi-\sqrt{2}}{3}}$. [En cartesianas las integrales son bastante complicadas].

12. $\boxed{x^{2/3}+y^{2/3}\le 1}$ Como $x^{2/3}+y^{2/3}=1$ es simétrica, tiene $y=\big[1-x^{2/3}\big]^{3/2}$ pendiente 0 en $(1,0)$, corta $y=x$ si $x=\sqrt{2}/4$, ... la región D es la del dibujo (este con Maple):



Jacobiano: $\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}c^3 & -3rc^2s\\ s^3 & 3rs^2c\end{vmatrix}=3r\,\mathrm{sen}^2\theta\cos^2\theta\big(\mathrm{sen}^2\theta+\cos^2\theta\big)=3r\,\mathrm{sen}^2\theta\cos^2\theta$.

Veamos que el conjunto $D^*=[0,1]\times\big[0,\frac{\pi}{2}\big]$ se transforma en nuestro recinto D :

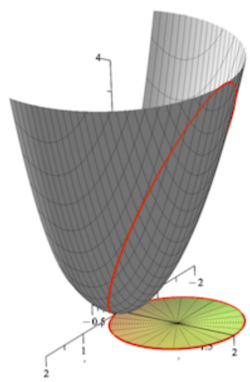
$\theta=0\to y=0$, $\theta=\frac{\pi}{2}\to x=0$; $x^{2/3}+y^{2/3}=r^{2/3}\big(\mathrm{sen}^2\theta+\cos^2\theta\big)=r^{2/3}\le 1$, $r\le 1$.

[Y en D^* es inyectivo, salvo en $(0,0)$, como las polares: $\theta=K$ constante lleva a distintas rectas $y=(\tan^3K)x$].

$$\text{Área}=\iint_D 1=\tfrac{3}{4}\int_0^{\pi/2}\!\!\int_0^1 r\,\mathrm{sen}^2 2\theta\,dr\,d\theta=\tfrac{3}{16}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta=\tfrac{3\pi}{32}-0=\boxed{\tfrac{3\pi}{32}}\,.$$

13. z positiva en el rectángulo: $V=\int_0^1\!\!\int_1^2(x^2+y)\,dy\,dx=\int_0^1x^2\,dx+\int_1^2y\,dy=\frac{1}{3}+\frac{4-1}{2}=\boxed{\frac{11}{6}}$.

14. Se puede describir el círculo $x^2+(y-1)^2\le 1$ con polares centradas en $(0,1)$:



$x=\rho\cos\phi$, $y=1+\rho\,\mathrm{sen}\,\phi$, con $\rho\le 1$ y $\phi\in[0,2\pi]$. [sigue siendo $J=\rho$].

Con ellas se tiene $z=\rho^2\cos^2\phi+\rho^2\mathrm{sen}^2\phi+2\rho\,\mathrm{sen}\,\phi+1$, y el volumen es:

$$V=\int_0^{2\pi}\!\!\int_0^1\big(\rho^3+\rho+2\rho^2\mathrm{sen}\,\phi\big)\,d\rho\,d\phi=\int_0^{2\pi}\big(\tfrac{3}{4}+\tfrac{2}{3}\,\mathrm{sen}\,\phi\big)\,d\phi=\boxed{\tfrac{3\pi}{2}}\,.$$

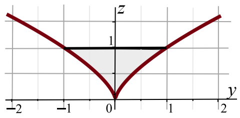
No sale mal en polares usuales: $r^2=2r\,\mathrm{sen}\,\theta$, $r=2\,\mathrm{sen}\,\theta$, $\theta\in[0,\pi]$. Y es $z=r^2$.

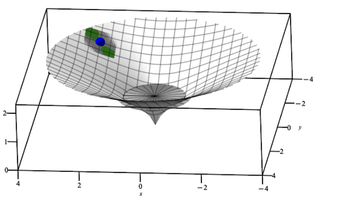
Así pues: $V=\int_0^\pi\!\!\int_0^{2\,\mathrm{sen}\,\theta}r^3\,dr\,d\theta=\int_0^\pi 4\,\mathrm{sen}^4\theta\,d\theta=\int_0^\pi\big(1-\cos 2\theta\big)^2d\theta=\boxed{\frac{3\pi}{2}}$.

En ambos órdenes de integración en cartesianas se complica. Por ejemplo:

$$\int_{-1}^1\!\!\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}(x^2+y^2)\,dz\,dy\,dx=2\int_0^1\Big[2x^2\sqrt{1-x^2}+\frac{(1+\sqrt{\cdot})^3-(1-\sqrt{\cdot})^3}{3}\Big]dx=\tfrac{8}{3}\int_0^1(x^2+2)\sqrt{1-x^2}\,dx=\cdots$$

15. $\boxed{g(x,y)=(x^2+y^2)^{1/3}}$ a) $g(x,0)=x^{2/3}$ y $g(0,y)=y^{2/3}$ no derivables en 0 $\Rightarrow$ no hay parciales $\Rightarrow$ **no diferenciable**.



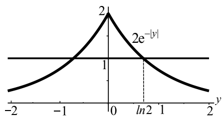


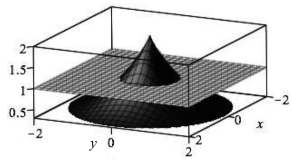
b) $g(2,-2)=2$, $\nabla g=\frac{2}{3(x^2+y^2)^{2/3}}(x,y)\xrightarrow{(2,-2)}\left(\frac13,-\frac13\right)$.

Plano tangente $z=2+\frac13(x-2)-\frac13(y+2)$, $\boxed{z=\frac13(2+x-y)}$

c) $z=1$ si $x^2+y^2=1$. $g(r,\theta)=r^{2/3}$. $J=r$. Volumen: $\int_0^{2\pi}\int_0^1 r\left(1-r^{2/3}\right)dr\,d\theta = 2\pi\left[\frac12 r^2-\frac38 r^{8/3}\right]_0^1=\pi\left[1-\frac34\right]=\boxed{\frac{\pi}{4}}$.

16. $\boxed{g(x,y)=2\,e^{-\sqrt{x^2+y^2}}}$ a) De revolución. $g(0,y)=2e^{-|y|}\Rightarrow g_y(0,0)$ no existe $\Rightarrow$ g **no diferenciable** en $(0,0)$.

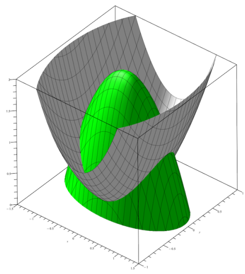




b) $2\,e^{-\sqrt{x^2+y^2}}=1 \Leftrightarrow r=\sqrt{x^2+y^2}=\log 2$. Polares-cilíndricas:

$$V=\int_0^{2\pi}\int_0^{\ln 2} r\left[2\,e^{-r}-1\right]dr\,d\theta = 2\pi\left[-2(r+1)\,e^{-r}-\tfrac12 r^2\right]_0^{\ln 2}=2\pi\left[1-\ln 2-\tfrac12(\ln 2)^2\right].$$

17. $\boxed{z=x^2+y^2}$ y $\boxed{z=2-x^2-7y^2}$ se cortan sobre la elipse $x^2+4y^2=1$, $y=\pm\frac12\sqrt{1-x^2}$.



El volumen vendrá dado por $V=8\int_0^1\int_0^{\sqrt{1-x^2}/2}\left(1-x^2-4y^2\right)dy\,dx=\frac83\int_0^1\left(1-x^2\right)^{2/3}dx$

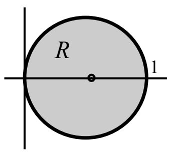
$$\stackrel{x=\text{sen}\,t}{=}\frac83\int_0^{\pi/2}\cos^4 t\,dt=\frac83\int_0^{\pi/2}\left(1+2\cos 2t+\tfrac{1+\cos 4t}{2}\right)dt=\frac{\pi}{2}.$$

O con cambio de las elipses: $x=r\cos\theta$, $y=\frac{r}{2}\,\text{sen}\,\theta$, $J=\frac{r}{2}\to V=\int_0^{2\pi}\int_0^1\left(r-r^3\right)dr\,d\theta=\frac{\pi}{2}$.

18. $M=2\int_0^{\pi/2}\int_0^{\cos\theta} r\cos\theta\,dr\,d\theta=\int_0^{\pi/2}(1-\text{sen}^2\theta)\cos\theta\,d\theta=\frac23$.

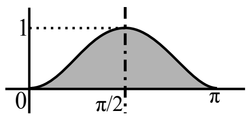
$\bar{x}=\frac{1}{M}\,2\int_0^{\pi/2}\int_0^{\cos\theta} r^2\cos^2\theta\,dr\,d\theta=\int_0^{\pi/2}\left(1-\text{sen}^2\theta\right)^2\cos\theta\,d\theta=\frac{8}{15}$.

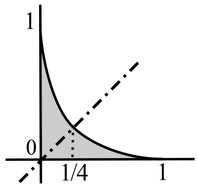
$\bar{y}=0$ por simetría de R y de la función densidad.



19. a) $\{0\le y\le \text{sen}^2 x,\ 0\le x\le\pi\}$. Por simetría, $\bar{x}=\frac{\pi}{2}$.

$$\bar{y}=\frac{1}{\int_0^\pi \text{sen}^2 x\,dx}\int_0^\pi\int_0^{\text{sen}^2 x} y\,dy\,dx=\frac{2}{\pi}\int_0^\pi\frac{\text{sen}^4 x}{2}dx=\frac38.$$



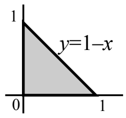


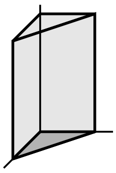
b) $\left\{\sqrt{x}+\sqrt{y}\le 1,\ x\le 0,\ y\ge 0\right\}$. $y=\left(1-\sqrt{x}\right)^2=1+x-2\sqrt{x}$.

$$\bar{x}=\frac{1}{\int_0^1(1+x-2\sqrt{x})\,dx}\int_0^1\int_0^{1+x-2\sqrt{x}} x\,dy\,dx=6\int_0^1\left(x+x^2-2x^{3/2}\right)dx=\frac15=\bar{y}\text{, por simetría.}$$

20. a) $\iiint_V(2x+3y+z)\,dx\,dy\,dz=2\int_1^2 2x\,dx+\int_{-1}^1 3y\,dy+2\int_0^1 z\,dz=2(4-1)+0+1=7$.

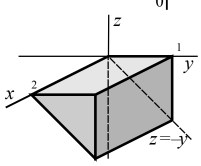
b) $\iiint_V x^2\cos z\,dx\,dy\,dz=\int_0^1\int_0^{1-x}\int_0^\pi x^2\cos z\,dz\,dy\,dx=\int_0^1\int_0^{1-x}0\,dy\,dx=0$.





c) $\int_0^2\int_0^1\int_{-y}^0 e^y\,dz\,dy\,dx=\int_0^2\int_0^1 y\,e^y\,dy\,dx=2\int_0^1 y\,e^y\,dy=2\left[(y-1)\,e^y\right]_0^1=2$.

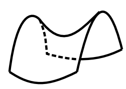
O bien: $\int_0^2\int_{-1}^0\int_{-z}^1 e^y\,dy\,dz\,dx=2\int_{-1}^0\left[e-e^{-z}\right]dz=2\,e+2\left[e^{-z}\right]_{-1}^0=2$.

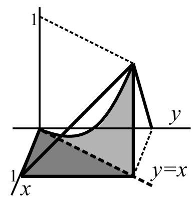


d) $\iiint_V xy^2z^3\,dx\,dy\,dz=\int_0^1\int_0^x\int_0^{xy}xy^2z^3\,dz\,dy\,dx=\frac14\int_0^1\int_0^x x^5y^6\,dy\,dx=\frac{1}{28}\int_0^1 x^{12}dx$

$=\frac{1}{364}$.

[$z=xy$ era un ‘paraboloide hiperbólico’ (silla de montar)].



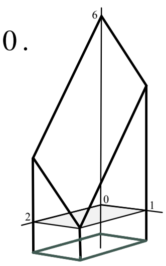


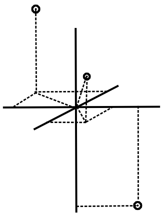
21. El plano tangente a $x^2+y^2+z=4$ es $\nabla F(1,1,2)\cdot(x-1,y-1,z-2)=2(x-1)+2(y-1)+(z-2)=0$.

Es decir, $z=6-2x-2y$.

La integral es $\iiint_V xy\,dx\,dy\,dz=\int_0^2\int_0^1\int_{-1}^{6-2x-2y}xy\,dz\,dy\,dx=\int_0^2\int_0^1 xy(7-2x-2y)\,dy\,dx$

$$=\int_0^2\left[\tfrac72 xy^2-x^2y^2-\tfrac23 xy^3\right]_0^1dx=\int_0^2\left(\tfrac{17}{6}x-x^2\right)dx=3.$$



22. 

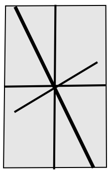
	cartesianas	polares	esféricas
P	$(0\,,2\,,-4)$	$(2,\frac{\pi}{2},-4)$	$(2\sqrt{5}\,,\frac{\pi}{2},\,\pi-\arctan\frac{1}{2})$
Q	$(-2,-2\sqrt{3}\,,\,3)$	$(4,\frac{4\pi}{3},3)$	$(5\,,\frac{4\pi}{3},\,\arctan\frac{4}{3})$
R	$(\frac{\sqrt{2}}{2}\,,\frac{\sqrt{2}}{2}\,,1)$	$(1\,,\frac{\pi}{4}\,,1)$	$(\sqrt{2}\,,\frac{\pi}{4},\frac{\pi}{4})$

23. $\{x=0\,,\ z=-2y\}$

$\{\theta=\frac{\pi}{2},\ z=-2r\ \text{ ó }\ \theta=-\frac{\pi}{2},\ z=2r\}$

$\{\theta=\frac{\pi}{2},\ \phi=\frac{5\pi}{6}\ \text{ ó }\ \theta=-\frac{\pi}{2},\ \phi=\frac{\pi}{6}\}$

recta por el origen

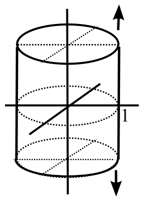


$\{x^2+y^2=1\}$

$\{r=1\}$

$\{\rho\,\text{sen}\,\theta=1\}$

cilindro vertical

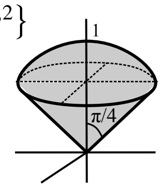


$\{x^2+y^2\le z^2\le 1-x^2-y^2\}$

$\{r\le z\le\sqrt{1-r^2}\,\}$

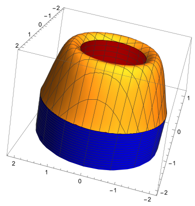
$\{\phi\le\frac{\pi}{4},\ \rho\le 1\}$

volumen entre cono y esfera



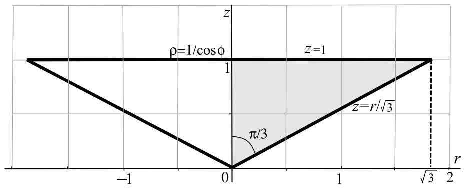
24. $\boxed{\iiint_V \frac{y^2}{x^2+y^2}\,dx\,dy\,dz}$ En cilíndricas, la función es $f(r,\theta,z)=\text{sen}^2\theta$, es $J=r$ y los límites son $1\le\rho\le 2$, $0\le\theta\le 2\pi$, $-2\le z\le \text{sen}\,r^2$, luego:

$$\iiint_V=\int_1^2\!\int_0^{2\pi}\!\int_{-2}^{\text{sen}\,r^2} r\,\text{sen}^2\theta\,dz\,d\theta\,dr=\Big[\int_1^2(2r+r\,\text{sen}\,r^2)dr\Big]\Big[\int_0^{2\pi}\text{sen}^2\theta\,d\theta\Big]=\tfrac{\pi}{2}\,(6-\cos 4+\cos 1)\,.$$



25. En cilíndricas $3z^2=x^2+y^2$ pasa a ser $z=\pm\frac{r}{\sqrt{3}}$ que corta $z=1$ si $r=\sqrt{3}$.

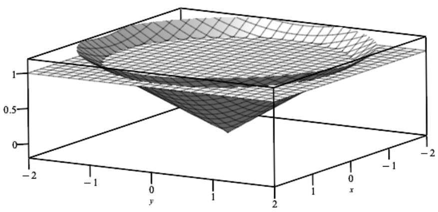
V es el cono generado al rotar el triángulo respecto al eje z.



La integral en cilíndricas se puede hacer de dos formas sencillas:

$$\iiint_V z=\int_0^{2\pi}\!\int_0^{\sqrt{3}}\!\int_{r/\sqrt{3}}^{1}\overset{J\downarrow}{r}\,z\,dz\,dr\,d\theta=\pi\int_0^{\sqrt{3}}\big[r-\tfrac{1}{3}r^3\big]dr=\pi\big[\tfrac{3}{2}-\tfrac{3}{4}\big]=\boxed{\tfrac{3\pi}{4}}\,.$$

$$\iiint_V z=\int_0^{2\pi}\!\int_0^{1}\!\int_0^{\sqrt{3}z} r\,z\,dr\,dz\,d\theta=\pi\int_0^1 3z^3\,dz=\boxed{\tfrac{3\pi}{4}}\,.$$



En esféricas, el plano se complica, pasando a ser $\rho=1/\cos\phi$.
Aunque el ángulo ϕ varía simplemente de 0 a $\frac{\pi}{3}$ $[\tan\frac{\pi}{3}=\sqrt{3}\,]$.

$$\iiint_V z=\int_0^{2\pi}\!\int_0^{\pi/3}\!\int_0^{1/\cos\phi}\rho^3\,\text{sen}\,\phi\cos\phi\,d\rho\,d\phi\,d\theta=\frac{2\pi}{4}\int_0^{\pi/3}\frac{\text{sen}\,\phi}{\cos^3\phi}\,d\phi$$
$$=\tfrac{\pi}{4}\big[\cos^{-2}\phi\big]_0^{\pi/3}=\tfrac{\pi}{4}\big[\tfrac{1}{1/4}-\tfrac{1}{1}\big]=\boxed{\tfrac{3\pi}{4}}\,.$$

En cartesianas es calculable, pero claramente más largo (la base D es el círculo $x^2+y^2=3$):

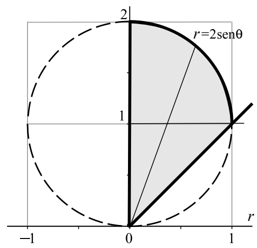
$$\iiint_V z\,dz\,dy\,dx=\int_{-\sqrt{3}}^{\sqrt{3}}\!\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\!\int_{\sqrt{x^2+y^2/3}}^{1} z\,dz\,dy\,dx=\tfrac{2}{3}\int_0^{\sqrt{3}}\!\int_0^{\sqrt{3-x^2}}\big[3-x^2-y^2\big]\,dy\,dx=\tfrac{4}{9}\int_0^{\sqrt{3}}(3-x^2)^{3/2}\,dx$$
$$\underset{x=\sqrt{3}\,\text{sen}\,t\uparrow}{=}4\int_0^{\pi/2}\cos^4t\,dt=\int_0^{\pi/2}\big(1+2\cos 2t+\tfrac{1+\cos 4t}{2}\big)\,dt=\boxed{\tfrac{3\pi}{4}}\,.$$

26. a) $y^2-2y+x^2=0\ \rightarrow\ y=1+\sqrt{1-x^2}\,,\ x=\sqrt{2y-y^2}\,.$

i) $\displaystyle\int_0^1\!\int_x^{1+\sqrt{1-x^2}} x\,dy\,dx=\int_0^1\big[x-x^2+x(2-x^2)^{1/2}\big]dx=\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{3}(1-x^2)^{3/2}\Big]_0^1=\boxed{\tfrac{1}{2}}\,.$

$\displaystyle\int_0^1\!\int_0^y x\,dx\,dy+\int_1^2\!\int_0^{\sqrt{2y-y^2}} x\,dx\,dy=\tfrac{1}{2}\int_0^1 y^2dy+\tfrac{1}{2}\int_1^2(2y-y^2)\,dy=\tfrac{1}{6}+\tfrac{1}{2}\big[y^2-\tfrac{y^3}{3}\big]_1^2=\boxed{\tfrac{1}{2}}\,.$

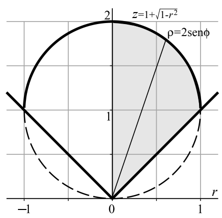
ii) $r^2=2r\,\text{sen}\,\theta\,.\ \displaystyle\int_{\pi/4}^{\pi/2}\!\int_0^{2\,\text{sen}\,\theta} r^2\cos\theta\,dr\,d\theta=\tfrac{8}{3}\int_{\pi/4}^{\pi/2}\text{sen}^3\theta\cos\theta\,d\theta=\tfrac{2}{3}\,\text{sen}^4\theta\Big]_{\pi/4}^{\pi/2}=\boxed{\tfrac{1}{2}}\,.$

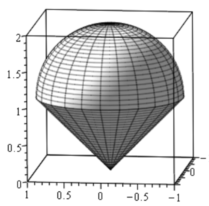


b) El cono $x^2+y^2=z^2$ y la esfera $x^2+y^2=2z-z^2$ se cortan si $z=0\,,1$.

V es el sólido de revolución del primer dibujo respecto al eje z.

i) Cilíndricas. La esfera es $z^2-2z-r^2=0$, $z=1+\sqrt{1-r^2}$ y el cono $z=r$.





$$\iiint_V 1=\int_0^{2\pi}\!\int_0^1\!\int_r^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\theta=2\pi\int_0^1\big[r-r^2+r(1-r^2)^{1/2}\,\big]dr$$
$$=2\pi\big[\tfrac{1}{2}r^2-\tfrac{1}{3}r^3-\tfrac{1}{3}(1-r^2)^{3/2}\big]_0^1=\boxed{\pi}\,.$$

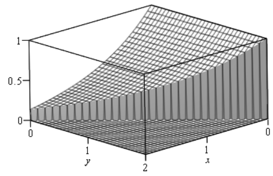
[Con el orden $dr\,dz$ aparecen dos integrales].

ii) En esféricas, la esfera toma la forma $\rho^2=2\rho\cos\phi$, $\rho=2\cos\phi$.

$$\iiint_V 1=\int_0^{2\pi}\!\int_0^{\pi/4}\!\int_0^{2\cos\phi}\rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta=\frac{2\pi}{3}\int_0^{\pi/4}8\cos^3\phi\,\text{sen}\,\phi\,d\phi=\tfrac{4\pi}{3}\big[-\cos^4\phi\big]_{\pi/4}^{\pi/2}=\tfrac{4\pi}{3}\big(1-\tfrac{1}{4}\big)=\boxed{\pi}\,.$$

27. a) $\int_0^2\int_0^{2-x}\int_0^{e^{-x}} z\,dz\,dy\,dx = \int_0^2\int_0^{2-x} \frac12 e^{-2x}dy\,dx = \int_0^2 \big(1-\frac{x}{2}\big)e^{-2x}dx$ = (partes)

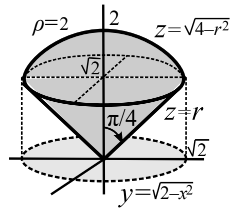
$= \big(\frac{x}{4}-\frac12\big)e^{-2x}\Big]_0^2 - \frac14\int_0^2 e^{-2x}dx = \frac12+\frac18\big[e^{-2x}\big]_0^2 = \boxed{\frac18\big(3+e^{-4}\big)}$.



[Pedía cartesianas. Algo más corto era haciendo $\int_0^2\int_0^{2-y}\int_0^{e^{-x}} z\,dz\,dx\,dy$].

b) El cono $z=\sqrt{x^2+y^2}$ y la esfera $x^2+y^2+z^2=4$ piden hacer la integral en esféricas

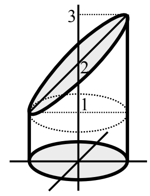
$$\iiint_V z = 2\pi\int_0^{\pi/4}\int_0^2 \rho^3 \operatorname{sen}\phi\cos\phi\,d\rho\,d\phi = \pi\big[\tfrac14\rho^4\big]_0^2\big[\operatorname{sen}^2\phi\big]_0^{\pi/4} = \boxed{2\pi}\,.$$



En cilíndricas: $2\pi\int_0^{\sqrt2}\int_r^{\sqrt{4-r^2}} rz\,dz\,dr = 2\pi\int_0^{\sqrt2} r\big(2-r^2\big)\,dr = 2\pi\big[r^2-\frac{r^4}{4}\big]_0^{\sqrt2} = 2\pi$.

En cartesianas: $\int_{-\sqrt2}^{\sqrt2}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx = \int_{-\sqrt2}^{\sqrt2}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\big(2-x^2-y^2\big)\,dy\,dx = \cdots$

28. a) $V=\iiint_V 1 = \int_0^{2\pi}\int_0^1\int_0^{r\operatorname{sen}\theta+2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2\operatorname{sen}\theta+2r)\,dr\,d\theta = \int_0^{2\pi}\big[\frac{\operatorname{sen}\theta}{3}+1\big]d\theta = 2\pi$.

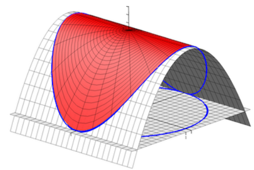


Peor en cartesianas: $V=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{y+2} dz\,dy\,dx = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(y+2)\,dy\,dx = 4\int_{-1}^1\sqrt{1-x^2}\,dx$

$= [\text{par},\ x=\operatorname{sen}t\,] = 8\int_0^{\pi/2}\cos^2 t\,dt = 4\int_0^{\pi/2}(1+\cos 2t)\,dt = 2\pi$.

[Sin integrar: volumen de cilindro de altura 1 más medio volumen de cilindro de altura 2].

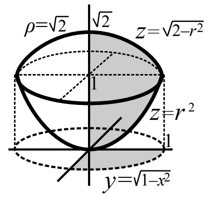
b) Sobre $x^2+y^2\le 1$ la gráfica de $z=1-x^2$ está por encima del plano $z=0$:



$$V=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2} dz\,dy\,dx = 2\int_{-1}^1 (1-x^2)^{3/2}dx \underset{x=\operatorname{sen}t}{=} 4\int_0^{\pi/2}\cos^4 t\,dt = \frac{3\pi}{4}\,.$$

Mejor cilíndricas: $V=\int_0^{2\pi}\int_0^1\int_0^{1-r^2\cos^2\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\big(r-r^3\cos^2\theta\big)dr\,d\theta = \int_0^{2\pi}\frac{3-\cos 2\theta}{8}\,d\theta = \frac{3\pi}{4}$.

c) Cilíndricas: $V=2\pi\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr = 2\pi\int_0^1\big(r\sqrt{2-r^2}-r^3\big)\,dr = \frac{\pi}{6}\big(8\sqrt2-7\big)$.



O bien: $V=\int_0^{2\pi}\int_0^1\int_0^{\sqrt z} r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^{\sqrt2}\int_0^{\sqrt{2-z^2}} r\,dr\,dz\,d\theta$

$= \pi\int_0^1 z\,dz + \pi\int_1^{\sqrt2}\big(2-z^2\big)\,dz = \frac{\pi}{2} + 2\pi\big(\sqrt2-1\big) - \frac{\pi}{3}\big(2\sqrt2-1\big) = \frac{\pi}{6}\big(8\sqrt2-7\big)$.

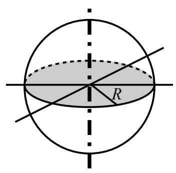
Esféricas: $V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt2}\rho^2\operatorname{sen}\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cos\phi/\operatorname{sen}^2\phi}\rho^2\operatorname{sen}\phi\,d\rho\,d\phi\,d\theta$ [de $\rho\cos\phi=\rho^2\operatorname{sen}^2\phi$]

$= 2\pi\,\frac{2\sqrt2}{3}\big[-\cos\phi\big]_0^{\pi/4} + \frac{2\pi}{3}\int_{\pi/4}^{\pi/2}\frac{\cos^3\phi}{\operatorname{sen}^5\phi}\,d\phi = \frac{4\pi}{3}\big(\sqrt2-1\big) + \frac{2\pi}{3}\int_{1/\sqrt2}^1\frac{1-t^2}{t^5}\,dt = \frac{\pi}{6}\big(8\sqrt2-7\big)$.

En cartesianas salen difíciles integrales: $V=4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx = \cdots$

29. $\iiint_V f\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{\rho^2\operatorname{sen}\phi}{\rho^3}\,d\rho\,d\phi\,d\theta = 2\pi\big[\log\rho\big]_1^2\int_0^{\pi}\operatorname{sen}\phi\,d\phi = \boxed{4\pi\log 2}$.

30. $M=\frac43\pi R^3\sigma$, con σ densidad constante.



$I_z=\iiint_V (x^2+y^2)\sigma\,dx\,dy\,dz = \sigma\int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^4\operatorname{sen}^3\phi\,d\rho\,d\phi\,d\theta = \frac{2\pi R^5\sigma}{5}\int_0^{\pi}(1-c^2)s\,d\phi = \frac{8\pi R^5\sigma}{15} = \frac{2MR^2}{5}$.

1. a) $\left(-t\,,\frac{1}{2}-t\right)$, $t\in[-1,0]$; $\left(t,\frac{1}{2}-t\right)$, $t\in[0,\frac{1}{2}]$; $\left(t,t-\frac{1}{2}\right)$, $t\in[\frac{1}{2},1]$

$\to\ L=\int_{-1}^{0}\sqrt{2}\,dt+\int_{0}^{1/2}\sqrt{2}\,dt+\int_{1/2}^{1}\sqrt{2}\,dt=2\sqrt{2}$ (¡claro!).

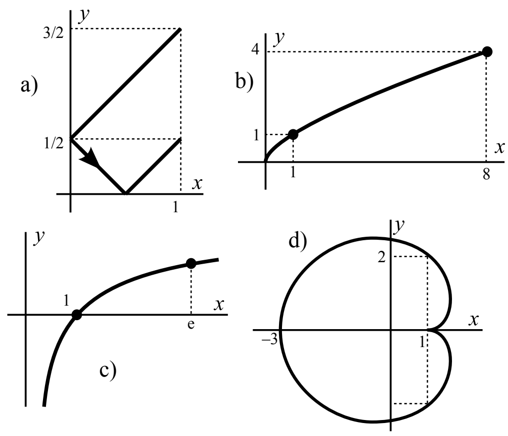
b) $L=\int_1^{e}\frac{\sqrt{x^2+1}}{x}\,dx\overset{u=x+\sqrt{x^2+1}}{=\cdots=}\sqrt{x^2+1}-\log\frac{1+\sqrt{x^2+1}}{x}\Big]_1^{e}\approx 2.0035$.

c) $L=\int_1^{8}\sqrt{1+\frac{4}{9x^{2/3}}}\,dx$ es más larga, pero parametrizando $x=y^{3/2}$,

$y\in[1,4]$, $L=\int_1^{4}\sqrt{1+\frac{9}{4}y}\,dy=\frac{8}{27}\left[10^{3/2}-\frac{1}{8}13^{3/2}\right]\approx 7.63$.

d) $\|2(\operatorname{sen}2t-\operatorname{sen}t,\cos t-\cos 2t)\|=2\sqrt{2}\sqrt{1-\cos t}=4\left|\operatorname{sen}\frac{t}{2}\right|$,

$L=8\int_0^{\pi}\operatorname{sen}\frac{t}{2}\,dt=-16\cos\frac{t}{2}\Big]_0^{\pi}=16$.

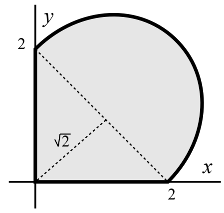


2. $r^2=2r\cos\theta+2r\operatorname{sen}\theta$, $x^2+y^2=2x+2y$, $(x-1)^2+(y-1)^2=2$, circunferencia.

Pasando de integrales: a) $A=\frac{\pi(\sqrt{2}\,)^2}{2}+\frac{2\cdot 1}{2}=\pi+2$; b) $L=2+2+\frac{2\pi\sqrt{2}}{2}=4+\pi\sqrt{2}$.

Con integrales: a) $\int_0^{\pi/2}\int_0^{2(\cos\theta+\operatorname{sen}\theta)}r\,dr\,d\theta=\int_0^{\pi/2}(2+4\operatorname{sen}\theta\cos\theta)\,d\theta=\pi+2$.

b) $r^2+(r')^2=8\ \to\ L=2+2+\int_0^{\pi/2}\sqrt{8}\,d\theta=4+\pi\sqrt{2}$.

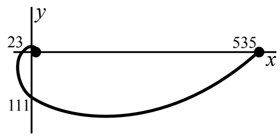


3. a) $f(x,y,z)=yz$, $\mathbf{c}(t)=(t,3t,2t)$, $\mathbf{c}'(t)=(1,3,2)$; $\int_{\mathbf{c}}f\,ds=\int_1^{3}6\,t^2\sqrt{14}\,dt=2\sqrt{14}\,t^3\Big]_1^{3}=52\sqrt{14}$.

b) $f(x,y,z)=x+z$, $\mathbf{c}(t)=\left(t,t^2,\frac{2}{3}t^3\right)$, $\mathbf{c}'(t)=(1,2t,2t^2)$; $\int_{\mathbf{c}}f\,ds=\int_0^{1}\left(t+\frac{2}{3}t^3\right)(1+2t^2)\,dt=\frac{1}{2}\left(t+\frac{2}{3}t^3\right)^2\Big]_0^{1}=\frac{25}{18}$.

4. $L=\int_0^{2\pi}\sqrt{e^{2\theta}+e^{2\theta}}\,d\theta=\sqrt{2}\int_0^{2\pi}e^{\theta}d\theta=\sqrt{2}\left[e^{2\pi}-1\right]\approx 756$.

$T_{\text{media}}=\frac{1}{L}\int_0^{2\pi}e^{\theta}\sqrt{e^{2\theta}+e^{2\theta}}\,d\theta=\frac{\sqrt{2}}{L}\int_0^{2\pi}e^{2\theta}d\theta=\frac{\sqrt{2}}{2L}\left[e^{4\pi}-1\right]=\frac{1}{2}\left[e^{2\pi}+1\right]\approx 268$.

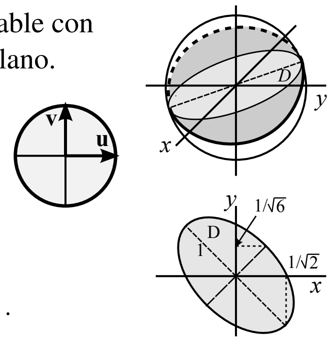


5. La intersección de $x^2+y^2+z^2=1$ y $x+y+z=0$ es una circunferencia C parametrizable con $\mathbf{c}(t)=\cos t\,\mathbf{u}+\operatorname{sen}t\,\mathbf{v}$, siendo $\mathbf{u}$, $\mathbf{v}$ vectores ortogonales unitarios contenidos en el plano.

Por ejemplo, $\mathbf{u}=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)$, $\mathbf{v}=\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$. $\|\mathbf{c}'(t)\|=1$.

$M=\int_0^{2\pi}\left(\frac{\cos t}{\sqrt{2}}+\frac{\operatorname{sen}t}{\sqrt{6}}\right)^2dt=\int_0^{2\pi}\left(\frac{\cos^2 t}{2}+\frac{\operatorname{sen}t\cos t}{\sqrt{3}}+\frac{\operatorname{sen}^2 t}{6}\right)^2dt=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$.

[La proyección de C sobre $z=0$ es $2x^2+2y^2+2xy=1$; de ella salen otras parametrizaciones con cálculos bastante más largos, por ejemplo: $\left(x,\frac{1}{2}\left[-x\pm\sqrt{2-3x^2}\,\right],\frac{1}{2}\left[x\pm\sqrt{2-3x^2}\,\right]\right)$, o parametrizando distinto la elipse: $\left(\frac{\sqrt{2}}{\sqrt{3}}\operatorname{sen}t,\frac{1}{\sqrt{2}}\cos t-\frac{1}{\sqrt{6}}\operatorname{sen}t,\left(\frac{1}{\sqrt{6}}-\frac{\sqrt{2}}{\sqrt{3}}\right)\operatorname{sen}t-\frac{1}{\sqrt{2}}\cos t\right)$].



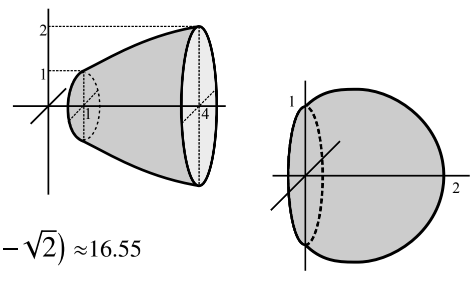
6. a) (y^2,y) , $y\in[1,2]$ ó $(x,x^{1/2})$, $x\in[1,4]$. Mejor con la primera:

$A=2\pi\int_1^{2}y\sqrt{1+4y^2}\,dy=\frac{\pi}{6}\left(1+4y^2\right)^{3/2}\Big]_1^{2}=\frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)\approx 30.8$.

b) $r=(1+\cos\theta)$, $\theta\in\left[0,\frac{\pi}{2}\right]$, $r'(\theta)=-\operatorname{sen}\theta$, $y=(1+\cos\theta)\operatorname{sen}\theta$,

$\|\mathbf{c}'(\theta)\|=\sqrt{r^2+(r')^2}=\sqrt{1+2\cos\theta+\cos^2\theta+\operatorname{sen}^2\theta}=\sqrt{2+2\cos\theta}$.

$A=2\sqrt{2}\,\pi\int_0^{\pi/2}(1+\cos\theta)^{3/2}\operatorname{sen}\theta\,d\theta=-\frac{4\sqrt{2}\,\pi}{5}(1+\cos\theta)^{5/2}\Big]_0^{\pi/2}=\frac{4\pi}{5}\left(8-\sqrt{2}\right)\approx 16.55$

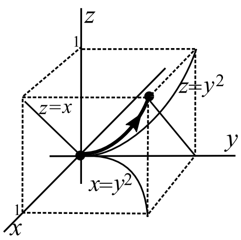


7. $\boxed{\mathbf{g}(x,y,z)=\left(3,z^2-1,2yz\right)}$, $\mathbf{c}(t)=\left(t^2,t,t^2\right)$. $\operatorname{div}\mathbf{g}=\boxed{2y}$. $\operatorname{rot}\mathbf{g}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ 3&z^2-1&2yz\end{vmatrix}=\boxed{\mathbf{0}}$.

$\|\mathbf{c}'\|=\sqrt{1+8t^2}$, $\operatorname{div}\bar{g}\left(\mathbf{c}(t)\right)=2t$. $\int_{\mathbf{c}}\operatorname{div}\mathbf{g}\,ds=\int_0^{1}2t\sqrt{1+8t^2}\,dt=\frac{1}{12}\left(1+8t^2\right)^{3/2}\Big]_0^{1}=\boxed{\frac{13}{6}}$.

$\operatorname{rot}\mathbf{g}=\bar{0}$, $\mathbf{g}\in C^1\ \Rightarrow$ hay U . $\begin{array}{l}U_x=3\ \to\ U=3x+p(y,z)\\ U_y=z^2-1\to U=yz^2-y+q(x,z)\\ U_z=2yz\ \to\ U=yz^2+r(x,y)\end{array}$, $U=3x-y+yz^2\ \Rightarrow$

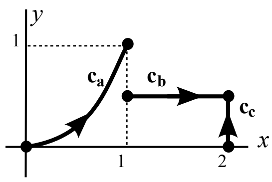
$\int_{\mathbf{c}}\mathbf{g}\cdot d\mathbf{s}=U(1,1,1)-U(0,0,0)=\boxed{3}\ \forall\mathbf{c}$.



O con la $\mathbf{c}$ dada: $\int_{\bar{c}}\mathbf{g}\cdot d\mathbf{s}=\int_0^{1}\left(3,t^4-1,2t^3\right)\cdot(2t,1,2t)\,dt=\int_0^{1}\left(6t+5t^4-1\right)dt=3t^2+t^5\Big]_0^{1}-1=\boxed{3}$.

O por este camino: $\bar{c}_*(t)=(t,t,t)$, $t\in[0,1]\to\int_{\mathbf{c}_*}\mathbf{g}\cdot d\mathbf{s}=\int_0^{1}\left(3,t^2-1,2t^2\right)\cdot(1,1,1)\,dt=\int_0^{1}\left(2+3t^2\right)dt=\boxed{3}$.

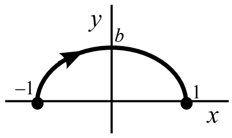
8. a) $\mathbf{c}(x) = (x, x^2)\,,\ 0 \le x \le 1\,,\ \int_C (x^2+y^2)dx + dy = \int_0^1 (x^2+x^4+2x)\,dx = \frac{23}{15}$.



b) $\mathbf{c}(x) = \left(x, \frac{1}{2}\right)\,,\ 1 \le x \le 2\,,\ \int_C (x^2+y^2)dx + dy = \int_1^2 (x^2+\frac{1}{4})\,dx = \frac{31}{12}$.

c) $\mathbf{c}(y) = (2, y)\,,\ 0 \le y \le \frac{1}{2}\,,\ \int_C (x^2+y^2)dx + dy = \int_0^{1/2} dy = \frac{1}{2}$.

9. $\mathbf{c}(t) = (\cos t, b \operatorname{sen} t)\,,\ t \in [0,\pi]$, recorre $b^2x^2+y^2=b^2$ en sentido opuesto.

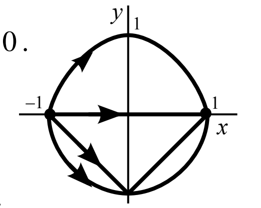


$T(b) = -\int_0^\pi (3b^2 \operatorname{sen}^2 t + 2, 16\cos t)\cdot(-\operatorname{sen} t, b\cos t)\,dt = 4b^2 - 8b\pi + 4$. $T'(b) = 8(b-\pi)$

$\qquad\qquad || \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$3b^2 \operatorname{sen} t(1-\cos^2 t) - 8b(1+\cos 2t) + 2\operatorname{sen} t$ $\qquad$ T mínimo si $b=\pi$.

10. $\mathbf{f}(x,y) = (xy, 0)$ entre $(-1,0)$ y $(1,0)$. Usamos en todas el parámetro x, $x\in[-1,1]$:



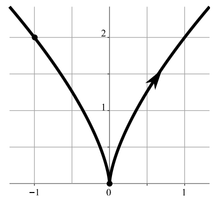
a) $\mathbf{c}(x) = (x,0)\,,\ \mathbf{c}' = (1,0)\,,\ \int_{-1}^1 0\,dx = 0$. $\quad$ b) $\mathbf{c}(x) = (x, 1-x^2)\,,\ \mathbf{c}' = (1,-2x)$. $\int_{-1}^1 (x-x^3)\,dx = 0$.

c) $\mathbf{c}(x) = (x, |x|-1)\,,\ \mathbf{c}' = \begin{cases} (1,-1)\,,\ x<0 \\ (1,1)\,,\ x>0 \end{cases}$. $\int_{-1}^0 (-x^2-x)\,dx + \int_0^1 (x^2-x)\,dx = 0$.

d) $\mathbf{c}(x) = \left(x, -\sqrt{1-x^2}\right)\,,\ \mathbf{c}' = (1, x(1-x^2)^{-1/2})$. $\int_{-1}^1 -x\sqrt{1-x^2}\,dx = 0$.

Pero no es gradiente de nigún campo U pues $f_y = x \not\equiv 0 = g_x$ (sobre otros caminos no se anula).

11. a) $\mathbf{c}(t) = \left(t^3, 2t^2\right)$. $\mathbf{c}'(t) = (3t^2, 4t)\,,\ \|\mathbf{c}'(t)\| = \sqrt{9t^4+16t^2} = |t|\sqrt{9t^2+16}$.

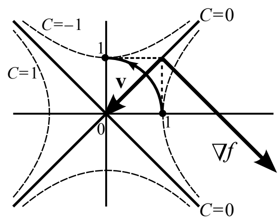


Longitud $L = \int_{-1}^0 (-t)\sqrt{9t^2+16}\,dt = \frac{1}{27}\left[-(9t^2+16)^{3/2}\right]_{-1}^0 = \frac{5^3-4^3}{27} = \frac{61}{27}$.

b) $h(x,y) = e^{2x+y}\,,\ \int_{\mathbf{c}} \nabla h\cdot d\mathbf{s} = h(1,2) - h(0,0) = e^4 - 1 = \int_0^1 (6t^2+4t)\,e^{2t^3+2t^2}dt = e^{2t^3+2t^2}\Big]_0^1$

(cálculo innecesario)

12. a] $f(x,y) = x^2-y^2 = 0 \to$ las rectas $y = \pm x$. [El resto, hipérbolas]. $\nabla f(x,y) = (2x,-2y)$,



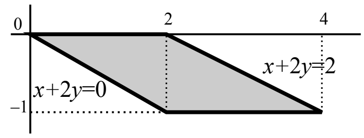
$\nabla f(1,1) = (2,-2)$. $D_{\mathbf{u}} f(1,1) = (2,-2)\cdot(-1,-1) = 0$. $\Delta f(x,y) = f_{xx} + f_{yy} = 2-2 = 0$.

b] Como $\mathbf{g} = (2x,-2y) = \nabla f$, será $\int_{\mathbf{c}} \mathbf{g}\cdot d\mathbf{s} = f(0,1) - f(1,0) = -2$. Directamente:

$\mathbf{c}(t) = (\cos t, \operatorname{sen} t)\,,\ t\in\left[0,\frac{\pi}{2}\right]\,,\ \int_{\mathbf{c}} \mathbf{g}\cdot d\mathbf{s} = \int_0^{\pi/2} (2c,-2s)\cdot(-s,c)\,dt = -2\operatorname{sen}^2 t\Big]_0^{\pi/2} = -2$.

O bien: $\mathbf{c}(t) = \left(t, \sqrt{1-t^2}\right)\,,\ t\in[1,0]\,,\ \int_{\mathbf{c}} \mathbf{g}\cdot d\mathbf{s} = \int_1^0 \left(2t, -2\sqrt{\cdot}\right)\cdot\left(1, \frac{-t}{\sqrt{\cdot}}\right)dt = \int_1^0 4t\,dt = -2$.

13. D limitado por $y=-2$, $y=0$, $x+2y=0$ y $x+2y=2$.



a) $\iint_D (x+2y)\,dx\,dy = \int_{-1}^0 \int_{-2y}^{2-2y} (x+2y)\,dx\,dy = \int_{-1}^0 \left(\left[\frac{x^2}{2}\right]_{-2y}^{2-2y} + 4y\right)dy = \int_{-1}^0 2\,dy = 2$.

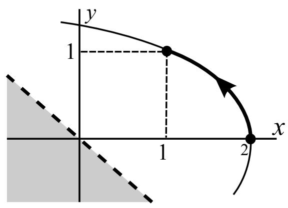
O bien: $\begin{matrix} u = x+2y\,,\ x = u-2v \\ v = y \end{matrix}$, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \to \int_0^2 \int_{-1}^0 u\,dv\,du = \int_0^2 u\,du = 2$.

b) Como $\mathbf{f}(x,y) = (1, \cos y)$ cumple $(1)_y = 0 = (\cos y)_x \Rightarrow$ deriva de potencial ($U = x + \operatorname{sen} y$) $\Rightarrow \oint_{\partial D} \mathbf{f}\cdot d\mathbf{s} = 0$.

Directamente (largo): $\mathbf{c_1} = \left(t, -\frac{t}{2}\right),\ t\in[0,2]$, $\mathbf{c_2} = (t,-1),\ t\in[2,4]$, $\mathbf{c_3} = \left(t, 1-\frac{t}{2}\right),\ t\in[4,2]$, $\mathbf{c_4} = (t,0),\ t\in[2,0]$,

$\int_0^2 [1 - \frac{1}{2}\cos\frac{t}{2}]\,dt + \int_2^4 dt + \int_4^2 [1 - \frac{1}{2}\cos(1-\frac{t}{2})]\,dt + \int_2^0 dt = -\frac{1}{4}\operatorname{sen} 1 + \frac{1}{4}\operatorname{sen} 1 = 0$.

14. Para $\boxed{\mathbf{f}(x,y) = \left(-\frac{y}{(y+x)^2}, \frac{x}{(y+x)^2}\right)}$ es $f_y = \frac{y-x}{(y+x)^3} = g_x$, y hallamos una U en $y+x>0$:



$\begin{matrix} U = \int \frac{-y\,dx}{(y+x)^2} = \frac{y}{y+x} + p(y) \\ U = \int \frac{x\,dy}{(y+x)^2} = \frac{y}{y+x} - 1 + q(x) \end{matrix} \Rightarrow U = \frac{y}{y+x} \Rightarrow \int_{\mathbf{c}} \mathbf{f}\cdot d\mathbf{s} = U(1,1) - U(2,0) = \frac{1}{2} - 0 = \frac{1}{2}$.

Directamente: $\mathbf{c} = \left(2-y^2, y\right)\,,\ y\in[0,1]\,,\ \int_{\mathbf{c}} \mathbf{f}\cdot d\mathbf{s} = \int_0^1 \frac{2+y^2}{(2+y-y^2)^2}\,dy = \cdots = \frac{y}{2+y-y^2}\Big]_0^1 = \frac{1}{2}$.

15. $\boxed{\mathbf{g}(x,y,z) = (2-y, -x, 1)}$ y $\boxed{\mathbf{c}(t) = (4\cos t, 4\operatorname{sen} t, 3t)}$. a) $\|\mathbf{c}'\| = \|(-4s, 4c, 3)\| = 5 \to L = \int_0^\pi 5\,dt = \boxed{5\pi}$.

b) $\operatorname{rot}\mathbf{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2-y & -x & 1 \end{vmatrix} = (0,0,-1+1) = \mathbf{0}$. Y como $\mathbf{g}\in C^1\left(\mathbf{R}^3\right)$ el campo deriva de un potencial.

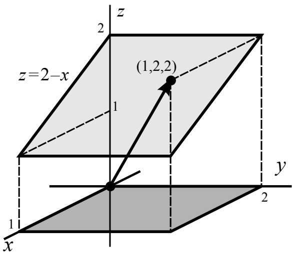
c) Definición: $\int_0^\pi (2-4s, -4c, 1)\cdot(-4s, 4c, 3)\,dt = \int_0^\pi \left(3 - 8\operatorname{sen} t - 16\cos 2t\right)dt = 3\pi + \left[8c - 8\operatorname{sen} 2t\right]_0^\pi = \boxed{3\pi - 16}$.

O bien $\begin{matrix} U_x = 2-y \to U = 2x - xy + p(y,z) \\ U_y = -x \to U = -xy + q(x,z) \\ U_z = 1 \to U = z + r(x,y) \end{matrix}$, $U = z + 2x - xy \Rightarrow \int_{\mathbf{c}} \mathbf{g}\cdot d\mathbf{s} = U(-4,0,3\pi) - U(4,0,0) = \boxed{3\pi - 8 - 8}\ \forall\,\mathbf{c}$.

O también podríamos ir por el segmento que une los puntos inicial y final: $(4-8t, 0, 3\pi t)\,,\ t\in[0,1]$.

16. $g(x,y,z)=y\,e^{2x-z}$. a) En $[0,1]\times[0,2]$ es $z=2-x > z=0$ (no se precisa el dibujo).

$\iiint_V g=\int_0^2\int_0^1\int_0^{2-x} y\,e^{2x-z}\,dz\,dx\,dy = 2\int_0^1\left[e^{2x}-e^{3x-2}\right]dx=\left[e^{2x}-\frac{2}{3}e^{3x-2}\right]_0^1=\frac{3e^2-3+2e+2e^{-2}}{3}$.



b) i) $\mathbf{c}(t)=(t,2t,2t)$, $\mathbf{c}'=(1,2,2)$, $\|\mathbf{c}'\|=3$. $\int_{\mathbf{c}} g\cdot ds=\int_0^1 g(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,dt=\int_0^1 6t\,dt=3$.

ii) Más fácil: $\int_{\mathbf{c}}\nabla g\cdot d\mathbf{s}=g(1,2,2)-g(0,0,0)=2$.

Directamente: $\nabla g=\left(2y\,e^{2x-z},e^{2x-z},-ye^{2x-z}\right)$, $\int_0^1(4t,1,-2t)\cdot(1,2,2)\,dt=\int_0^1 2\,dt=2$.

17. $\boxed{\mathbf{f}(x,y,z)=(xy,2x,-yz)}$. $\operatorname{div}\mathbf{f}=y+0-y=\boxed{0}$. $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ xy&2x&-yz\end{vmatrix}=-z\,\mathbf{i}+0\,\mathbf{j}+(2-x)\,\mathbf{k}=\boxed{(-z,0,2-x)}$.

[no deriva de un potencial].

$\int_{\mathbf{c}}\mathbf{f}\cdot d\bar{s}=\int_0^\pi(cs,2c,-s)\cdot(-s,c,0)\,dt=\int_0^\pi(2\cos^2 t-\operatorname{sen}^2 t\cos t)\,dt=\pi+\left[\frac{1}{2}\operatorname{sen}2t-\frac{1}{3}\operatorname{sen}^3 t\right]_0^\pi=\boxed{\pi}$.

Como $\|\mathbf{c}'\|=\sqrt{\operatorname{sen}^2 t+\cos^2 t+0}=1$, la longitud de la curva es $L=\int_0^\pi 1\,dt=\boxed{\pi}$.

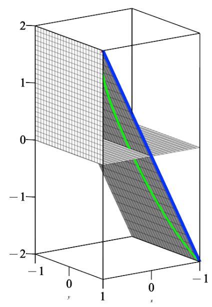
18. $\boxed{\mathbf{F}(x,y,z)=(x,y,z)}$. a) $\mathbf{c}(t)=(t,t,t)$, $t\in[0,1]$; $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=\int_0^1(t,t,t)\cdot(1,1,1)\,dt=\int_0^1 3t\,dt=\boxed{\frac{3}{2}}$.

b) $\mathbf{c}(t)=(\cos t,\operatorname{sen}t,0)$, $t\in[0,2\pi]$; $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=\int_0^{2\pi}(\cos t,\operatorname{sen}t,0)\cdot(-\operatorname{sen}t,\cos t,0)\,dt=\int_0^{2\pi}0\,dt=\boxed{0}$.

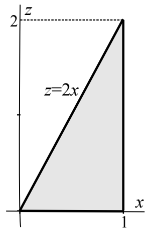
Como claramente $\mathbf{F}$ es el gradiente de $U=\frac{1}{2}\left(x^2+y^2+z^2\right)$, el a) es ya inmediato y el b) ya es obvio.

19. $\mathbf{F}(x,y,z)=\left(2xz+y,x,x^2\right)$. a) $\operatorname{div}\mathbf{F}=2z$, $\operatorname{rot}\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial_x&\partial_y&\partial_z\\ 2xz+y&x&x^2\end{vmatrix}=(0,2x-2x,1-1)=\mathbf{0}$

y $\mathbf{F}\in C^1\Rightarrow\exists U$ potencial.



b) $\iiint_V 2z\,dz\,dy\,dx=\int_0^1\int_{-1}^1\int_0^{2x}2z\,dz\,dy\,dx=\int_0^1 8x^2\,dx=\boxed{\frac{8}{3}}$.



c) Hallando el potencial U: $\begin{array}{l}U=x^2z+xy+\cdots\\ U=xy+\cdots\\ U=x^2z+\cdots\end{array}\Rightarrow U(x,y,z)=x^2z+xy$.

Por tanto: $\int_{\mathbf{c}}\mathbf{F}\cdot d\bar{s}=U(1,1,2)-U(-1,1,-2)=3-(-3)=\boxed{6}$.

Como es conservativo y la integral no depende de camino, se podría ir por el segmento que une los puntos:

$\mathbf{c}_*(t)=(t,1,2t)$, $t\in[-1,1]$. $\int_{-1}^1\left(4t^2+1,t,t^2\right)\cdot(1,0,2)\,dt=2\int_0^1(6t^2+1)\,dt=\boxed{6}$.

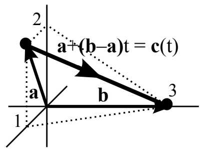
Usando la definición con la $\mathbf{c}$ dada se complica el cálculo y surge una integral que parece no calculable:

$\int_{-1}^1\left(4t^2+e^{t^2-1},t,t^2\right)\cdot\left(1,2te^{t^2-1},2\right)dt=2\int_0^1\left[6t^2+(2t^2+1)\,e^{t^2-1}\right]dt=\left[2t^3+te^{t^2-1}\right]_0^1=\boxed{6}$.

derivada de $\uparrow te^{t^2-1}$

d) $\mathbf{c}(1)=(1,1,2)$, $\mathbf{c}'(1)=(1,2,2)$. $\mathbf{x}=(1+t,1+2t,2+2t)$ corta $z=0$ en $(0,-1,0)$

20. $\mathbf{F}(x,y,z)=(1,2yz,y^2)$, $\mathbf{c}(t)=(1,0,2)+t(-1,3,-2)=(1-t,3t,2-2t)$, $t\in[0,1]$.



$\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=\int_0^1\left(1,12t-12t^2,9t^2\right)\cdot(-1,3,-2)\,dt=\int_0^1(-1+36t-54t^2)\,dt=-1$.

Como $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ 1&2yz&y^2\end{vmatrix}=(2y-2y)\,\mathbf{i}+0\,\mathbf{j}+0\,\mathbf{k}=\mathbf{0}$ y $\mathbf{F}\in C^1(\mathbf{R}^3)$ hay potencial y la integral será -1 para toda curva.

$\begin{array}{l}U=x+p(y,z)\\ U=y^2z+q(x,z)\\ U=y^2z+r(x,y)\end{array}\Rightarrow U=x+y^2z$, $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}=U(0,3,0)-U(1,0,2)=0-1=-1$ (de otra forma).

21. $\mathbf{f}(x,y,z)=\left(z^2,2y,cxz\right)$. a) $\operatorname{div}\mathbf{f}=2+cx$. $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ z^2&2y&cxz\end{vmatrix}=(0,2z-cz,0)$.

b) Para $c=2$ es $\operatorname{rot}\mathbf{f}=\mathbf{0}$ y como $\mathbf{f}\in C^1(\mathbf{R}^3)$, existe el potencial: $\begin{array}{l}U_x=z^2\to U=xz^2+p(y,z)\\ U_y=2y\to U=y^2+q(x,z)\\ U_z=2xz\to U=xz^2+r(x,y)\end{array}$, $U=xz^2+y^2$.

c) Por tanto, $\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(1,0,1)-U(0,0,0)=1$, sin necesidad de hacer ninguna integral de línea.

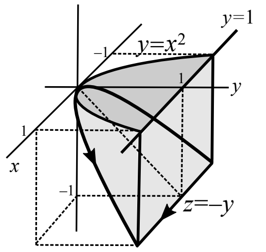
d) Una parametrización: $\mathbf{c}(t)=(t,0,t)$, $t\in[0,1]\to\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=\int_0^1(t^2,0,ct^2)\cdot(1,0,1)\,dt=\int_0^1(c+1)t^2dt=\frac{c+1}{3}$.

22. $\mathbf{f}(x,y,z)=\left(e^{-z},1,-xe^{-z}\right)$. a) $\mathbf{c}(t)=(1,1-t,3t)$, $t\in[0,1]\to$

$$\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=\int_0^1\left(e^{-3t},1,-e^{-3t}\right)\cdot(0,-1,3)\,dt=\int_0^1\left(-1-3e^{-3t}\right)dt=\boxed{e^{-3}-2}\,.$$

b) $\operatorname{rot}\mathbf{f}=\mathbf{0}$, $\mathbf{f}\in C^1\Rightarrow$ hay potencial. $\begin{array}{l}U=xe^{-z}+p(y,z)\\ U=y+q(x,z)\\ U=xe^{-z}+r(x,y)\end{array}$, $\boxed{U=xe^{-z}+y}$ $\left[\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(1,0,3)-U(1,1,0)=e^{-3}-2\right]$.

23. a) $\iiint_V y=\int_{-1}^{1}\int_{x^2}^{1}\int_{-y}^{0}y\,dz\,dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}y^2\,dy\,dx=\frac{2}{3}\int_0^1\left(1-x^6\right)dx=\frac{2}{3}\left[1-\frac{1}{7}\right]=\boxed{\frac{4}{7}}$.



O bien: $\iiint_V y=\int_0^1\int_{-y}^{0}\int_{-\sqrt{y}}^{\sqrt{y}}y\,dx\,dz\,dy=\int_0^1\int_{-y}^{0}2y^{3/2}\,dz\,dy=\int_0^1 2y^{5/2}dy=\boxed{\frac{4}{7}}$.

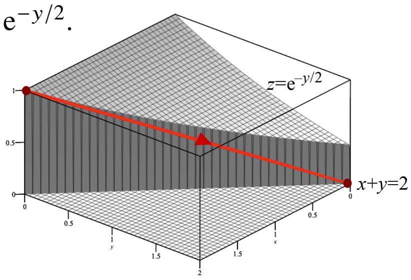
b) $\mathbf{f}(x,y,z)=(y,x,z)$, $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ y&x&z\end{vmatrix}=(0,0,1-1)=\mathbf{0}$ y $\mathbf{f}\in C^1$ en $\mathbf{R}^3$
$\Rightarrow$ existe función potencial.

$\begin{array}{l}U_x=y\ \to\ U=xy+p(y,z)\\ U_y=x\ \to\ U=xy+q(x,z)\\ U_z=z\ \to\ U=\frac{1}{2}z^2+r(x,y)\end{array}$, $\boxed{U=xy+\frac{1}{2}z^2}$ $\Rightarrow\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=U(1,1,-1)-U(-1,1,-1)=\boxed{2}$ para todo camino.

O bien: $\mathbf{c}(x)=\left(x,x^2,-x^2\right)$, $x\in[-1,1]\to\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=\int_{-1}^{1}\left(x^2,x,-x^2\right)\cdot(1,2x,-2x)\,dt=\int_{-1}^{1}\left(3x^2+2x^3\right)dx=\boxed{2}$.

O por el camino más simple: $\mathbf{c}(x)=(x,1,-1)$, $x\in[-1,1]\to\int_{\mathbf{c}}\mathbf{f}\cdot d\mathbf{s}=\int_{-1}^{1}(1,x,-1)\cdot(1,0,0)\,dx=\int_{-1}^{1}dx=\boxed{2}$.

24. $\boxed{f(x,y,z)=2xz}$, $\mathbf{c}(t)=\left(2-t,t,1-\frac{t}{2}\right)$, V acotado por $x,y,z=0$, $x+y=2$, $z=e^{-y/2}$.



a) $\int_0^2\int_0^{2-x}\int_0^{e^{-y/2}}2xz\,dz\,dy\,dx=\int_0^2\int_0^{2-x}xe^{-y}dy\,dx=\int_0^2\left(x-xe^{x-2}\right)dx=$ (partes)

$=2-xe^{x-2}\Big]_0^2+\int_0^2 e^{x-2}dx=\boxed{1-e^{-2}}$.

$\left[\text{Un poco más largo haciendo }\int_0^2\int_0^{2-y}\int_0^{e^{-y/2}}2xz\,dz\,dx\,dy=\frac{1}{2}\int_0^2(y-2)^2e^{-y}dy\right]$.

b) Es $\mathbf{c}'(t)=\left(-1,1,-\frac{1}{2}\right)$, $\|\mathbf{c}'\|=\sqrt{1+1+\frac{1}{4}}=\frac{3}{2}$, $f\left(\mathbf{c}(t)\right)=(t-2)^2$.

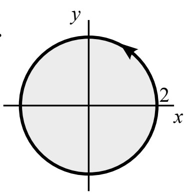
Por tanto, $\int_{\bar{c}}F\,ds=\int_0^2\frac{3}{2}(t-2)^2dt=\frac{1}{2}(t-2)^3\Big]_0^2=\boxed{4}$.

c) La integral de un gradiente es inmediata: $\int_{\mathbf{c}}\nabla f\cdot d\mathbf{s}=f\left(\bar{c}(2)\right)-f\left(\bar{c}(0)\right)=f(0,2,0)-f(2,0,1)=\boxed{-4}$.

Perdiendo el tiempo podríamos utilizar la definición hallando $\nabla f=(2z,0,2x)$ y usando la parametrización dada:

$$\int_{\mathbf{c}}\nabla f\cdot d\mathbf{s}=\int_0^2(2-t,0,4-2t)\cdot\left(-1,1,-\tfrac{1}{2}\right)dt=\int_0^2(2t-4)\,dt=4-8=\boxed{-4}\,.$$

25. $\mathbf{g}(x,y)=\left(1,xy^2\right)$. $g_x-f_y=y^2$. No deriva de un potencial. i) $\mathbf{c}(t)=(2\cos t,2\operatorname{sen}t)$, $t\in[0,2\pi]$.

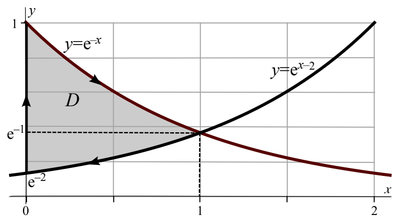


$\to\int_{\mathbf{c}}\mathbf{g}\cdot d\mathbf{s}=\int_0^{2\pi}\left(1,8cs^2\right)\cdot(-2s,2c)\,dt=2\cos t\Big]_0^{2\pi}+\int_0^{2\pi}4\operatorname{sen}^2 2t\,dt=\int_0^{2\pi}(2-2\cos 4t)\,dt=4\pi$.

ii) Green: $\iint_{\circ}y^2\,dx\,dy=\int_0^{2\pi}\int_0^2 r^3\operatorname{sen}^2\theta\,dr\,d\theta=\left[\frac{1}{4}r^4\right]_0^2\cdot\frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta=4\pi$.

$\left[\text{En cartesianas mucho más largo: }\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}y^2\,dy\,dx=\frac{2}{3}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx=\cdots\right]$.

26. a) $\int_0^1\int_{e^{x-2}}^{e^{-x}}x\,e^x\,dy\,dx=\int_0^1\left[x-xe^{2x-2}\right]dx=\frac{x^2}{2}-\frac{x}{2}e^{2x-2}\Big]_0^1+\frac{1}{2}\int_0^1 e^{2x-2}dx=\frac{1-e^{-2}}{4}$.



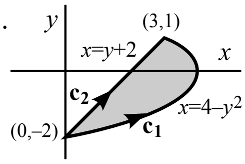
b) $g_x-f_y=-x\,e^x$. Según Green, la $\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}$ vale lo de arriba. Directamente:

$\mathbf{c}_1(t)=(0,t)$, $y\in\left[e^{-2},1\right]$; $\mathbf{c}_2(t)=\left(t,e^{-t}\right)$, $t\in[0,1]$; $\mathbf{c}_3(t)=\left(t,e^{t-2}\right)$, $t\in[1,0]$.

$$\int_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{e^{-2}}^{1}(0,1)\cdot(0,1)\,dt+\int_0^1(t,1)\cdot\left(1,-e^{-t}\right)dt+\int_1^0\left(te^{2t-2},1\right)\cdot\left(1,e^{t-2}\right)dt$$

$$=1-e^{-2}+\int_0^1\left(t-e^{-t}\right)dt+\int_1^0\left(te^{2t-2}+e^{t-2}\right)dt=\frac{3}{2}-e^{-2}+\left[e^{-t}\right]_0^1+\left[\frac{t}{2}e^{2t-2}-\frac{1}{4}e^{2t-2}+e^{t-2}\right]_1^0=\frac{1-e^{-2}}{4}\,.$$

27. a) $\mathbf{f}(x,y)=(y^2,2x)$, $\mathbf{c_1}=(4-t^2,t)$, $t\in[-2,1]$, $\mathbf{c_2}=(t,t-2)$, $t\in[0,3]$ (sentido opuesto).

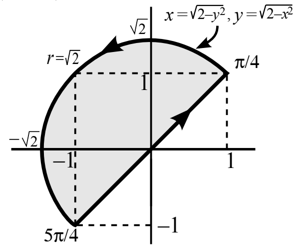


$$\int_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{-2}^{1}(-2t^3+8-2t^2)\,dt-\int_0^3(t^2-2t+4)\,dt=26-\tfrac12-12=\tfrac{27}{2}.$$

$$g_x-f_y=2-2y\,,\ \int_{-2}^{1}\int_{y+2}^{4-y^2}(2-2y)\,dx\,dy=\int_{-2}^{1}(4-6y+2y^3)\,dy=\tfrac{27}{2}.$$

b) $\mathbf{f}(x,y)=(y^2,xy)$. $\iint_D -y\,dx\,dy=-\int_{\pi/4}^{5\pi/4}\int_0^{\sqrt2}r^2\,\mathrm{sen}\,\theta=\left[\tfrac13 r^3\right]_0^{\sqrt2}\left[\cos\theta\right]_{\pi/4}^{5\pi/4}=\tfrac{2\sqrt2}{3}\left[-\tfrac{1}{\sqrt2}-\tfrac{1}{\sqrt2}\right]=-\tfrac43$.

En cartesianas (de las dos formas) es más complicado. Por ejemplo:



$$\iint_D f=-\int_{-\sqrt2}^{-1}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}y\,dy\,dx-\int_{-1}^{1}\int_{-x}^{\sqrt{2-x^2}}y\,dy\,dx=0+\int_{-1}^{1}\left[x^2-1\right]dx=-\tfrac43.$$

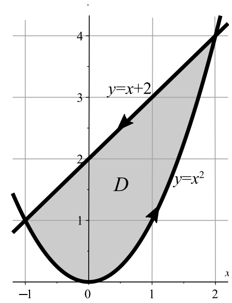
Parametrizaciones sencillas: $\mathbf{c_1}(t)=(t,t)$, $t\in[-1,1]$ (en sentido correcto).

$\mathbf{c_2}(t)=(\sqrt2\cos t,\sqrt2\,\mathrm{sen}\,t)$, $t\in\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$ (también en buen sentido).

$$\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{-1}^{1}(t^2,t^2)\cdot(1,1)\,dt+\int_{\pi/4}^{5\pi/4}\left(2\,\mathrm{sen}^2t,2\,\mathrm{sen}\,t\cos t\right)\cdot\left(-\sqrt2\,\mathrm{sen}\,t,\sqrt2\cos t\right)$$

$$=\int_{-1}^{1}2t^2\,dt+2\sqrt2\int_{\pi/4}^{5\pi/4}\left[sc^2-s^3\right]dt=\tfrac43+2\sqrt2\left[\cos t-\tfrac23\cos^3t\right]_{\pi/4}^{5\pi/4}=\tfrac43+2\left[1+1-\tfrac13-\tfrac13\right]=-\tfrac43.$$

c) $\mathbf{f}(x,y)=(-xy,y)$. $g_x-f_y=x$. $\iint_D x=\int_{-1}^{2}\int_{x^2}^{2+x}x\,dy\,dx=\int_{-1}^{2}(2x+x^2-x^3)\,dx=\frac94$.



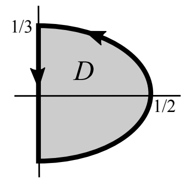
O peor: $\int_0^1\int_{-\sqrt y}^{\sqrt y}x\,dx\,dy+\int_1^4\int_{y-2}^{\sqrt y}x\,dx\,dy=0+\frac12\int_1^4(5y-y^2-4)\,dy=\frac94$.

Parametrizamos así ∂D: $\mathbf{c}_1(x)=(x,x^2)$, $x\in[-1,2]$. $\mathbf{c}_2(x)=(x,x+2)$, $x\in[2,-1]$

$$\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{-1}^{2}(-x^3,x^2)\cdot(1,2x)\,dx+\int_{2}^{-1}(-x^2-2x,2+x)\cdot(1,1)\,dx$$

$$=\int_{-1}^{2}x^3\,dx+\int_{-1}^{2}(x^2+x-2)\,dx=\tfrac{15}{4}+\tfrac93+\tfrac32-6=\tfrac{15}{4}-\tfrac32=\tfrac94.$$

d) $\mathbf{f}(x,y)=(x,x^2)$. $g_x-f_y=2x$. $\iint_D 2x=\int_{-1/3}^{1/3}\int_0^{\sqrt{1-9y^2}/2}2x\,dx\,dy=\frac12\int_0^{1/3}(1-9y^2)\,dy=\frac19$.

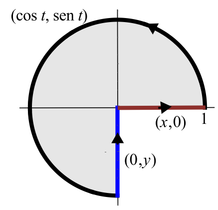


O con $x=\frac r2\cos\theta$, $y=\frac r3\,\mathrm{sen}\,\theta$, $J=\frac r6$, $\frac16\int_0^1\int_{-\pi/2}^{\pi/2}r^2\cos\theta\,d\theta\,dr=\frac16\cdot\frac12\cdot2=\frac19$.

Sobre $x=0$ es $\mathbf{f}=\mathbf{0}$ y la elipse viene dada por: $\mathbf{c}(t)=\left(\frac12\cos t,\frac13\,\mathrm{sen}\,t\right)$, $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$:

$$\to\ \oint_{\partial D}\bar{\mathrm{f}}\cdot d\bar{s}=\int_{-\pi/2}^{\pi/2}\left(\tfrac c2,\tfrac{c^2}{4}\right)\cdot\left(-\tfrac s2,\tfrac c3\right)dt=\tfrac16\int_0^{\pi/2}(1-s^2)c\,dt=\tfrac16\left[s-\tfrac13s^3\right]_0^{\pi/2}=\tfrac19.$$

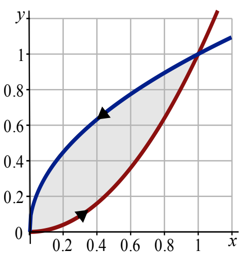
e) $\mathbf{f}(x,y)=(xy,2-x^2)$, $g_x-f_y=-3x$. $\iint_D -3x=-\int_0^{3\pi/2}\int_0^1 3r^2\cos\theta\,dr\,d\theta=1$.



$$\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_0^{3\pi/2}\left(\cos t\,\mathrm{sen}\,t,2-\cos^2t\right)\cdot(-\mathrm{sen}\,t,\cos t)\,dt+\int_{-1}^{0}(0,2)\cdot(0,1)\,dy+\int_0^1 0\,dx$$

$$=\int_0^{3\pi/2}\cos t\left(2-\cos^2t-\mathrm{sen}^2t\right)dt+2=\mathrm{sen}\,t\Big]_0^{3\pi/2}+2=1.$$

f) $\mathbf{f}(x,y)=(x^3,x^2y)$. $g_x-f_y=2xy$. $\int_0^1\int_{x^2}^{\sqrt x}2xy\,dy\,dx=\int_0^1 x\left(x-x^4\right)dx=\frac13-\frac16=\frac16$.

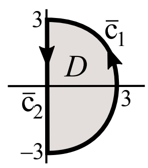


Parametrizamos las curvas: $\bar{c}_1=(x,x^2)$, $x\in[0,1]$. $\bar{c}_2=(y^2,y)$, $y\in[1,0]$.

$$\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\int_{\mathbf{c_1}}\mathbf{f}\cdot d\mathbf{s}+\int_{\mathbf{c_2}}\mathbf{f}\cdot d\mathbf{s}=\int_0^1(x^3,x^4)\cdot(1,2x)\,dx-\int_0^1(y^6,y^5)\cdot(2y,1)\,dy$$

$$=\int_0^1\left(x^3+2x^5\right)dx-\int_0^1\left(2y^7+y^5\right)dy=\tfrac16.$$

g) $\mathbf{f}(x,y)=(y^2,x^2)$. $g_x-f_y=2(x-y)$. $2\int_{-\pi/2}^{\pi/2}\int_0^3 r^2(\underbrace{\cos\theta}_{\text{par}}-\underbrace{\mathrm{sen}\,\theta}_{\text{impar}})\,dr\,d\theta=36\int_0^{\pi/2}\cos\theta\,d\theta=36$.



O bien: $2\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(x-y)\,dy\,dx=4\int_0^3 x\left(9-x^2\right)^{1/2}dx=-\frac43\left(9-x^2\right)^{3/2}\Big]_0^3=\frac{4\cdot27}{3}=36$.

O bien: $\int_{-3}^{3}\int_0^{\sqrt{9-y^2}}(2x-2y)\,dx\,dy=\int_{-3}^{3}\left(9-y^2-\underbrace{2y\sqrt{9-y^2}}_{\text{impar}}\right)dy=54-2\left[\frac{y^3}{3}\right]_0^3=54-18=36$.

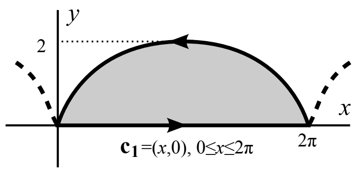
La ∂D. Semicircunferencia: $\bar{c}_1=(3\cos t,3\,\mathrm{sen}\,t)$, $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Segmento: $(0,y)$, $y\in[3,-3]$.

$$\int_{\bar{c}_1}\mathbf{f}\cdot d\mathbf{s}=27\int_{-\pi/2}^{\pi/2}\underbrace{(s^2,c^2)\cdot(-s,c)}_{c^3-s^3\ \text{(par–impar)}}\,dt=54\int_0^{\pi/2}\underbrace{\cos^3t}_{c-s^2c}\,dt=54\left[\mathrm{sen}\,t-\tfrac13\,\mathrm{sen}^3t\right]_0^{\pi/2}=36.$$

$\Big[$En cartesianas: $\bar{c}_*=\left(\sqrt{9-y^2},y\right)$, $y\in[-3,3]$. $\int_{\bar{c}_*}\mathbf{f}\cdot d\mathbf{s}=\int_{-3}^{3}(y^2,9-y^2)\cdot\left(\frac{-y}{\sqrt{\cdot}},1\right)dx=2\int_0^3\left(9-y^2-\underbrace{\frac{y^3}{\sqrt{\cdot}}}_{\text{impar}}\right)dy\Big]$.

Para el segmento: $\int_{\bar{c}_2}\mathbf{f}\cdot d\mathbf{s}=\int_3^{-3}\underbrace{(y^2,0)\cdot(0,1)}_{=0}\,dy=0$. Por tanto, $\oint_{\partial D}\mathbf{f}\cdot d\mathbf{s}=\boxed{36}$, como la doble.

28. $x = t - \operatorname{sen} t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$. Utilizando Green:

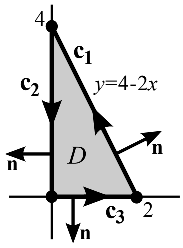


$$A = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} 0 - \frac{1}{2}\int_0^{2\pi}\big[(t-\operatorname{sen} t)\operatorname{sen} t - (1-\cos t)^2\big]\,d\theta$$

$$= \int_0^{2\pi}\big(1-\cos t - \tfrac{1}{2}t\operatorname{sen} t\big)\,dt = 3\pi .$$

29. $\mathbf{g}(x,y) = \big(x^2, -2xy\big)$. **Green**. $\iint_D (g_x - f_y) = \int_0^2\int_0^{4-2x}(-2y)\,dy\,dx = -\int_0^2 \big(4-2x\big)^2\,dx = \frac{1}{6}(4-2x)^3\big]_0^2 = -\frac{32}{3}$.

La ∂D está formada por 3 segmentos fáciles de parametrizar. Por ejemplo:



$\mathbf{c_1}(t) = (t, 4-2t)$, $t \in [2,0]$ [para ir en sentido antihorario].

$\mathbf{c_2}(t) = (0,t)$, $t \in [4,0]$. $\mathbf{c_3}(t) = (t,0)$, $t \in [0,2]$. Entonces:

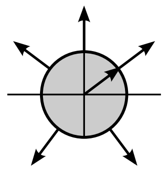
$$\oint_{\partial D} \mathbf{g}\cdot d\mathbf{s} = \int_{\mathbf{c_1}} + \int_{\mathbf{c_2}} + \int_{\mathbf{c_3}} = \int_4^0 \big(t^2, 4t^2-8t\big)\cdot(1,-2)\,dt + \int_4^0 0\,dt + \int_0^2 \big(t^2,0\big)\cdot(1,0)\,dt$$

$$= \int_2^0 (16t - 7t^2)\,dt + 0 + \int_0^2 t^2\,dt = -32 + \frac{56}{3} + \frac{8}{3} = -\frac{32}{3} .$$

Divergencia. $\iint_D \operatorname{div}\mathbf{f}\,dx\,dy = \iint_D (2x-2x)\,dx\,dy = 0$. Las normales unitarias exteriores $\mathbf{n}$ a cada segmento son, respectivamente: $\big(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\big)$, $(-1,0)$ y $(0,-1)$. Sus normas $\|\mathbf{c}_\mathbf{k}'(t)\|$ son: $\sqrt{5}$, 1 y 1 . Por tanto:

$$\oint_{\partial D} \mathbf{g}\cdot\mathbf{n}\,ds = \int_0^4 \big(t^2, 4t^2-8t\big)\cdot(2,1)\,dt + \int_0^4 0\,dt + \int_0^2 \big(t^2,0\big)\cdot(0,-1)\,dt = \int_0^2 (6t^2 - 8t)\,dt - \int_0^2 0\,dt = 16 - 16 = 0 .$$

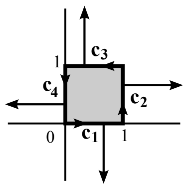
30. i) $\mathbf{f}(x,y) = (x,y)$, $D = \{x^2+y^2 \le 1\}$. $\mathbf{f}\cdot\mathbf{n} = 1$ (dos vectores unitarios en el mismo sentido).

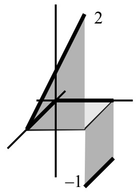


$$\oint_{\partial D} 1\,ds \underset{\uparrow}{=} 2\pi = \iint_D 2\,dx\,dy \underset{\uparrow}{=} 2\pi .$$

longitud — 2 veces el área

ii) $\mathbf{f}(x,y) = (2xy, -y^2)$, $\operatorname{div}\mathbf{f} = 0$, $\iint_D \operatorname{div}\mathbf{f} = 0$.



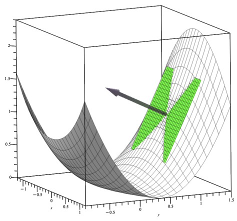


Los cuatro $\mathbf{n}$ exteriores son: $\mathbf{n_1} = (0,-1)$, $\mathbf{n_2} = (1,0)$, $\mathbf{n_3} = (0,1)$, $\mathbf{n_4} = (-1,0)$.

Luego: $\oint_{\partial D} \mathbf{f}\cdot\mathbf{n}\,ds = \int_{\mathbf{c_1}} y^2 ds + \int_{\mathbf{c_2}} 2xy\,ds - \int_{\mathbf{c_3}} y^2 ds - \int_{\mathbf{c_4}} 2xy\,ds$

$= 0 + \int_0^1 2y\,dy - \int_0^1 1\,dx - 0 = 0$, tomando $\mathbf{c_2} = (1,y)$, $y \in [0,1]$, $\mathbf{c_3} = (1-x,1)$, $x \in [0,1]$.

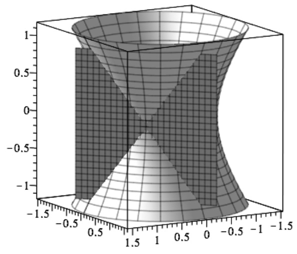
[Hasta aquí el control 2].



1. $\boxed{\mathbf{r}(u,v)=(2u,u^2+v,v^2)}$. a) $\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2u&0\\0&1&2v\end{vmatrix}=2(2uv,-2v,1)\overset{u=0,\,v=1}{\longrightarrow}2(0,-2,1)$.

Plano tangente: $(0,-2,1)\cdot(x,y-1,z-1)=0\;\to\;z=2y-1$.

b) $u=\frac{x}{2}$, $v=y-\frac{x^2}{4}$, $z=\left(y-\frac{x^2}{4}\right)^2$. $z_x(0,1)=0$, $z_y(0,1)=2$, $\nwarrow z=1+0(x-1)+2(x-1)$.

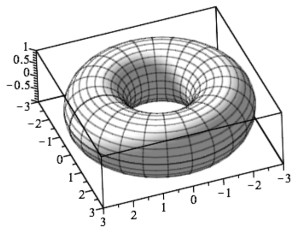


2. $\boxed{\mathbf{r}(u,v)=(\operatorname{ch}u\cos v,\operatorname{ch}u\operatorname{sen}v,\operatorname{sh}u)}$ $\begin{matrix}u\in\mathbf{R}\\v\in[0,2\pi]\end{matrix}$ $\quad x^2+y^2-z^2=\operatorname{ch}^2u-\operatorname{sh}^2u=1$. $\;x^2+y^2=\operatorname{ch}^2u$ curvas de nivel.

$\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\operatorname{sh}u\cos v&\operatorname{sh}u\operatorname{sen}v&\operatorname{ch}u\\-\operatorname{ch}u\operatorname{sen}v&\operatorname{ch}u\cos v&0\end{vmatrix}=\operatorname{ch}u(-\operatorname{ch}u\cos v,-\operatorname{ch}u\operatorname{sen}v,\operatorname{sh}u)\overset{u=0,\,v=\pi/4}{\longrightarrow}-\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)$.

$\mathbf{r}(0,\frac{\pi}{4})=\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)$. Plano tangente: $-\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)\cdot\left(x-\frac{1}{\sqrt2},y-\frac{1}{\sqrt2},z\right)=0$, $x+y=\sqrt2$.

Partiendo de $F(x,y,z)=x^2+y^2-z^2=1$. $\nabla F=(2x,2y,-2z)\big|_{(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0)}=(\sqrt2,\sqrt2,0)^{\uparrow}$



3. $\boxed{\mathbf{r}(\theta,\phi)=\big((2+\cos\phi)\cos\theta,(2+\cos\phi)\operatorname{sen}\theta,\operatorname{sen}\phi\big)}$, $\theta,\phi\in[0,2\pi]$.

$\mathbf{r}_\theta\times\mathbf{r}_\phi=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-(2+\cos\phi)\operatorname{sen}\theta&(2+\cos\phi)\cos\theta&0\\-\operatorname{sen}\phi\cos\theta&-\operatorname{sen}\phi\operatorname{sen}\theta&\cos\phi\end{vmatrix}=(2+\cos\phi)\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-\operatorname{sen}\theta&\cos\theta&0\\-\operatorname{sen}\phi\cos\theta&-\operatorname{sen}\phi\operatorname{sen}\theta&\cos\phi\end{vmatrix}$,

$=(2+\cos\phi)(\cos\theta\cos\phi,\operatorname{sen}\theta\cos\phi,\operatorname{sen}\phi)$. $\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|=2+\cos\phi$.

Área $=\iint_S dS=\iint_D\|\mathbf{r}_\theta\times\mathbf{r}_\phi\|\,d\theta\,d\phi=\int_0^{2\pi}\!\!\int_0^{2\pi}(2+\cos\phi)\,d\theta\,d\phi=8\pi^2$.

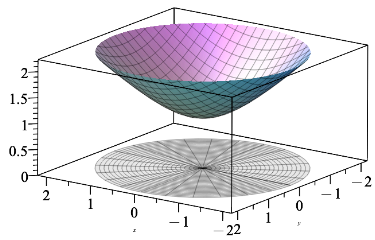
4. $\boxed{\iint_S(x^2+y^2)\,dS}$, $x^2+y^2+z^2=4\;\to\;\mathbf{r}(u,v)=(2\operatorname{sen}u\cos v,2\operatorname{sen}u\operatorname{sen}v,2\cos u)$, $\|\mathbf{r}_u\times\mathbf{r}_v\|=4\operatorname{sen}u$

$$\iint_S(x^2+y^2)\,dS=16\int_0^{2\pi}\!\!\int_0^{\pi}\operatorname{sen}^3u\,du\,dv=32\pi\int_0^{\pi}\operatorname{sen}u(1-\cos^2u)\,du=32\pi\left[\tfrac13\cos^3u-\cos u\right]_0^{\pi}=\tfrac{128}{3}\pi\,.$$

Con $\mathbf{r}(x,y)=\left(x,y,\sqrt{4-x^2-y^2}\right)$ [integrando y recinto simétricos en z, basta esta y multiplicar por 2 la integral].

$$\|\mathbf{r}_x\times\mathbf{r}_y\|=\sqrt{(f_x)^2+(f_y)^2+1}=\left[\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1\right]^{1/2}=\frac{2}{\sqrt{4-x^2-y^2}}\;\Rightarrow$$

$$2\iint_S(x^2+y^2)\,dS=4\iint_B\frac{x^2+y^2}{\sqrt{4-x^2-y^2}}\,dx\,dy\underset{\text{polares}\uparrow}{=}4\int_0^{2\pi}\!\!\int_0^2\frac{r^3}{\sqrt{4-r^2}}\,dr\,d\theta=4\pi\int_0^4\frac{s\,ds}{\sqrt{4-s}}=\frac{128}{3}\pi\,.$$



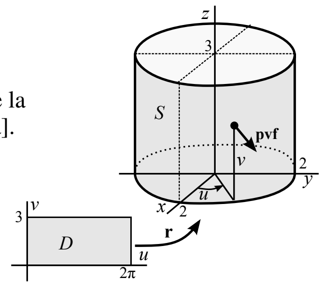
5. $\boxed{f(x,y,z)=z}$, $z^2=1+x^2+y^2$. $\mathbf{r}(r,\theta)=\left(r\cos\theta,r\operatorname{sen}\theta,\sqrt{1+r^2}\right)$. $\theta\in[0,2\pi]$, $r\in[0,2]$ (es $5=1+r^2$ si $r=2$).

$\mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\c&s&r/\sqrt{\cdot}\\-rs&rc&0\end{vmatrix}=\left(-\frac{r^2c}{\sqrt{\cdot}},-\frac{r^2s}{\sqrt{\cdot}},r\right)$. $\|\mathbf{r}_r\times\mathbf{r}_\theta\|=\frac{r\sqrt{1+2r^2}}{\sqrt{1+r^2}}$.

Por tanto: $\iint_S z\,dS=\int_0^{2\pi}\!\!\int_0^2 r\sqrt{1+2r^2}\,dr\,d\theta=\frac{2\pi}{6}\left(1+2r^2\right)^{3/2}\Big]_0^2=\boxed{\frac{26\pi}{3}}$.

O bien: $\mathbf{r}(x,y)=\left(x,y,\sqrt{1+x^2+y^2}\right)$, $(x,y)\in B$ círculo de centro (0,0) y radio 2 .

$\mathbf{r}_x\times\mathbf{r}_y=\left(-\frac{x}{\sqrt{\cdot}},-\frac{y}{\sqrt{\cdot}},1\right)\overset{\|\ \|}{\longrightarrow}\frac{\sqrt{1+2x^2+2y^2}}{\sqrt{1+x^2+y^2}}$. $\iint_S z\,dS=\iint_B\sqrt{1+2x^2+2y^2}\,dx\,dy=\cdots$ (usando polares acabamos arriba).



6. Parametrización de $x^2+y^2=4$: $\left.\begin{matrix}x=2\cos u\\y=2\operatorname{sen}u\\z=v\end{matrix}\right\}$ $\begin{matrix}u\in[0,2\pi]\\v\in[0,3]\end{matrix}$ $\;\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2s&2c&0\\0&0&1\end{vmatrix}=2(\cos u,\operatorname{sen}u,0)$.

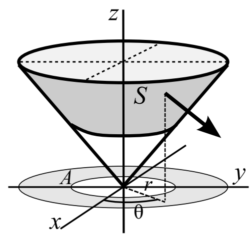
a) área de $S=\iint_S 1\,dS=\iint_D\|\mathbf{r}_u\times\mathbf{r}_v\|\,du\,dv=\int_0^{2\pi}\!\!\int_0^3 2\,dv\,du=12\pi$ [claro, longitud de la base por la altura].

b) i) $\boxed{f(x,y,z)=x^2}$ $\iint_S f\,dS=\int_0^{2\pi}\!\!\int_0^3 8\cos^2u\,dv\,du=12\int_0^{2\pi}(1+\cos 2v)\,dv=24\pi$.

ii) $\boxed{\mathbf{f}(x,y,z)=(xz,yz,2)}$ $\mathbf{f}\big(\mathbf{r}(u,v)\big)=2(v\cos u,v\operatorname{sen}u,1)$.

$$\iint_S\mathbf{f}\cdot d\mathbf{S}=\iint_D\mathbf{f}\big(\mathbf{r}(u,v)\big)\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv=\int_0^{2\pi}\!\!\int_0^3 4v\,dv\,du=36\pi\,.$$

7. $\boxed{z^2=x^2+y^2}$. a) $\left.\begin{array}{l} x=r\cos\theta \\ y=r\,\mathrm{sen}\,\theta \\ z=r\end{array}\right\}$ $\begin{array}{l} r\in[1,2] \\ \theta\in[0,2\pi]\end{array}$ $\mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ c&s&1\\ -r\,s&r\,c&0\end{vmatrix}=(-r\cos\theta,-r\,\mathrm{sen}\,\theta,r)$. [apunta hacia interior]



$\|\mathbf{r}_r\times\mathbf{r}_\theta\|=\sqrt{2}\,r$, área $=2\pi\int_1^2\|\cdot\|=3\pi\sqrt{2}$.

$\iint_S\mathbf{f}\cdot d\mathbf{S}=-\iint_D(r\cos\theta,r\,\mathrm{sen}\,\theta,1)\cdot(-r\cos\theta,-r\,\mathrm{sen}\,\theta,r)\,dr\,d\theta=\big[\,\mathbf{f}(x,y,z)=(x,y,1)\,\big]$

$=\int_0^{2\pi}\int_1^2[r^2-r]\,dr\,d\theta=2\pi\big[\frac{r^3}{3}-\frac{r^2}{2}\big]_1^2=\boxed{\frac{5}{3}\pi}$. [es >0 pues $\mathbf{f}$ apunta también hacia exterior]

De otra forma: $\mathbf{r}(x,y)=\big(x,y,\sqrt{x^2+y^2}\,\big)$, $\mathbf{r}_x\times\mathbf{r}_y=\big(-x\,(x^2+y^2)^{-1/2},-y\,(x^2+y^2)^{-1/2},1\big)$.

$-\iint_A(x,y,1)\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dx\,dy=-\iint_A\big[1-\sqrt{x^2+y^2}\,\big]\,dx\,dy=\int_0^{2\pi}\int_1^2[r^2-r]\,dr\,d\theta=\cdots$ como antes.

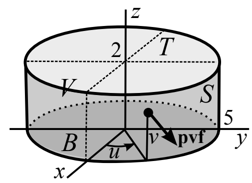
b) $\mathrm{rot}\,\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ x&y&1\end{vmatrix}=(0,0,0)$ y $\mathbf{f}\in C^1(\mathbf{R}^3)$ $\Rightarrow$ $\mathbf{f}$ es conservativo. [De hecho la $U=xy+z$ se ve a ojo].
Su integral sobre toda línea cerrada es 0, sobre esa circunferencia en particular.

8. $\boxed{\mathbf{F}(x,y,z)=(x^2+y^2+z^2)(x,y,z)}$ $\mathrm{div}\,\mathbf{F}=2x^2+2y^2+2z^2+3(x^2+y^2+z^2)=5(x^2+y^2+z^2)=5\rho^2$. $\mathrm{rot}\,\mathbf{F}=\mathbf{0}$.

$\iiint_V\mathrm{div}\,\mathbf{F}\,dx\,dy\,dz=\int_0^1\int_0^{2\pi}\int_0^{\pi}5\rho^4\,\mathrm{sen}\,\phi\,d\phi\,d\theta\,d\rho=2\pi\big[-\cos\phi\big]_0^{\pi}=4\pi$.

$\mathbf{r}=(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)$, $\phi\in[0,\pi]$
$\mathbf{r}_\phi\times\mathbf{r}_\theta=\mathrm{sen}\,\phi\;\mathbf{r}(\phi,\theta)$, $\theta\in[0,2\pi]$, $\iint_{\partial V}\underbrace{\mathbf{F}\cdot\mathbf{n}}_{=1}\,dS=\int_0^{2\pi}\int_0^{\pi}\mathrm{sen}\,\phi\,d\phi\,d\theta=4\pi$ (= superficie de S).

9. a) $\boxed{\mathbf{f}(x,y,z)=\big(4x,4y,z^2\big)}$ en el cilindro $x^2+y^2\le 25$, $0\le z\le 2$. $\mathrm{div}\,\mathbf{f}=8+2z$.



$\iiint_V\mathrm{div}\,\mathbf{f}=$ [cilíndricas] $=\int_0^{2\pi}\int_0^5\int_0^2 r(8+2z)\,dz\,dr\,d\theta=2\pi\cdot\frac{25}{2}(16+4)=\boxed{500\pi}$.

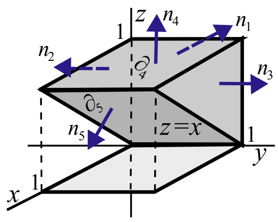
S dada por $\mathbf{r}=(5\cos\theta,5\,\mathrm{sen}\,\theta,z)$, $\mathbf{r}_\theta\times\mathbf{r}_z=5(\cos\theta,\mathrm{sen}\,\theta,0)$, $\theta\in[0,2\pi]$, $z\in[0,2]$.

Sobre ella: $\mathbf{f}(\mathbf{r}(\theta,z))=(20\cos\theta,20\,\mathrm{sen}\,\theta,z^2)$, $\iint_S\mathbf{f}\cdot d\mathbf{S}=\int_0^{2\pi}\int_0^2 100\,dz\,d\theta=400\pi$.

En la tapa superior T: $\mathbf{f}\cdot\mathbf{n}=(4x,4y,4)\cdot(0,0,1)=2\ \rightarrow\ \iint_{B_5}4=\pi\,5^2\,4=100\pi$.

En base B: $\mathbf{f}\cdot\mathbf{n}=(4x,4y,0)\cdot(0,0,-1)=0\ \rightarrow\ \iint_{B_5}0=0$. Por tanto, $\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}=400\pi+100\pi+0=\boxed{500\pi}$.

b) $\boxed{\mathbf{f}(x,y,z)=(0,0,z^2)}$ $\iiint_V\mathrm{div}\,\mathbf{f}=\int_0^1\int_0^1\int_x^1 2z\,dz\,dy\,dx=\int_0^1\big(1-x^2\big)\,dx=1-\frac{1}{3}=\boxed{\frac{2}{3}}$.



El flujo sobre 3 de las 5 caras del borde ∂V es nulo:

Sobre $x=0$ es $\mathbf{n}_1=(-1,0,0)\Rightarrow\mathbf{f}\cdot\mathbf{n}_1=0$, como su integral en el cuadrado trasero.

Sobre $y=0,1$, $\mathbf{n}_{2,3}=(0,\mp 1,0)\ \Rightarrow$ se anula la integral de $\mathbf{f}\cdot\mathbf{n}$ en ambos triángulos.

En la superior $z=1$, $\mathbf{n}_4=(0,0,1)\ \Rightarrow\ \mathbf{f}\cdot\mathbf{n}_4=1$, $\iint_{\partial_4}\mathbf{f}\cdot d\mathbf{S}=$ área del cuadrado $=1$.

Solo falta el flujo sobre el cuadrado inclinado ∂_5 que parametrizamos en la forma:

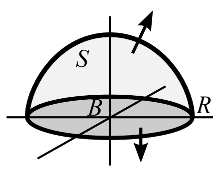
$r(x,y)=(x,y,x)$, $(x,y)\in[0,1]\times[0,1]$, con producto vectorial fundamental $r_x\times r_y=(-1,0,1)$.

Como este vector apunta hacia el interior, el flujo hacia el exterior es

$$\iint_{\partial_5}\mathbf{f}\cdot d\mathbf{S}=-\int_0^1\int_0^1(0,0,x^2)\cdot(-1,0,1)\,dx\,dy=-\int_0^1x^2dx=-\frac{1}{3}\,.$$

Con lo que el flujo sobre toda la frontera es $\iint_{\partial V}\mathbf{f}\cdot d\bar{S}=0+0+0+1-\frac{1}{3}=\boxed{\frac{2}{3}}$ como debía.

10. $\boxed{\mathbf{f}(x,y,z)=(\,y,-x,1)}$ Flujo $\Phi=\Phi_S+\Phi_B$ a través de la superficie cerrada $\partial V=S\cup B$ (semisuperficie esférica superior de radio R junto a su círculo base):



Parametrización $\mathbf{r}(\phi,\theta)$ de S: $\left.\begin{array}{l} x=R\,\mathrm{sen}\,\phi\cos\theta \\ y=R\,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta \\ z=R\cos\phi\end{array}\right\}$ $\begin{array}{l}\phi\in\big[0,\frac{\pi}{2}\big]\\ \theta\in[0,2\pi]\end{array}$ $\begin{array}{l}\mathbf{r}_\phi=(R\,c\phi\,c\theta,R\,c\phi\,s\theta,-R\,s\phi)\\ \mathbf{r}_\theta=(-R\,s\phi\,s\theta,R\,s\phi\,c\theta,0)\end{array}$

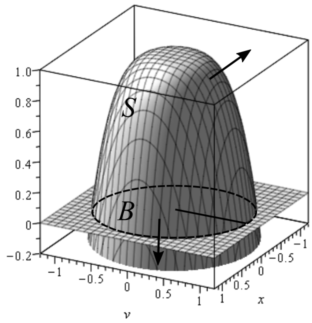
$\mathbf{r}_\phi\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ R\,c\phi\,c\theta&R\,c\phi\,s\theta&-R\,s\phi\\ -R\,s\phi\,s\theta&R\,s\phi\,c\theta&0\end{vmatrix}=R^2(s^2\phi\,c\theta,s^2\phi\,s\theta,s\phi\,c\phi)=R^2\,\mathrm{sen}\,\phi\,(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)$
$=R\,\mathrm{sen}\,\phi\,\mathbf{r}(\phi,\theta)$ (apunta hacia afuera).

$$\Phi_S=\iint_S\mathbf{f}\cdot d\mathbf{S}=\int_0^{\pi/2}\int_0^{2\pi}R^2\,\mathrm{sen}\,\phi\cos\phi\,d\theta\,d\phi=2\pi R^2\big[\tfrac{1}{2}\,\mathrm{sen}^2\phi\big]_0^{\pi/2}=\pi R^2\,.$$

Para B es $\mathbf{n}=(0,0,-1)$, $\mathbf{f}\cdot\mathbf{n}=-1\ \Rightarrow\ \Phi_B=\iint_B\mathbf{f}\cdot\mathbf{n}\,dS=-\iint_B dS=-\pi R^2$ [$=-$área]. $\Phi=\Phi_S+\Phi_B=\boxed{0}$.

Comprobamos el teorema de Gauss. Como $\nabla\cdot\mathbf{f}=0$: $\iiint_V\nabla\cdot\mathbf{f}\,dV=\iiint_V 0\,dV=\boxed{0}=\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}$.

11. $\boxed{\mathbf{f}(x,y,z)=(x,1,y)}$ $z=1-(x^2+y^2)^2$ corta $z=0$ en la circunferencia unidad.



$\operatorname{div}\mathbf{f}=1$. Cilíndricas: $\iiint_V \operatorname{div}\mathbf{f}=\int_0^{2\pi}\!\int_0^1\!\int_0^{1-r^4} r\,dz\,dr\,d\theta = 2\pi\int_0^1 (r-r^5)\,dr = \boxed{\frac{2\pi}{3}}$.

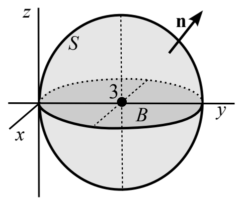
Sobre la base B: $(x,y,0)$, $\iint_{\bigcirc}(x,1,y)\cdot(0,0,-1)\,dx\,dy=-\iint_{\bigcirc} y\,dx\,dy=0$ (impar).

Sobre la tapa S: $\mathbf{r}(x,y)=\big(x,y,1-(x^2+y^2)^2\big)\to \mathbf{r}_x\times\mathbf{r}_y=\big(4x(),4y(),1\big)$ (exterior).

$$\iint_{\bigcirc}\big(\underbrace{4x^2(x^2+y^2)}_{\text{impar}}+\underbrace{4y(x^2+y^2)}_{\text{impar}}+y\big)\,dx\,dy=\int_0^{2\pi}\!\int_0^1 \underset{2r^5(1+\cos 2\theta)}{\underset{\|}{4r^5\cos^2\theta}}\,dr\,d\theta=\tfrac{1}{3}\cdot 2\pi=\iiint_{\partial V}\mathbf{f}\cdot d\mathbf{S}.$$

$\big[$O bien, $\mathbf{r}(r,\theta)=(r\cos\theta, r\,\mathrm{sen}\,\theta, 1-r^4)$, $r\in[0,1]$, $\theta\in[0,2\pi]$, ... $\big]$.

12. $\boxed{\mathbf{f}(x,y,z)=3yz\,\mathbf{i}+2xz\,\mathbf{j}+(z+xy)\,\mathbf{k}}$ $x^2-6x+y^2+z^2=0 \Leftrightarrow (x-3)^2+y^2+z^2=3^2$ [esfera de radio 3 centrada en $(3,0,0)$].



Como $\operatorname{div}\mathbf{f}=1$, según Gauss: $\iint_{\partial V}\mathbf{f}\cdot d\mathbf{S}=\iiint_V 1\,dx\,dy\,dz=$ volumen de $V=\frac{4}{3}\pi 3^3=36\pi$.

Calcularlo directamente sería largo, tanto usando esféricas centradas en el punto:

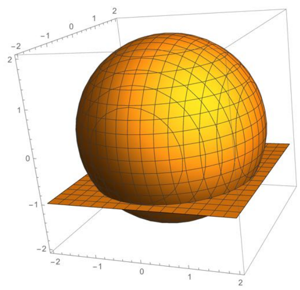
$\mathbf{r}(\phi,\theta)=(3+3\,\mathrm{sen}\,\phi\cos\theta, 3\,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta, 3\cos\phi)$, $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$,

como las parametrizaciones cartesianas: $\big(x,y,\pm\sqrt{6x-x^2-y^2}\big)$, $(x,y)\in B$ círculo de radio 3 centrado en $(3,0)$.

13. $\boxed{\mathbf{F}(x,y,z)=(-y,x,2)}$ a) Como $\operatorname{div}\mathbf{f}=0$ la $\iiint_V=\boxed{0}$. Hallemos el flujo hacia el exterior del volumen dado. La ∂V está compuesta por dos partes: S_1 sobre la esfera y S_2 en el plano $z=-1$.

En la esfera, si $\mathbf{r}(\phi,\theta)=2(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)$, la normal exterior es $\mathbf{r}_\phi\times\mathbf{r}_\theta=2\,\mathrm{sen}\,\phi\;\mathbf{r}(\phi,\theta)$.

Los ángulos varían: $0\le 2\pi$, $\cos\phi=-\frac{1}{2}\to 0\le\phi\le\frac{2\pi}{3}$. El flujo sobre la esfera es:



$$8\int_0^{2\pi/3}\!\int_0^{2\pi}(-\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\mathrm{sen}\,\phi\cos\theta,1)\cdot(\mathrm{sen}\,\phi\cos\theta,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta,\cos\phi)\,\mathrm{sen}\,\phi\,d\theta\,d\phi$$
$$=8\int_0^{2\pi/3}\!\int_0^{2\pi}\big(\cos\phi-\underbrace{2\,\mathrm{sen}\,\theta\cos\theta\,\mathrm{sen}^2\phi}_{\text{se anula}}\big)\,\mathrm{sen}\,\phi\,d\theta\,d\phi=8\pi\big[\mathrm{sen}^2\phi\big]_0^{2\pi/3}=6\pi.$$

El flujo en $z=-1$ es a través del círculo $x^2+y^2\le 3$, con vector normal $-\mathbf{k}$:

$\mathbf{F}\cdot(-\mathbf{k})=-2$, $\iint_{S_2}\mathbf{F}\cdot d\mathbf{s}=-2\iint_{x^2+y^2\le 3}dx\,dy=-2\,\big[\underbrace{\pi(\sqrt{3})^2}_{\text{área del círculo}}\big]=-6\pi$. $\boxed{6\pi-6\pi=0}$.

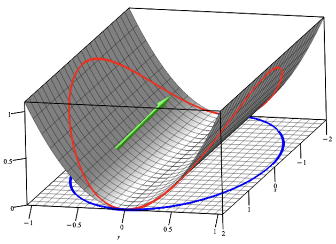
b) Para Stokes calculamos $\operatorname{rot}\mathbf{F}$: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial_x&\partial_y&\partial_z\\ -y&x&2\end{vmatrix}=2\mathbf{k}$. Su flujo a través de S_1 (hacia el exterior) es:

$$\iint_S \operatorname{rot}\mathbf{F}\cdot d\mathbf{S}=\int_0^{2\pi}\!\int_0^{2\pi/3}2\mathbf{k}\cdot 2\mathbf{r}\;\mathrm{sen}\,\phi\,d\phi\,d\theta=16\pi\int_0^{2\pi/3}\mathrm{sen}\,\phi\cos\phi\,d\phi=-8\pi\big[\cos^2\phi\big]_0^{2\pi/3}=\boxed{6\pi}.$$

En ∂S, circunferencia en $z=-1$ con centro en $(0,0,-1)$ y radio $\sqrt{3}$ (pues $x^2+y^2=4-1=3$) la integral de línea de $\mathbf{F}$ es (en el sentido correcto para que el vector normal vaya hacia el exterior):

$\mathbf{c}(t)=(\sqrt{3}\cos t,\sqrt{3}\,\mathrm{sen}\,t,-1)$, $\mathbf{c}'(t)=(-\sqrt{3}\,\mathrm{sen}\,t,\sqrt{3}\cos t,0)$, $\int_0^{2\pi}3\,(-s,c,2)\cdot(-s,c,0)\,dt=3\int_0^{2\pi}dt=\boxed{6\pi}$.

14. $\boxed{\mathbf{g}(x,y,z)=(x,xz,1)}$ En cartesianas $\mathbf{r}(x,y)=(x,y,y^2)$, con $(x,y)\in D$ elipse dada. $\mathbf{r}_x\times\mathbf{r}_y=(-f_x,-f_y,1)=(0,-2y,1)$ (que apunta hacia arriba).

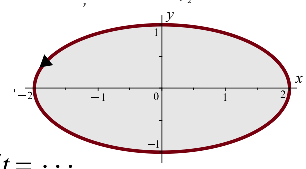


$\operatorname{rot}\mathbf{g}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial_x&\partial_y&\partial_z\\ x&xz&1\end{vmatrix}=(-x,0,z)$. $\operatorname{rot}\mathbf{g}\big(\mathbf{r}(x,y)\big)=(-x,0,y^2)$.

$$\iint_S \operatorname{rot}\mathbf{g}\cdot d\mathbf{S}=\iint_D(-x,0,y^2)\cdot(0,-2y,1)\,dx\,dy=\iint_D y^2\,dx\,dy.$$

Lo más directo es usar el cambio de las elipses:

$\begin{matrix}x=2r\cos\varphi\\ y=r\,\mathrm{sen}\,\varphi\end{matrix}$ con $J=2r\to\int_0^{2\pi}\!\int_0^1 2r^3\,\mathrm{sen}^2\varphi\,dr\,d\varphi=\frac{1}{4}\int_0^{2\pi}(1-\cos 2\varphi)\,d\varphi=\boxed{\frac{\pi}{2}}$.



En cartesianas $\int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}y^2\,dy\,dx\underset{\text{paridades}\uparrow}{=}4\int_0^2\!\int_0^{\sqrt{4-x^2}/2}y^2\,dy\,dx=\frac{1}{6}\int_0^2(4-x^2)^{3/2}dx$

$\underset{x=2\,\mathrm{sen}\,t\;\to}{}=\frac{8}{3}\int_0^{\pi/2}\cos^4 t\,dt=\frac{2}{3}\int_0^{\pi/2}(1+2\cos 2t+\cos^2 2t)\,dt=\cdots$

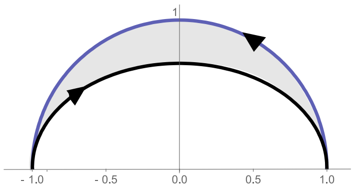
O más corto: $\int_{-1}^{1}\int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}}y^2\,dx\,dy=8\int_0^1 y^2\sqrt{1-y^2}\,dy=8\int_0^{\pi/2}\mathrm{sen}^2 t\cos^2 t\,dt=2\int_0^{\pi/2}\mathrm{sen}^2 2t\,dt=\frac{\pi}{2}$.

[O directamente: $\mathbf{r}(r,\varphi)=(2r\cos\varphi, r\,\mathrm{sen}\,\varphi, r^2\,\mathrm{sen}^2\varphi)$, $r\in[0,1]$, $\varphi\in[0,2\pi]$. $\mathbf{r}_r\times\mathbf{r}_\varphi=(0,-4r^2 s,2r)$ ...].

Prametrizamos ahora la ∂S en el sentido adecuado al $\mathbf{n}$: $\mathbf{c}(t)=(2\cos t,\mathrm{sen}\,t,\mathrm{sen}^2 t)$, $t\in[-\pi,\pi]\to$

$$\oint_{\partial S}\mathbf{g}\cdot d\mathbf{s}=\int_{-\pi}^{\pi}(2c,2cs^2,1)\cdot(-2s,c,2sc)\,dt=\int_{-\pi}^{\pi}\big[\underbrace{2\,\mathrm{sen}^2 t\cos^2 t}_{\text{par}}-\underbrace{2sc}_{\text{impar}}\big]dt=\int_0^{\pi}\mathrm{sen}^2 2t\,dt=\boxed{\frac{\pi}{2}}.$$

15. $\boxed{\mathbf{F}(x,y,z)=\big(y,2x+z,e^x\big)}$ $\quad\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial_x&\partial_y&\partial_z\\ y&2x+z&e^x\end{vmatrix}=\big(-1,-e^x,1\big)$ y $\mathbf{n}=\mathbf{k}$.



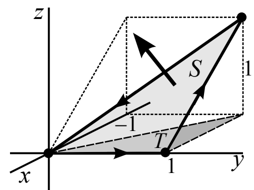
$(\nabla\times\mathbf{F})\cdot\mathbf{k}=1$. El flujo es el área (medio círculo, $\frac{\pi}{2}$, menos media elipse, $\frac{\pi}{2\sqrt2}$):

$$\iint_D dx\,dy=\int_{-1}^{1}\int_{\frac{1}{\sqrt2}\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\,dx=\Big(1-\tfrac{1}{\sqrt2}\Big)\int_{-1}^{1}\sqrt{1-x^2}\,dx\underset{x=\operatorname{sen}t}{=}\Big(1-\tfrac{1}{\sqrt2}\Big)\tfrac{\pi}{2}.$$

La circulación en el círculo: $\int_0^\pi(\operatorname{sen}t,2\cos t,e^{\cos t})\cdot(-\operatorname{sen}t,\cos t,0)\,dt=\int_0^\pi\big(2\cos^2t-\operatorname{sen}^2t\big)dt=\pi-\frac12\pi=\frac12\pi$.

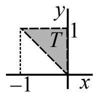
En la elipse (en otro sentido): $\int_\pi^0\big(\frac{1}{\sqrt2}\operatorname{sen}t,2\cos t,e^{\cos t}\big)\cdot\big(-\operatorname{sen}t,\frac{1}{\sqrt2}\cos t,0\big)dt=\frac{1}{\sqrt2}\int_\pi^0\big(2\cos^2t-\operatorname{sen}^2t\big)dt=\frac{-\pi}{2\sqrt2}$.

16. $\boxed{\mathbf{f}(x,y,z)=(yz,e^y,1)}$ $\to$ $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ yz&e^y&1\end{vmatrix}=(0,y,-z)$. vector normal: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 0&1&0\\ -1&1&1\end{vmatrix}=\mathbf{i}+\mathbf{k}$.



Triángulo sobre el plano $(1,0,1)\cdot(x,y,z)=x+z=0\to\mathbf{r}(x,y)=(x,y,-x)$ en T, $\mathbf{r}_x\overset{\uparrow}{\times}\mathbf{r}_y$

$$\iint_S\operatorname{rot}\mathbf{f}\cdot\mathbf{n}\,dS=\int_{-1}^{0}\int_{-x}^{1}(0,y,x)\cdot(1,0,1)\,dy\,dx=\int_{-1}^{0}(x+x^2)\,dx=\Big[\tfrac{x^2}{2}+\tfrac{x^3}{3}\Big]_{-1}^{0}=\boxed{-\tfrac16}.$$



Parametrizamos ∂S:

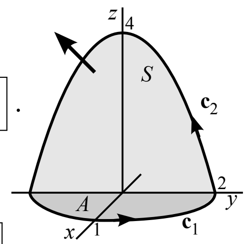
$\mathbf{c}_1(t)=(0,t,0),\ t\in[0,1]$ $\quad\mathbf{c}_1'=(0,1,0)$ $\quad\mathbf{f}(\mathbf{c}_1)=(0,e^t,1)$

$\mathbf{c}_2(t)=(-t,1,t),\ t\in[0,1]$ $\quad\mathbf{c}_2'=(-1,0,1)$ $\quad\mathbf{f}(\mathbf{c}_2)=(t,e,1)$ $\quad\to$

$\mathbf{c}_3(t)=(-t,t,t),\ t\in[1,0]$ $\quad\mathbf{c}_3'=(-1,1,1)$ $\quad\mathbf{f}(\mathbf{c}_3)=(t^2,e^t,1)$

$$\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_0^1e^t\,dt+\int_0^1(1-t)\,dt+\int_1^0(1-t^2+e^t)\,dt=e-1+1-\tfrac12-1+\tfrac13+1-e=\boxed{-\tfrac16}$$ como debía.

17. $\mathbf{f}(x,y,z)=(3,x^2,y)$, $\mathbf{r}(x,y)=(x,y,4-4x^2-y^2)$ $\to$ $\operatorname{rot}\mathbf{f}=(1,0,2x)$, $\mathbf{r}_x\times\mathbf{r}_y=(-f_x,-f_y,1)=(8x,2y,1)$.



$$\iint_S\operatorname{rot}\mathbf{f}\cdot d\mathbf{S}=\iint_A(1,0,2x)\cdot(8x,2y,1)\,dx\,dy=\int_{-2}^{2}\int_0^{\sqrt{1-y^2/4}}10x\,dx\,dy=\int_{-2}^{2}\Big[5-\tfrac54y^2\Big]dy=\boxed{\tfrac{40}{3}}.$$

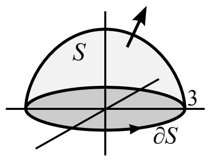
$\mathbf{c}_1(t)=(\cos t,2\operatorname{sen}t,0),\ t\in\big[-\frac{\pi}{2},\frac{\pi}{2}\big]$ $\quad\mathbf{c}_1'=(-\operatorname{sen}t,2\cos t,0)$ $\quad\mathbf{f}(\mathbf{c}_1)=(3,\cos^2t,2\operatorname{sen}t)$

$\mathbf{c}_2(t)=(0,-t,4-t^2),\ t\in[-2,2]$ $\quad\mathbf{c}_2'=(0,-1,-2t)$ $\quad\mathbf{f}(\mathbf{c}_2)=(3,0,-t)$ $\quad\to$

$$\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_{-\pi/2}^{\pi/2}\Big[2\cos t(1-\operatorname{sen}^2t)-\underset{\text{impar}}{3\operatorname{sen}t}\Big]dt+\int_{-2}^{2}2t^2dt=\Big[4\operatorname{sen}t-\tfrac43\operatorname{sen}^3t\Big]_0^{\pi/2}+\tfrac{32}{3}=\boxed{\tfrac{40}{3}}.$$

[Sale más largo parametrizando $\mathbf{c}_1(t)=\big((1-\frac{t^2}{4})^{1/2},t,0\big)$, $t\in[-2,2]$].

18. $\boxed{\mathbf{F}(x,y,z)=-y\,\mathbf{i}+2x\,\mathbf{j}+(x+z)\,\mathbf{k}}$. $\operatorname{rot}\mathbf{f}=(0,-1,3)$. Superficie $x^2+y^2+z^2=9$, $z\geq0$:



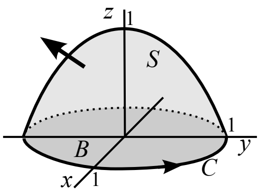
$\mathbf{r}(\phi,\theta)=\big(3\operatorname{sen}\phi\cos\theta,3\operatorname{sen}\phi\operatorname{sen}\theta,3\cos\phi\big)$, $\phi\in[0,\frac{\pi}{2}]$, $\theta\in[0,2\pi]$,

$\mathbf{r}_\phi\times\mathbf{r}_\theta=9(\operatorname{sen}^2\phi\cos\theta,\operatorname{sen}^2\phi\operatorname{sen}\theta,\operatorname{sen}\phi\cos\phi)$.

$$\iint_S\operatorname{rot}\mathbf{F}\cdot d\mathbf{S}=9\int_0^{2\pi}\int_0^{\pi/2}(3\operatorname{sen}\phi\cos\phi-\underset{0\swarrow}{\operatorname{sen}^2\phi\operatorname{sen}\theta})\,d\phi\,d\theta=27\pi\big[\operatorname{sen}^2\phi\big]_0^{\pi/2}=\boxed{27\pi}.$$

Además: $\mathbf{c}(t)=(3\cos t,3\operatorname{sen}t,0)$, $t\in[0,2\pi]$ $\to\oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}=9\int_0^{2\pi}\big[\operatorname{sen}^2t+2\cos^2t\big]dt=9\int_0^{2\pi}\frac{3+\cos t}{2}\,dt=\boxed{27\pi}$.

19. $\boxed{\mathbf{f}(x,y,z)=(x,xy,2z)}$ $\operatorname{rot}\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ x&xy&2z\end{vmatrix}=(0,0,y)$. $\quad S$: $\mathbf{r}(x,y)=(x,y,1-x^2-y^2)$. $\mathbf{r}_x\times\mathbf{r}_y=(2x,2y,1)$.



$$\iint_S\operatorname{rot}\mathbf{f}\cdot d\mathbf{S}=\iint_B y\,dx\,dy=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}y\,dy\,dx=\boxed{0}$$ [o impar y recinto simétrico].

$\mathbf{c}(t)=(\cos t,\operatorname{sen}t,0)$, $t\in[0,2\pi]$ $\to\oint_{\partial S}\mathbf{f}\cdot d\mathbf{s}=\int_0^{2\pi}(-\operatorname{sen}t\cos t+\cos^2t\operatorname{sen}t)\,dt=\boxed{0}$.

$\operatorname{div}\mathbf{f}=3+x$. $\iiint_V\operatorname{div}\mathbf{f}=[\text{cilíndricas}]=\int_0^{2\pi}\int_0^1\int_0^{1-r^2}r(3+r\cos\theta)\,dz\,dr\,d\theta=6\pi\int_0^1(r-r^3)\,dr=\boxed{\tfrac32\pi}$.

$$\iint_S\mathbf{f}\cdot d\mathbf{S}=\iint_B\underbrace{(x,xy,2-2x^2-2y^2)\cdot(2x,2y,1)}_{=2(1+xy-y^2)}\,dx\,dy=2\int_0^{2\pi}\int_0^1\big(r+r^3\cos\theta\operatorname{sen}\theta-r^3\operatorname{sen}^2\theta\big)\,dr\,d\theta=\tfrac32\pi.$$

$\mathbf{r}_B(x,y)=(x,y,0)$, $(x,y)\in B$, $\iint_B\mathbf{f}\cdot\mathbf{n}\,dS=-\iint_B(x,xy,0)\cdot(0,0,-1)\,dx\,dy=0$. $\quad\iint_{S^*}=\iint_S+\iint_B=\boxed{\tfrac32\pi}$.

20. a) $u\,\Delta u=u\big(u_{xx}+u_{yy}\big)=\operatorname{div}\big(uu_x,uu_y\big)-\big(u_x^2+u_y^2\big)=\operatorname{div}\big(u\nabla u\big)-\|\nabla u\|^2$ (y casi igual para $n=3$).

b) $\iint_D u\,\Delta u\,dx\,dy=\iint_D\operatorname{div}\big(u\nabla u\big)dx\,dy-\iint_D\operatorname{div}\|\nabla u\|^2dx\,dy\overset{\text{TD}}{=}\oint_{\partial D}u\,u_{\mathbf{n}}\,ds-\iint_D\|\nabla u\|^2dx\,dy$, pues $u\nabla u\cdot\mathbf{n}=u\,u_{\mathbf{n}}$.

c) Gauss $\to\iiint_V u\,\Delta u\,dx\,dy\,dz=\iint_{\partial V}u\,u_{\mathbf{n}}\,dS-\iiint_V\|\nabla u\|^2dx\,dy\,dz$. d) En $\mathbf{R}$: $\int_a^b uu''dx=uu'\big]_a^b-\int_a^b(u')^2dx$ (partes).